Exercise: Endler, *Valuation Theory*, (3.9). *Let (K, v) be a non-Archimedean valued field of residue characteristic $p > 0$. [Remark: The word "residue" was omitted when this problem was first posted, which makes the problem almost trivial.] Suppose that x and y both lie in the valuation ring. Show that $v(x - y) > 0$ implies $v(x^p - y^p) > v(x - y)$.*

[stankewicz] Following a hint, let's recall that $x = y + (x - y)$ so

$$\begin{aligned}
x^p - y^p &= (y + (x - y))^p - y^p \\
&= y^p + \binom{p}{1} y^{p-1}(x - y) + \cdots + \binom{p}{p-1} y(x - y)^{p-1} + (x - y)^p - y^p \\
&= (x - y)\left((x - y)^{p-1} + \sum_{i=1}^{p-1} \binom{p}{i} y^{p-i}(x - y)^{i-1}\right) \\
&= (x - y)\left((x - y)^{p-1} + p\sum_{i=1}^{p-1} \binom{p-1}{i} y^{p-i}(x - y)^{i-1}\right)
\end{aligned}$$

Define c to be $\sum_{i=1}^{p-1} \binom{p-1}{i} y^{p-i}(x - y)^{i-1}$ and note that c is in the valuation ring (this holds for any commutative ring since we've only really used the binomial theorem and the distributive law so far) because x, y and $\left\{\binom{p-1}{i}\right\}_{i=1}^{p-1}$ are.

Thus we have shown that $x^p - y^p = (x - y)\left((x - y)^{p-1} + pc\right)$. Thus by the nonarchimedean assumption

$$\begin{aligned}
v(x^p - y^p) &= v\left((x - y)\left((x - y)^{p-1} + pc\right)\right) \\
&= v(x - y) + v\left((x - y)^{p-1} + pc\right) \\
&\geq v(x - y) + \min\left\{(p - 1)v(x - y), v(pc)\right\}
\end{aligned}$$

Now recall that $v(pc) = v(p) + v(c) \geq v(p)$ since c lies in the valuation ring. Moreover, the residue characteristic is p which implies among other things that p lies in the maximal ideal of of the valuation ring and thus has positive valuation. Therefore $v(pc) > 0$. Moreover, $v(x - y) > 0$ by assumption, so $\min\left\{(p - 1)v(x - y), v(pc)\right\} > 0$.

Combining all this together, we find that

$$v(x^p - y^p) \geq v(x - y) + \min\left\{(p - 1)v(x - y), v(pc)\right\} > v(x - y)$$

as claimed. ■