Extension of a valuation in an arbitrary field extension

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The purpose of this note is to prove the following result:

Let (K,v) be a valued field and L/K be an arbitrary extension. Then there is a valuation on L extending v.

The idea of the proof is to use the known case of a finite extension, and to build on the results of exercises 2.2 and 2.3. The argument is a standard application of Zorn's lemma.

Let Σ be the collection of all valued fields (F, w) such that F is an intermediate extension $K \subseteq F \subseteq L$ and w extends v. We define a partial order on Σ by $(F_1, w_1) \leq (F_2, w_2)$ if $F_1 \subseteq F_2$ and w_2 extends w_1. Clearly this makes $(\Sigma, \leq)$ into a nonempty partially ordered set. Suppose now that $\{(F_a, v_a)\}$ is a totally ordered subset of Σ and let $f = \bigcup_a F_a$. Then $K \subseteq f \subseteq L$ and by exercise 2.2 there is a valuation w on f extending each v_a. Since each v_a extends v, then w is an extension of v. Therefore, $(f, w) \in \Sigma$ is an upper bound for the chain $\{(F_a, v_a)\}$.

By Zorn's lemma, there is a maximal element $(F, W) \in \Sigma$. We claim that $F = L$. To see this, let $t \in L$. The valuation W can be extended to $F(t)$ (by exercise 2.3 if t is transcendental over F, and by known results if t is algebraic), which implies that $F(t) = F$ by maximality of (F, W), and hence $t \in F$. This shows that $F = L$ and we conclude that the valuation v can be extended to L.