CLASSROOM NOTES

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THE IMPOSSIBILITY OF A DIVISION ALGEBRA OF VECTORS IN THREE DIMENSIONAL SPACE

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When the student discovers that plane vectors, under the identification $(a, b)=a+bi$, satisfy all the field axioms (i.e. are a commutative division algebra over the reals), he ought to wonder whether something similar might be done with vectors in space. When he meets the dot and cross products, he ought to ask why the latter (in spite of its plausible interpretation) does not specialize to the plane and whether something more like the algebra of numbers could be constructed.

The historical record attests to the naturalness and importance of such questions. Indeed they seem to have troubled mathematicians as soon as the nature of complex numbers began to become clear. After many years of pondering imaginaries and of experimenting with operations in space, Gauss in 1831 wrote (in [1]): "The writer has reserved for himself . . . the question why the relations between things that make up a manifold of more than two dimensions cannot provide quantities admissible in universal arithmetic." In 1833 Hamilton (see [2]) arrived at the first really modern treatment of complex numbers as ordered pairs of reals and immediately attempted a generalization to triples. After ten years of intensive and futile experimentation, he took the leap to quaternions without proving the impossibility in 3-space. Grassmann [3] began with efforts to develop an algebra of space vectors and went on to his very general algebras without being able to generalize complex numbers to three dimensions or to prove the impossibility of doing so. De Morgan and others experimented with multiple algebras. Weierstrass in his lectures from 1861 on discussed the Gaussian question ([4] p. 361; [5] p. 312; [6] pp. 24–27), but Hankel in 1867 first printed a proof that no hypercomplex number system could satisfy all the laws of algebra. He wrote ([7] pp. 106–108) " . . . thus is answered the question whose solution was promised but not given by Gauss." Another answer was given after ten years by Frobenius [8], C. S. Peirce [9], E. Cartan [10], and Grisseman [11], who proved that only one more division algebra, namely quaternions, is made possible by dropping the commutativity of multiplication. As recently as 1958, Milnor, Bott, and others [12] proved that the only possible division algebras over the reals (without assuming either the commutative or associative laws of multiplication) are of dimension 1 (the reals), 2 (complex numbers), 4 (quaternions), and 8 (Cayley numbers). The sagacity of Cartan's remark of 1908 ([4] p. 362) that " . . . a definitive answer, if one exists, can only be given by the whole ulterior development of algebra and analysis," may be verified by observing the role played by the Gaussian question in general algebra. (See, for example, [13] Chs. 3–6, [14] Ch. V.)

Although the results cited above make abundantly clear the restrictions on algebra in three dimensions, the literature does not contain an accessible and understandable answer to the question most naturally asked by a student of elementary calculus today, namely, whether multiplication of vectors in 3-space can be so defined that there results a natural generalization of complex numbers just as these were the natural generalization of real numbers. More precisely, we consider the set of triples of reals (x, y, z) with the identifications $(x, 0, 0)=x$ and $(x, y, 0)=(x, y)=x+iy$, with the usual definitions of addition and multiplication by a scalar, and (hopefully) with a multiplication of vectors somehow defined so that all the field properties hold.

That these demands are very inconsistent the student can easily discover for himself if he experiments without getting bogged down (as did Hamilton!) in trying particular definitions of multiplication. For example, letting $(x, y, z)=x+yi+zj$, closure under multiplication implies that for some real a, b, c

$$ij = a + bi + cj. \tag{1}$$

If we multiply both members by i and substitute from (1) in the result, we find

$$(ac - b) + (a + bc)i + (c^2 + 1)j = 0. \tag{2}$$

Hence, $c^2+1=0$ and c is not real. (Compare [8], [9].)

Since the above argument does not use commutativity, we have proved the impossibility in three dimensions even if that property is not required. Associativity was used when we multiplied (1) by i, but the following example of a zero product neither of whose factors is zero shows that even with neither associativity nor commutativity the other properties of a division algebra cannot hold in three dimensions.

$$(i - c)(ac - b + (bc + a)i + (c^2 + 1)j) = 0. \tag{3}$$

A slightly more complicated example in a more general context was given by Dickson in 1935 ([15] pp. 113–115). Others may be found by examining the equation $(A+Bi+Cj)(D+Ei+Fj)=0$.

There are other questions suitable for student investigation. Would a new definition of multiplication of complex numbers make a difference? (See [4] pp. 351–353 or [16]). What about dimensions higher than three? Complex coefficients? (Consider $j^2=\alpha+\beta i+\gamma j$.) Peirce's proof in [9] is readable without extensive background and may lead the student toward deeper study. The Gaussian question, which inspired so much of the development of modern algebra, might be used to good effect as motivation in teaching today.

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A NEW MODEL OF THE HYPERBOLIC PLANE

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On entering a course in hyperbolic geometry many a student expects to encounter hyperbolas and to have much to do with them. If the course follows a metric rather than projective approach, he will find that its development does not involve hyperbolas directly at all, his closest contact with these curves being via the hyperbolic functions. Nor will he find that there are any hyperbolas in any of the familiar models of the hyperbolic plane. Under these circumstances it is of interest to note that in the model to be described presently hyperbolas play a fundamental role: the branches of a certain set of hyperbolas correspond to the straight lines in hyperbolic plane geometry. As this suggests, the model, unlike familiar models, utilizes the entire Euclidean plane rather than some special part of it.

As a first step in describing the model let us consider the mapping in the Euclidean plane with equations

$$(1) \qquad x' = \frac{x}{\sqrt{(x^2 + y^2 + 1)}}, \qquad y' = \frac{y}{\sqrt{(x^2 + y^2 + 1)}}.$$