SOLUTION TO EXERCISE 5.3

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Let K be a finite extension of $\mathbb{Q}_p$ with ramification index e. Let v be the usual integer-valued discrete valuation on K, $R = \{x \in K | v(x) \geq 0\}$ the valuation ring, and $\mathfrak{m} = \{x \in K | v(x) > 0\}$ the unique maximal ideal.

Let $\mathcal{U}^n = 1 + \mathfrak{m}^n$. As $\mathcal{U}^n$ is the kernel of the map on unit groups induced by the quotient $R \to R/\mathfrak{m}^n$, it is, in particular, a multiplicative group.

Claim. *For $n \geq \max\{\frac{2e}{p-1}, e\log_p 2\}$, there is a canonical isomorphism of groups $L : (\mathcal{U}^n, \cdot) \to (\mathfrak{m}^n, +)$.*

Proof. We will give this isomorphism and its inverse explicitly, via "logarithmic" and "exponential" maps, which we first introduce below as formal power series.

$$L(t) := \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(t-1)^k}{k}, \ E(t) := \sum_{k=0}^{\infty} \frac{t^k}{k!} \in \mathbb{Q}[[t]].$$

We recognize L as the usual Taylor expansion for $\log t$ around $t = 1$, and E as the expansion of e^t around $t = 0$. As functions on $\mathbb{C}$, L and E are inverses on small discs around 1 and 0, respectively, so in particular, we must have the equality $L(E(t)) = E(L(t)) = t$ as formal power series with $\mathbb{Q}$-coefficients (and hence also as formal power series with coefficients in any field of characteristic 0). Similarly, we have the formal identities required to eventually assert that L and E define appropriate group homomorphisms, namely $L(xy) = L(x) + L(y)$ and $E(x+y) = E(x)E(y)$. With these facts in mind, it will suffice to show the following for $n \geq \max\{\frac{2e}{p-1}, e\log_p 2\}$:

i) If $x \in \mathcal{U}^n$, then $L(x)$ converges as an infinite sum in K, hence defining a homomorphism $L : (\mathcal{U}^n, \cdot) \to (K, +)$.
ii) $L(\mathcal{U}^n) \subset \mathfrak{m}^n$.
iii) If $x \in \mathfrak{m}^n$, then $E(x)$ converges as an infinite sum in K, which therefore, assuming all values are nonzero, defines a homomorphism $E : (\mathfrak{m}^n, +) \to (K^\times, \cdot)$.
iv) $E(\mathfrak{m}^n) \subset \mathcal{U}^n$.

For i), we see that if $x \in \mathcal{U}^n$, then $v((x-1)^k) \geq nk$, whereas $v(k) \leq e\log_p k$. Therefore $v(\frac{(x-1)^k}{k}) \to \infty$, so $\frac{(x-1)^k}{k} \to 0$, and hence $L(x)$ converges.

For ii), however, we need that $v(\frac{(x-1)^k}{k}) \geq n$ for ALL k, for which it would suffice to have $nk - e\log_p k \geq n$ for all k. One can check that this inequality is guaranteed for all k if we merely insist it for $k = 2$, yielding $n \geq e \log_p 2$.

For iii), we see that if $x \in \mathfrak{m}^n$, then $v(x^k) \geq nk$, whereas

$$v(k!) \leq v(p)(\frac{k}{p} + \frac{k}{p^2} + ...) = ek(\frac{\frac{1}{p}}{1 - \frac{1}{p}}) = \frac{ek}{p-1} \leq \frac{nk}{2},$$

provided $n \geq \frac{2e}{p-1}$. Therefore, $v(\frac{x^k}{k!}) \to \infty$, so $\frac{x^k}{k!} \to 0$, hence $E(x)$ converges.

For iv), we need that

$$v(\frac{x^k}{k!}) \geq nk - \frac{ek}{p-1} \geq n \text{ for all } x \in \mathfrak{m}^n, k \geq 1,$$

which again holds provided $n \geq \frac{2e}{p-1}$.

□

Note that in the special case of a totally unramified extension, in particular $\mathbb{Q}_p$ itself, this isomorphism is valid for all $n \in \mathbb{N}$ when $p \geq 3$, and for $n \geq 2$ for $p = 2$.

Remark [PLC]: This result is as false as it gets for local fields of positive characteristic (c.f. Exercise 5.4): there is no nontrivial group homomorphism from the unit group $\mathbb{F}_q[[t]]^\times$ to the additive group $\mathbb{F}_q[[t]]$, since the former has no elements of order p and the latter is a p-torsion group. This may be viewed as an analytic indication of the fact that the additive group and the multiplicative group are distinct **formal groups** in positive characteristic, which has implications in the theory of abelian varieties. Also note that the proof does not have a chance to get started because the formal power series which define the logarithm and the exponential are not well-defined in positive characteristic. Nevertheless, one sometimes encouters the **Artin-Hasse exponential** $\sum_{n=0}^{p-1} \frac{t^n}{n!}$ when doing arithmetic geometry in positive characteristic...