Solution to exercise 2.2??

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Let \((K, |\cdot|)\) be a normed field. Fix a positive integer \(n\) and let \(P_n\) denote the set of all degree \(n\) polynomials over \(K\) which have distinct roots in \(\overline{K}\). There is a natural injection \(P_n \hookrightarrow K^{n+1}\) given by \(\sum_{i=0}^{n} a_i x^i \mapsto (a_0, \ldots, a_n)\). We denote by \(D(n)\) the image of \(P_n\) under this map.

We will denote by \(A^{n+1}\) the set \(K^{n+1}\) endowed with the Zariski topology, and we will reserve the notation \(K^{n+1}\) for the same set endowed with the product topology.

(a) The set \(D(n)\) is open in \(A^{n+1}\).

If \(p(x) \in K[x]\) is any polynomial, we may consider its discriminant \(\Delta(p)\), which is an element of \(K\) having the property that \(\Delta(p) = 0\) if and only if \(p\) has a repeated root. Moreover, there is an explicit formula for \(\Delta(p)\) as a polynomial in the coefficients of \(p\). We therefore have a polynomial \(\Delta \in K[t_0, \ldots, t_n]\) such that \(A^{n+1} \setminus D(n)\) is the union of the zero set of \(\Delta\) with the zero set of the last coordinate function. Since this union is clearly a closed set, it follows that \(D(n)\) is open.

(b) The set \(D(n)\) is open in \(K^{n+1}\).

This follows immediately from the fact that the product topology is finer than the Zariski topology. To see this, suppose that \(Z\) is closed in the Zariski topology, so that \(Z\) is the zero set of a collection of polynomials \(f_i \in K[t_0, \ldots, t_n]\). Each \(f_i\), viewed as a map \(f_i : K^{n+1} \to K\), is continuous (this follows easily from the axioms of a normed field). Therefore, its zero set \(Z(f_i)\) is closed, being the inverse image of the closed set \(\{0\}\) under \(f_i\). Since \(Z = \bigcap_i Z(f_i)\), then clearly \(Z\) is closed in \(K^{n+1}\).
We introduce some notation: for a polynomial \( f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0 \) we set \( |f| = \max |a_i| \).

A simple consequence of the triangle inequality which we shall need is that the absolute value of every root of \( f \) is bounded above by the number \( \max \left( 1, \sum_{i=0}^{n-1} |a_i|/|a_n| \right) \). (Note that we are implicitly using the fact that the norm on \( K \) extends to \( \overline{K} \)).

(c) Let \((K, |\cdot|)\) be a normed field and let \( f \in K[t] \) have degree \( n \). Then for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that if \( g \in K[t] \) has degree \( n \) and \( |f - g| < \delta \) then every root of \( g \) is within \( \varepsilon \) of a root of \( f \) and vice versa.

Let \( f = a_n t^n + \cdots + a_0 = a_n (t - \alpha_1) \cdots (t - \alpha_n) \) and let \( \varepsilon > 0 \) be arbitrary. Choose any \( \delta > 0 \) such that \( \delta < \min \left( \frac{|a_n|}{2}, \frac{|a_n| \varepsilon^n}{\sum_{i=0}^{n-1} M_i \cdot 2 \sum_{i=0}^{n-1} N_i} \right) \), where \( M = \sum_{i=0}^{n-1} (1 + 2 |a_i|/|a_n|) \) and \( N = \max \left( 1, \sum_{i=0}^{n-1} |a_i|/|a_n| \right) \).

Suppose that \( g = b_n t^n + \cdots + b_0 \in K[t] \) satisfies \( |f - g| < \delta \), and let \( \beta \) be any root of \( g \). Then we know \( |eta| \leq \max \left( 1, \sum_{i=0}^{n-1} |b_i|/|b_n| \right) \), and it is easy to see that \( |b_i|/|b_n| \leq 1 + 2 |a_i|/|a_n| \), so \( |eta| \leq M \).

We thus have that
\[
|f(\beta)| = |f(\beta) - g(\beta)| \leq \sum_{i=0}^{n} |a_i - b_i||\beta|^i < \delta \sum_{i=0}^{n} M^i < |a_n| \varepsilon^n.
\]

Therefore, \( |a_n| \prod_{i=1}^{n} |\beta - \alpha_i| < |a_n| \varepsilon^n \), so \( \prod_{i=1}^{n} |\beta - \alpha_i| < \varepsilon^n \) and hence one of the factors \( |\beta - \alpha_i| \) must be smaller than \( \varepsilon \). This shows that \( \beta \) is within \( \varepsilon \) of a root of \( f \).

Now let \( \alpha \) be any root of \( f \). Then \( |\alpha| \leq N \) so arguing as above we see that \( |g(\alpha)| < \delta \sum_{i=0}^{n} N^i < |a_n| \varepsilon^n \), so we conclude that there is some root \( \beta \) of \( g \) such that \( |\beta - \alpha| < \varepsilon \).

(d) Suppose that \( f \in K[t] \) has degree \( n \) and has \( n \) distinct roots. Then there is a \( \delta > 0 \) such that if \( g \in K[t] \) has degree \( n \) and \( |f - g| < \delta \) then \( g \) also has \( n \) distinct roots.

Let \( \alpha_1, \ldots, \alpha_n \) be the roots of \( f \) and choose \( \varepsilon > 0 \) such that the balls \( B(\alpha_i, \varepsilon) \) are pairwise disjoint.

Let \( \delta \) be as in part (c) and suppose \( g \in K[t] \) has degree \( n \) and satisfies \( |f - g| < \delta \). Then by part (c), every root of \( f \) is within \( \varepsilon \) of a root of \( g \), so \( g \) must have a root \( \beta_i \in B(\alpha_i, \varepsilon) \). Since these balls are disjoint, the roots \( \beta_1, \ldots, \beta_n \) are distinct, and therefore \( g \) has \( n \) distinct roots.