7. Steven G. Krantz, *Complex Analysis: The Geometry Viewpoint*, Carus Mathematical Monographs, 23, Math. Assoc. of Amer., Washington, DC, 1990.
8. Serge Lang, *Complex Analysis*, 3rd ed. Springer, New York, 1993, p. 213.
9. Arlan Ramsay and Robert D. Richtmyer, *Introduction to Hyperbolic Geometry*, Springer, New York, 1995.
10. Hans Schwerdtfeger, *Geometry of Complex Numbers, Circle Geometry, Möbius Transformations, Non-Euclidean Geometry*, Dover, New York, 1979, p. 146.
11. William P. Thurston, *Three-Dimensional Geometry and Topology*, Vol. 1, ed. by Silvio Levy, Princeton Univ. Press, New Jersey, 1997, p. 81.
12. Abraham A. Ungar, The holomorphic automorphism group of the complex disk, *Aequat. Math.* **47** (1994) 240–254.
13. Abraham A. Ungar, Thomas precession: its underlying gyrogroup axioms and their use in hyperbolic geometry and relativistic physics, *Found. Phys.* **27** (1997) 881–951.
14. Abraham A. Ungar, From Pythagoreas to Einstein: The Hyperbolic Pythagorean Theorem, *Found. Phys.* **28** (1998) 1283–1321.
15. Edward C. Wallace and Steven F. West, *Roads to Geometry*, 2nd ed., Prentice Hall, NJ, 1998, pp. 362–363.

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Is the Composite Function Integrable?

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It is well known that the composition of two continuous functions is continuous and hence Riemann integrable. However, the composition of two Riemann integrable functions may or may not be Riemann integrable. For example, let

$$f(y) = \begin{cases} 1 & \text{when } y \neq 0, \\ 0 & \text{when } y = 0, \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{when } x \text{ is an irrational number}, \\ \frac{1}{p} & \text{when } x = \frac{q}{p}, \text{ where } p \text{ and } q \text{ are two coprime integers}. \end{cases}$$

Then

$$f \circ g(x) = \begin{cases} 0 & \text{when } x \text{ is an irrational number}, \\ 1 & \text{when } x = \frac{q}{p}, \text{ where } p \text{ and } q \text{ are two coprime integers}. \end{cases}$$

Both f and g are Riemann integrable on $[0,1]$, but the composition $f \circ g$ is not. Therefore, it is natural to ask whether the composition of two functions is still Riemann integrable, when one is Riemann integrable and the other is continuous.

In what follows, we let f be a function defined on the interval $[a, b]$, and let g be a function defined on the interval $[c, d]$ with its range contained in $[a, b]$.

Question 1. If f is continuous on $[a, b]$ and g is Riemann integrable on $[c, d]$, is the composition $f \circ g$ Riemann integrable on $[c, d]$?

The answer is yes.

Since f is continuous on the closed interval $[a, b]$, it is uniformly continuous on $[a, b]$. Hence, for each $\varepsilon > 0$, there exists a $\delta > 0$, such that for any ξ_1 and ξ_2 in $[a, b]$ with $|\xi_1 - \xi_2| < \delta$ we have

$$|f(\xi_1) - f(\xi_2)| < \frac{\varepsilon}{2(d-c)}. \tag{1}$$

Moreover, f is bounded on $[a, b]$; say, $|f(y)| \le M$ for all $y \in [a, b]$.

Since g is Riemann integrable on $[c, d]$, for the above $\delta > 0$, there exists an $\eta > 0$ such that for any division T of $[c, d]$ with norm $|T| < \eta$, the following relation always holds:

$$\sum_{\alpha} \omega_\alpha \Delta x_\alpha < \frac{\varepsilon\delta}{4M}, \tag{2}$$

where Δx_α is the length of the interval I_α in the division T and

$$\omega_\alpha = \max_{x, y \in I_\alpha} \{|g(x) - g(y)|\}$$

is the oscillation of g on I_α. We recall that the norm $|T|$ is the maximum length of the intervals in T.

Now we consider the composition $f \circ g$. For the division T, let M_α be the oscillation of $f \circ g$ on I_α. Divide all the intervals of the division T into two parts. The first part contains all the intervals on which the oscillation of g is not less than δ, and the second part contains the rest of the intervals. Then we have

$$\sum_{\alpha} M_\alpha \Delta x_\alpha = \sum_{\omega_j \ge \delta} M_j \Delta x_j + \sum_{\omega_i < \delta} M_i \Delta x_i. \tag{3}$$

From (1), we know that for any interval I_i in the second part, $M_i < \varepsilon/2(d-c)$. Thus

$$\sum_{\omega_i < \delta} M_i \Delta x_i < \left(\sum_{\omega_i < \delta} \Delta x_i\right) \cdot \frac{\varepsilon}{2(d-c)} \le \frac{\varepsilon}{2}, \tag{4}$$

but

$$\sum_{\alpha} \omega_\alpha \Delta x_\alpha \ge \sum_{\omega_j \ge \delta} \omega_j \Delta x_j > \delta \sum_{\omega_j \ge \delta} \Delta x_j. \tag{5}$$

Combining (5) with (2), we obtain

$$\sum_{\omega_j \ge \delta} \Delta x_j < \frac{\varepsilon}{4M}.$$

Then

$$\sum_{\omega_j \ge \delta} M_j \Delta x_j < 2M \cdot \sum_{\omega_j \ge \delta} \Delta x_j < 2M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}. \tag{6}$$

Combining (3) with (4) and (6), we have

$$\sum_{\alpha} M_\alpha \Delta x_\alpha < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That is to say, $f \circ g$ is Riemann integrable on $[c, d]$.

Thus we have proved the following result, which can also be found in [**1**, p. 197].

Proposition 1. *If f is continuous on $[a, b]$ and g is Riemann integrable on $[c, d]$ with its range in $[a, b]$, then $f \circ g$ is Riemann integrable on $[c, d]$.*

Question 2. If f is Riemann integrable on $[a, b]$ and g is continuous on $[c, d]$, is $f \circ g$ always Riemann integrable on $[c, d]$?

The answer is negative, as shown by the following counterexample. Let

$$f(y) = \begin{cases} 0 & \text{when } y = 0, \\ 1 & \text{when } y \neq 0, \end{cases}$$

on $[a, b] = [0, 1]$, and define g inductively as follows.

First, let $g_0(x) = 0$, $x \in [0, 1]$. Next, construct g_1 based on g_0. Divide $[0, 1]$ into three sections, say, I_1, I_2, I_3 in proper order, such that the centre of I_2 is $\frac{1}{2}$ and the length of I_2 is $\frac{1}{3}$. Modifying the function g_0 on I_2 appropriately, we obtain a function g_1, that satisfies the following conditions:

- $g_1(x) = g_0(x)$ for x in I_1 and I_3;
- g_1 is continuous on $[0, 1]$;
- $g_1(x)$ is always greater than zero for any x in the interior of I_2;
- the maximum value of g_1 on I_2 is $\frac{1}{2}$.

Once g_{n-1} is defined, we construct g_n as follows. First, divide all the intervals on which g_{n-1} is always zero into three sections, such that the centre of the middle section is the centre of the original interval and the length of the middle section is $1/3^n \cdot 2^{n-1}$. Second, modify the values of g_{n-1} only on the middle sections of them and obtain a function g_n, such that g_n is still continuous on $[0, 1]$, but in the interior of each modified intervals, g_n is always greater than zero and the maximum is 2^{-n}. We note that there are 2^{n-1} intervals in which g_n and g_{n-1} have different values. Thus the total length of them is 3^{-n}.

Continuing this process gives a sequence of functions $\{g_n\}$ that satisfy the following conditions:

- g_n is continuous on $[0, 1]$;
- $|g_n(x) - g_{n-1}(x)| \leq \frac{1}{2^n}$, for any $x \in [0, 1]$;
- the total length of all the intervals in which g_n is not zero is

$$S_n = \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right).$$

Thus, for any positive integers $n > m$ we have

$$\begin{aligned} |g_n(x) - g_m(x)| &\leq |g_n(x) - g_{n-1}(x)| + \cdots + |g_{m+1}(x) - g_m(x)| \\ &\leq \frac{1}{2^n} + \cdots \frac{1}{2^{m+1}} < \frac{1}{2^m}. \end{aligned}$$

For any $\varepsilon > 0$, there is a positive integer N, say $N > \ln \varepsilon^{-1}/\ln 2$ when $\varepsilon < 1$. Then for any integers $n > m > N$, we have $|g_n(x) - g_m(x)| < 2^{-N} < \varepsilon$ for any $x \in [0, 1]$. That is to say, $g_n(x)$ is uniformly convergent on $[0, 1]$. Let $g_n(x)$ be uniformly convergent to $g(x)$ on $[0, 1]$. Then g satisfies:

- g is continuous on $[0, 1]$;
- $g(x)$ is not identically zero on any subinterval of $[0, 1]$;
- the total length of all the intervals in which $g(x)$ is not zero is

$$S = \lim_{n \to \infty} \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^n\right) = \frac{1}{2}.$$

We now prove that $f \circ g$ is not Riemann integrable on [0, 1].

Let T be a division of [0, 1]. Divide T into two parts. The first part T_1 contains all the intervals in which $g(x)$ is non-zero and the second part T_2 contains the rest. The total length of all the intervals in T_1 is at most $\frac{1}{2}$; hence the total length of all the intervals in T_2 is at least $\frac{1}{2}$. But in any interval I_i of T_2, we can always find two points ξ_i and ζ_i such that $g(\xi_i) = 0$ and $g(\zeta_i) \neq 0$. Obviously, $f \circ g(\xi_i) = 0$ and $f \circ g(\zeta_i) = 1$. Thus the oscillation M_i of $f \circ g$ on I_i is 1.

Let M_α be the oscillation of $f \circ g$ on any interval I_α of T, and Δx_α be the length of the interval I_α. Then

$$\sum_\alpha M_\alpha \Delta x_\alpha = \sum_{T_1} M_j \Delta x_j + \sum_{T_2} M_i \Delta x_i \geq \sum_{T_2} M_i \Delta x_i = \sum_{T_2} \Delta x_i \geq \frac{1}{2}.$$

Thus $f \circ g$ is not Riemann integrable on [0, 1].

The discussion can be continued by asking for conditions on g to ensure that $f \circ g$ is Riemann integrable, provided that f is Riemann integrable. The following result provides one answer to this question. The proof is left to the reader.

Proposition 2. *Let f be a Riemann integrable function defined on $[a, b]$ and let g be a differentiable function with continuous and non-zero derivative on $[c, d]$. If the range of g is contained in $[a, b]$, then $f \circ g$ is Riemann integrable on $[c, d]$.*

REFERENCE

1. Jonathan Lewin and Myrtle Lewin, *An Introduction to Mathematical Analysis*, Random House, New York, 1988.

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On the Generalized "Lanczos' Generalized Derivative"

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This short note is an extrapolation of Groetsch's interesting article [**1**], and may lead to a clearer understanding of Lanczos' derivative. Only a minimal familiarity with random variables is required.

Lanczos' generalized derivative is defined by

$$D_h f(x) = \frac{3}{2h^3} \int_{-h}^{h} t f(x + t)\, dt,$$

where h is a parameter that can be assumed positive. It generalizes the ordinary derivative in the following two senses:

(1) Suppose $f(x)$ is locally C^4 at x_0. Then $D_h\, f(x_0) = f'(x_0) + O(h^2)$.