Solution to exercise 2.2??

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Let $(K, |\cdot|)$ be a normed field. Fix a positive integer n and let P_n denote the set of all degree n polynomials over K which have distinct roots in $\overline{K}$. There is a natural injection $P_n \hookrightarrow K^{n+1}$ given by $\sum_{i=0}^{n} a_i x^i \mapsto (a_0, \ldots, a_n)$. We denote by $D(n)$ the image of P_n under this map.

We will denote by $\mathbb{A}^{n+1}$ the set K^{n+1} endowed with the Zariski topology, and we will reserve the notation K^{n+1} for the same set endowed with the product topology.

(a) *The set $D(n)$ is open in $\mathbb{A}^{n+1}$.*

If $p(x) \in K[x]$ is any polynomial, we may consider its discriminant $\Delta(p)$, which is an element of K having the property that $\Delta(p) = 0$ if and only if p has a repeated root. Moreover, there is an explicit formula for $\Delta(p)$ as a polynomial in the coefficients of p. We therefore have a polynomial $\Delta \in K[t_0, \ldots, t_n]$ such that $\mathbb{A}^{n+1} \backslash D(n)$ is the union of the zero set of Δ with the zero set of the last coordinate function. Since this union is clearly a closed set, it follows that $D(n)$ is open.

(b) *The set $D(n)$ is open in K^{n+1}.*

This follows immediately from the fact that the product topology is finer than the Zariski topology. To see this, suppose that Z is closed in the Zariski topology, so that Z is the zero set of a collection of polynomials $f_i \in K[t_0, \ldots, t_n]$. Each f_i, viewed as a map $f_i : K^{n+1} \to K$, is continuous (this follows easily from the axioms of a normed field). Therefore, its zero set $Z(f_i)$ is closed, being the inverse image of the closed set $\{0\}$ under f_i. Since $Z = \bigcap_i Z(f_i)$, then clearly Z is closed in K^{n+1}.

We introduce some notation: for a polynomial $f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0$ we set $|f| = \max |a_i|$. A simple consequence of the triangle inequality which we shall need is that the absolute value of every root of f is bounded above by the number $\max\left(1, \sum_{i=0}^{n-1} \frac{|a_i|}{|a_n|}\right)$. (Note that we are implicitly using the fact that the norm on K extends to $\overline{K}$.)

(c) *Let $(K, |\cdot|)$ be a normed field and let $f \in K[t]$ have degree n. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $g \in K[t]$ has degree n and $|f - g| < \delta$ then every root of g is within ε of a root of f and vice versa.*

Let $f = a_n t^n + \cdots + a_0 = a_n(t - \alpha_1)\cdots(t - \alpha_n)$ and let $\varepsilon > 0$ be arbitrary. Choose any $\delta > 0$ such that $\delta < \min\left(\frac{|a_n|}{2}, \frac{|a_n|\varepsilon^n}{\sum_{i=0}^n M^i}, \frac{|a_n|\varepsilon^n}{2\sum_{i=0}^n N^i}\right)$, where $M = \sum_{i=0}^{n-1}\left(1 + 2\frac{|a_i|}{|a_n|}\right)$ and $N = \max\left(1, \sum_{i=0}^{n-1} \frac{|a_i|}{|a_n|}\right)$. Suppose that $g = b_n t^n + \cdots + b_0 \in K[t]$ satisfies $|f - g| < \delta$, and let β be any root of g. Then we know $|\beta| \leq \max\left(1, \sum_{i=0}^{n-1} \frac{|b_i|}{|b_n|}\right)$, and it is easy to see that $\frac{|b_i|}{|b_n|} \leq 1 + 2\frac{|a_i|}{|a_n|}$, so $|\beta| \leq M$.

We thus have that

$$|f(\beta)| = |f(\beta) - g(\beta)| \leq \sum_{i=0}^{n} |a_i - b_i||\beta|^i < \delta \sum_{i=0}^{n} M^i < |a_n|\varepsilon^n.$$

Therefore, $|a_n| \prod_{i=1}^n |\beta - \alpha_i| < |a_n|\varepsilon^n$, so $\prod_{i=1}^n |\beta - \alpha_i| < \varepsilon^n$ and hence one of the factors $|\beta - \alpha_i|$ must be smaller than ε. This shows that β is within ε of a root of f.

Now let α be any root of f. Then $|\alpha| \leq N$ so arguing as above we see that $|g(\alpha)| < \delta \sum_{i=0}^n N^i < \frac{|a_n|\varepsilon^n}{2} < |b_n|\varepsilon^n$, so we conclude that there is some root β of g such that $|\beta - \alpha| < \varepsilon$.

(d) *Suppose that $f \in K[t]$ has degree n and has n distinct roots. Then there is a $\delta > 0$ such that if $g \in K[t]$ has degree n and $|f - g| < \delta$ then g also has n distinct roots.*

Let $\alpha_1, \ldots, \alpha_n$ be the roots of f and choose $\varepsilon > 0$ such that the balls $B(\alpha_i, \varepsilon)$ are pairwise disjoint. Let δ be as in part (c) and suppose $g \in K[t]$ has degree n and satisfies $|f - g| < \delta$. Then by part (c), every root of f is within ε of a root of g, so g must have a root $\beta_i \in B(\alpha_i, \varepsilon)$. Since these balls are disjoint, the roots $\beta_1, \ldots, \beta_n$ are distinct, and therefore g has n distinct roots.