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The $p$-torsion of elliptic curves is uniformly bounded. (Russian)

This paper is a very important contribution to the arithmetic on elliptic curves. Let $X$ be an elliptic curve defined over the field of complex numbers $C$. Let $D$ be a cyclic subgroup of $X$ of order $p^m$. The absolute invariants $j(X)$ and $j(X/D)$ are connected by the modular equation of level $p^m F_m[j(X), j(X/D)] = 0$. Let $K$ be a number field. The above correspondence commutes with the action of $\text{Gal}(\overline{K}/K)$.

The author shows that for large values of $m$ the curve $F_m(X, Y) = 0$ contains a finite number of $K$-rational points. This plus the fact that the torsion of curves with fixed invariant $j \in K$ is bounded [V. A. Dem'janenko, Mat. Zametki 3 (1968), 271–278; MR0227166 (37 #2751)] gives the main result of the paper: If $K$ is a number field, then there exists a constant $c$ such that the order of the $p$-torsion group of the $K$-rational points of an elliptic curve defined over $K$ does not exceed $c$.

The proof of the finiteness of the number of $K$-rational points on $F_m(X, Y) = 0$ is based on the following result: Let $X (X(K), A(K))$ denote sets of $K$-rational points on $X$ and $A$, respectively) be a curve and $A$ a $K$-simple abelian variety contained in the jacobian of $X$ with multiplicity $m(X, A)$ (to within isogeny). If $m(X, A) > \text{rk} A(k)/\text{rk End}_k A$, then the set $X(K)$ is finite.

The above is obtained as a corollary to the following important theorem proved in this paper: Let $A(X)$ be the group of $K$-homomorphisms of a normal projective variety $X$ into an abelian variety $A$ taking a fixed point $x \in X(K)$ into zero. Assume that the rank of the Néron-Severi group of $X$ is 1. If $\text{rk} A(X) > \text{rk} A(K)$, then the set $X(K)$ is finite.

The author also proves the following: Let $X$ be an elliptic curve over $k$ that has no complex multiplication. Then on the set of its $K$-forms the order of a maximal cyclic $p$-subgroup rational over $K$ is bounded.

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