THE EUCLIDEAN ALGORITHM IN ALGEBRAIC NUMBER FIELDS

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Abstract. This article, which is an update of a version published 1995 in Expo. Math., intends to survey what is known about Euclidean number fields; we will do this from a number theoretical (and number geometrical) point of view. We have also tried to put some emphasis on the open problems in this field.

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Contents

1. Prehistory

The Euclidean Algorithm is a method used by Euclid to compute the greatest common divisor of two numbers; from today's perspective it is the founding stone of the number theory in Euclid's book. How close Euclid came to understand the unique factorization property of the integers is open to debate: using his 'geometric language', he only could formulate it for products of three different primes.

During the middle ages, Arabic and later European mathematicians studied the prime factors of a given number in connection with the problem of amicable numbers, and realized that the list of all factors of a number n can be produced from its prime factorization; the first clear statement of unique factorization, however, is due to Gauss in 1801. Gauss's proof of the Unique Factorization Domain is not built directly on the Euclidean algorithm, although he refers expressis verbis to Euclid's famous proposition that if a prime divides a product, it must divide one of the factors. In a paper published in 1832, Gauss proved that the ring $\mathbb{Z}[i]$ admits a Euclidean algorithm, and that it has unique factorization, but the proof of unique factorization in $\mathbb{Z}[i]$ is accomplished by 'pulling it back to $\mathbb{Z}$'.

The first mathematician who emphasized that the existence of a Euclidean algorithm implied unique factorization was Dirichlet, and he did that as late as 1847!

2. Definitions and General Properties

An integral domain R is called *Euclidean* with respect to a given function $f : R \to \mathbb{N}$ if f has the following properties:

$$f(\alpha) = 0 \iff \alpha = 0 \tag{1}$$

$$\text{for all } \alpha, \beta \in R \setminus \{0\} \text{ there is a } \gamma \in R \text{ such that } f(\alpha - \beta\gamma) < f(\beta). \tag{2}$$

We call such an f a *Euclidean function* on R. There are equivalent definitions of Euclidean rings and functions, most of which are studied in [120]. For example, a function $f : R \to \mathbb{R}_{\geq 0}$ satisfying (1) and (2) is called Euclidean if it also satisfies

$$\text{For every } \kappa > 0 \text{ the set } \{f(\alpha) : \alpha \in R,\ f(\alpha) < \kappa\} \text{ is finite.} \tag{3}$$

It is easily seen that an integral domain which is Euclidean with respect to a real-valued function is also Euclidean with respect to a suitably chosen integer-valued function. Variants of Euclidean functions have been studied by Picavet [164], Lenstra [120], and Hiblot [97].

2.1. Euclidean minima.

For any integral domain R we can define the Euclidean minimum $M(R, f)$ of R with respect to a given integer-valued function f satisfying (1) by

$$M(R, f) = \inf \{\kappa > 0 : \text{ for all } \alpha, \beta \in R \setminus \{0\} \text{ there exists } \gamma \in R \text{ such that } f(\alpha - \beta\gamma) < \kappa \cdot f(\beta)\}.$$

Obviously R is (resp. is not) Euclidean with respect to f if $M(R, f) < 1$ (resp. $M(R, f) > 1$). If $M(R, f) = 1$, both possibilities actually occur. If $\beta \neq 0$ is a non-unit in R, then we have $M(R, f) \geq f(\beta)^{-1}$.

Let S be an integral domain contained in R; then

$$M(R/S, f) = \inf \{\kappa > 0 : \text{ for all } \alpha, \beta \in S \setminus \{0\} \text{ there exists } \gamma \in R \text{ such that } f(\alpha - \beta\gamma) < \kappa \cdot f(\beta)\}$$

is called the relative Euclidean minimum of S in R. It would be interesting to find non-trivial inequalities relating $M(R, f)$, $M(S, f)$ and $M(R/S, f)$, especially if $S = \mathcal{O}_K$ and $R = \mathcal{O}_L$ are the rings of integers in an extension L/K of number fields and f is the absolute value of the norm (cf. Sect. 3).

2.2. **S-Euclidean Rings.** Stein [181] introduced the following idea for computing gcd's in the ring $\mathbb{Z}$: using the following rules, $\gcd(a, b)$ can be computed by repeated addition, substraction, and division by 2:

- $\gcd(2a, 2b) = 2\gcd(a, b)$,
- $\gcd(2n + 1, 2b) = \gcd(2n + 1, b)$
- $\gcd(a, b) = \gcd(a, \frac{a-b}{2})$ if $a \equiv b \equiv 1 \bmod 2$.

More generally, let R be a UFD, $S = \{p_1, \ldots, p_r\}$ a subset of elements of R, and $f : R \longrightarrow \mathbb{N}$ some function with the property that $f(a) = 0$ for $a \in R$ if and only if $a = 0$. We say that R is S-Euclidean with respect to f if for every pair $a, b \in R \setminus \{0\}$ there are $q, c \in R$ such that $a - qb = cp_i$ for some $p_i \in S$ and $f(c) < f(b)$. Then $\gcd(a, b) = \gcd(a, cp_i) = \delta \gcd(b, c)$, where $\delta = 1$ if $p_i \nmid a$ and $\delta = p_i$ otherwise.

A ring is S-Euclidean with respect to $S = \{1\}$ if and only if it is Euclidean in the usual sense.

It is also clear that if R is S-Euclidean for some f and if $T \subset R$ is any finite set containing S, then R is also T-Euclidean with respect to f.

An easy exercise shows

Proposition 2.1. *Let p be a prime in $\mathbb{Z}$, and put $S = \{p\}$. Then $\mathbb{Z}$ is S-Euclidean with respect to the usual absolute value.*

The algorithm corresponding to the set $S = \{2\}$ is called the binary gcd-algorithm. Generalizations to k-ary algorithms were studied by Sorenson [180].

The binary gcd-algorithm was generalized to $\mathbb{Z}[i]$ by Weilert [196, 197] (see also Collins [48]), to $\mathbb{Z}[\zeta_3]$ by Damgard & Frandsen [52], and to the rings of integers in the complex quadratic fields with discriminant $-7, -8, -11$ and -19 by Agarwal & Frandsen [1]. The last example shows that S-Euclidean rings are not necessarily Euclidean. I do not know whether Euclidean rings are S-Euclidean for suitably chosen sets S.

2.3. **Motzkin Sets.** If R is Euclidean, the function f_{min} defined by

$$f_{\min}(\alpha) = \min \{f(\alpha) : f \text{ is a Euclidean function on } R\}$$

is called the minimal Euclidean function on R. It is easily seen that $f_{\min}$ is in fact a Euclidean function on R. For any integral domain R, define the Motzkin sets $E_k, k \geq 0$, by

$$\begin{aligned} E_0 &= \{0\}, \\ E_1 &= \{0\} \cup R^*, \text{ the unit group of } R \text{ and, generally,} \\ E_k &= \{0\} \cup \{\alpha \in R : \text{ each residue class } \bmod \alpha \text{ contains a } \beta \in E_{k-1}\}, \\ E_\infty &= \bigcup_{k \geq 0} E_k \end{aligned}$$

The Motzkin sets of $R = \mathbb{Z}$ are easily computed:

$$E_0 = \{0\}, E_1 = \{0, \pm 1\}, E_2 = \{0, \pm 1, \pm 2, \pm 3\}, \ldots, E_k = \{0, \pm 1 \ldots, \pm(2^k - 1)\}.$$

The following observation is due to Motzkin [147]:

Proposition 2.2. *R is Euclidean if and only if $E_\infty = R$. If $E_\infty = R$, then the function f_M defined by $f_M(\alpha) = \min\{k \in \mathbb{N} : \alpha \in E_k\}$ coincides with the minimal Euclidean function on R.*

The minimal Euclidean algorithm for the rings $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{-2}\,]$ was implemented by Fuchs [78]. Since the minimal Euclidean function is submultiplicative ($f(ab) \ge f(a)$ for all $a, b \in R \setminus \{0\}$), every Euclidean ring admits a submultiplicative Euclidean function. Whether this is also true for multiplicative functions ($f(ab) = f(a)f(b)$ for $a, b \in R$) is not known.

Our next result provides us with examples of Euclidean functions f such that $M(R, f) = 1$:

Proposition 2.3. *Let R be an integral domain; then*

(1) *$R = E_1$ if and only if R is a field;*
(2) *if R is not a field, then $R \neq E_k$ for all $k \in \mathbb{N}$; if, moreover, R is Euclidean, then $M(R, f_{\min}) = 1$.*

2.4. ***k*-stage Euclidean Rings.** In 1976, Cooke [49] introduced the following more general concept: let R be an integral domain. A sequence of equations (with $\alpha, \beta, \gamma_i, \rho_i \in R$)

$$\begin{aligned} \alpha &= \beta\gamma_1 + \rho_1, \\ \beta &= \rho_1\gamma_2 + \rho_2, \\ &\vdots \\ \rho_{k-2} &= \rho_{k-1}\gamma_k + \rho_k \end{aligned}$$

is called a *k-stage division chain* starting from the pair (α, β); we say that R is *quasi-Euclidean*, if we can find a function $f : R \to \mathbb{N}$ with the properties

(Q1) $f(\alpha) = 0 \iff \alpha = 0$,
(Q2) for every pair $\alpha, \beta \in R \setminus \{0\}$ there exists a k-stage division chain for some $k \in \mathbb{N}$ such that $f(\rho_k) < f(\beta)$.

If we can replace (2.4) by the stronger condition

(Q'2) there is a $k \in \mathbb{N}$ such that for every pair $\alpha, \beta \in R \setminus \{0\}$ there exists an n-stage division chain for some $n \le k$ with $f(\rho_k) < f(\beta)$,

then R is called *k-stage Euclidean* with respect to f. We also can introduce k-Euclidean minima in an obvious way. Several equivalent definitions of quasi-Euclidean rings have been studied by Cooke [49], Bougaut [15, 16, 17], Décoste [63, 64] and Leutbecher [131]. See also some papers on Nagata's pairwise algorithm by Chen & Leu [36] and Nagata [149, 150, 151, 152, 153].

2.5. **Euclidean Ideal Classes.** Lenstra [127], inspired by papers of Fontené [76] and Cahen [20], introduced *Euclidean ideal classes*; they generalize Euclidean rings because the trivial ideal class $[R]$ is Euclidean if and only if R is Euclidean. Euclidean ideal classes have been investigated by van der Linden [140, 141]. Non-trivial Euclidean ideal classes seem to occur very rarely: if K is a real quadratic field which contains a non-trivial Euclidean ideal class, then disc $K = 40, 60, 85$. The known examples in degree ≥ 3 are:

- the cubic field with $\operatorname{disc} K = -283$ and $h(K) = 2$ (van der Linden),
- the cubic field with $\operatorname{disc} K = -331$ and $h(K) = 2$ (Lemmermeyer),

- the quartic field $\mathbb{Q}(\sqrt{-3}, \sqrt{13}\,)$ with $h(K) = 2$ (Lenstra).

Schulze [176] defined Euclidean systems; they generalize Euclidean ideal classes, and the simplest Euclidean systems correspond to the Dedekind-Hasse-test (cf. [92]):

Proposition 2.4. *R is a principal ideal ring if and only if there is a function $f : R \to \mathbb{N}$ satisfying (E1) with the following property: for every $\alpha, \beta \in R$ such that $\beta \nmid \alpha$ there exist $\lambda, \mu \in R$ such that $0 < f(\lambda\alpha - \mu\beta) < f(\beta)$.*

A different notion of a Euclidean system was introduced by Treatman in his thesis [186].

3. The Norm as a Euclidean Function

Let K be an algebraic number field and $\mathcal{O}_K$ its ring of integers. If the absolute value of the norm is a Euclidean function, $\mathcal{O}_K$ (or, by abuse of language, K) is called norm-Euclidean. The Euclidean minimum of K with respect to the norm is called norm-Euclidean minimum and will be denoted by $M(K)$. More generally, for a set S of primes in $\mathcal{O}_K$, let $\mathcal{O}_S$ denote its ring of S-integers. We can define the S-norm (or simply norm) S_N by $N_S\mathfrak{a} = (\mathcal{O}_S : \mathfrak{a})$ for any non-zero ideal $\mathfrak{a}$ in $\mathcal{O}_S$ as usual and put $N_S\alpha := N_S(\alpha\mathcal{O}_S)$. The first example of a norm-Euclidean ring $\mathcal{O}_S$ was apparently given by Wedderburn[1] [195].

The following theorem of Weinberger [198] (whose proof builds on previous work by Hooley) suggested strongly the existence of number fields that are Euclidean with respect to functions different from the norm (GRH denotes a certain set of generalized Riemann hypotheses):

Proposition 3.1. *Assume that GRH holds; then every number field K with unit rank ≥ 1 has class number 1 if and only if K is Euclidean with respect to a suitably chosen function f.*

On the other hand, the work of O'Meara [159] and Vaserstein [192] (cf. Cooke [49, 50]) shows unconditionally

Proposition 3.2. *Every number field K with unit rank ≥ 1 has class number 1 if and only if it is k-stage norm-Euclidean for some $k \in \mathbb{N}$.*

3.1. Euclidean Minima.

For every $\xi \in K$, define $M(\xi) = \inf\,\{|N_{K/\mathbb{Q}}(\xi - \eta)| : \eta \in \mathcal{O}_K\}$. $M(\xi)$ is called the *Euclidean minimum* at ξ, and we have $M(K) = \sup\,\{M(\xi) : \xi \in K\}$. Obviously, $M(\xi) = M(\xi - \eta)$ for every $\eta \in \mathcal{O}_K$, i.e. $M(\xi)$ only depends on the class of ξ in $K/\mathcal{O}_K$. Now let

$$C_1 = \{\xi \in K/\mathcal{O}_K : M(\xi) = M(K)\}$$

and define the *second Euclidean minimum* of K by

$$M_2(K) = \sup\,\{M(\xi) : \xi \in (K/\mathcal{O}_K) \setminus C_1\}.$$

Obviously $M_2(K) \le M(K) = M_1(K)$, and if this inequality is strict, we say that $M_1(K)$ is *isolated*. The Euclidean minima $M_k(K), k \ge 2$, are defined in a similar way. There are number fields with an infinite sequence of strictly decreasing Euclidean minima, and fields whose second minimum is not isolated. Barnes & Swinnerton-Dyer [5, 6, 7] showed

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[1] I thank Keith Dennis for bringing this to my attention.

Proposition 3.3. *If K is a number field with unit rank ≥ 1 and if C_1 is finite, then the minimum $M_1(K)$ is isolated.*

In order to prove that a given number field K is norm-Euclidean, we choose a $\mathbb{Q}$-basis $\{\alpha_1, ..., \alpha_n\}$ of K and let $\phi : \alpha = \sum a_i\alpha_i \to (a_1, \ldots, a_n) \in \mathbb{R}^n$; after identifying K and $\phi(K)$, we find that $\mathcal{O}_K$ is a lattice in $\overline{K} = \mathbb{R}^n$, and that K is dense in $\overline{K}$. We extend the norm $N_{K/\mathbb{Q}}\alpha = \prod_\sigma \alpha^\sigma$ on K (here σ runs through all $n = (K : \mathbb{Q})$ embeddings of K into $\mathbb{C}$) to a continuous function

$$N : \mathbb{R}^n \to \mathbb{R} : (\xi_1, \ldots, \xi_n) \mapsto N(x) = \prod_\sigma \Big(\sum_{j=1}^n \xi_j\alpha_j^\sigma\Big).$$

Obviously K is norm-Euclidean if and only if for all $\xi \in K$ we can find $\eta \in \mathcal{O}_K$ such that $|N(\phi(\xi) - \phi(\eta))| < 1$; we see that it suffices to show that for every real $\xi \in \overline{K}$ we have $|N(\xi - \phi(\eta))| < 1$ for a suitably chosen $\eta \in \mathcal{O}_K$.

Therefore we define the Euclidean minimum at $x \in \overline{K}$ by

$$M(x) = \inf\ \{|N(x - \phi(\eta))| : \eta \in \mathcal{O}_K\},$$

and call $M(\overline{K}) = \sup\ \{M(x)|x \in \overline{K}\}$ the inhomogeneous minimum of K; it is clear by definition that $M(K) \leq M(\overline{K})$. Let $x \in \overline{K}$ and a real $\varepsilon > 0$ be given; it follows from the definition of $M(\overline{K})$ that we can find $\eta \in \mathcal{O}_K$ with $|N(x-\phi(\eta))| < M(\overline{K}) + \varepsilon$. If we can satisfy the stronger inequality $|N(x - \phi(\eta))| \leq M(\overline{K})$ for every $x \in \overline{K}$ we shall say that the minimum $M(\overline{K})$ is *attained*.

Proposition 3.4. *We have $M(K) = M(\overline{K})$ for every number field K with unit rank 1, and there exist $x \in \overline{K}$ with $M(x) = M(\overline{K})$.*

This equality has been observed by Barnes & Swinnerton-Dyer [5]; they proved it for $n = 2$, and van der Linden [140, 141] gave a proof for fields with unit rank 1. Computations seem to suggest the following conjectures for number fields K with unit rank ≥ 1:

(1) $M(K)$ is isolated even if C_1 is not finite;
(2) $M(K)$ is always rational;
(3) $M(K) = M(\overline{K})$ for every number field with unit rank ≥ 1;
(4) in Prop. 3.4, some $x = \sum a_j\alpha_j$ has coordinates $a_i \in K$;
(5) in Prop. 3.4, x can be chosen from the dense subset K (i.e. x can be chosen to have rational coordinates a_i; such x are called rational points in $\overline{K}$).

Call $ESp(K) = \{M(x)|x \in \mathbb{R}^n\}$ the *Euclidean spectrum* of K; $ESp(K)$ is known to be closed as a subset of the reals (Theorem L of Barnes & Swinnerton-Dyer). Let $\partial\ ESp(K)$ be the boundary of $ESp(K)$ (with respect to the usual topology on $\mathbb{R}$). Another question is

(6) Is $\partial\ ESp(K) \subset K$ if K is totally real?

For related questions, we refer the reader to Berend & Moran [12]. The background necessary for the computation of Euclidean minima has been provided by Barnes & Swinnerton-Dyer; although the presentation of some of the proofs given in their papers [5, 6] can be simplified, these articles still are worth reading, and they are recommended to anyone interested in computing minima of number fields of small degree.

The inequality $M(K) \leq 2^{-n}\sqrt{d}$ for totally real number fields of degree n and absolute value of discriminant d is called the "Minkowski conjecture" (cf. O. Keller, *Geometrie der Zahlen,* Enzyklop. d. math. Wiss. I 2, 2. Aufl.); it is known to hold for $n \le 5$, and Chebotarev could prove that $M(K) \le 2^{-n/2}\sqrt{d}$. Similar results (not even a conjecture) for fields with mixed signature are not known except for a theorem of Swinnerton-Dyer [182] concerning complex cubic fields (see Sect. 5).

3.2. **Lower Bounds for $M(K)$.** There are several methods for getting bounds on $M(K)$, and in particular for showing that a given number field is not norm-Euclidean. The simplest criterion uses totally ramified primes:

Proposition 3.5. *Let K/k be a finite extension of number fields, and suppose that the prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ is completely ramified in K/k, i.e. that $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}^2$. If $\beta \equiv \alpha^n \bmod \mathfrak{p}$ for some $\alpha, \beta \in \mathcal{O}_K \setminus \mathfrak{p}$, and if there do not exist $b \in \mathcal{O}_K$ such that*

1. *$b \equiv \beta \bmod \mathfrak{p}$;*
2. *$b = N_{K/k}\delta$ for some $\delta \in \mathcal{O}_K$;*
3. *$|N_{k/\mathbb{Q}}\, b| < N\mathfrak{p}$;*

then K is not Euclidean.

In the special case $k = \mathbb{Q}$ and $\mathfrak{p} = p\mathbb{Z}$, there are only two $b \in \mathbb{Z}$ satisfying (1) and (3), because $|N_{k/\mathbb{Q}}\, b| = |b|$ and $|N\mathfrak{p}| = p$. Moreover, if K is totally complex, only positive $b \in \mathbb{Z}$ can be norms from K.

Our next result exploits the action of the unit group E_K on the factor group $K/\mathcal{O}_K$; it is easy to see that $\mathrm{Orb}(\overline{\xi}) = \{\varepsilon\overline{\xi} : \varepsilon \in E_K\}$ is finite for every class $\overline{\xi} = \xi + \mathcal{O}_K \in K/\mathcal{O}_K$. The following theorem is essentially due to Barnes & Swinnerton-Dyer:

Theorem 3.6. *Let $K = \mathbb{Q}(\alpha)$ be a number field with unit group E_K. If, given a $\xi \in K$ and a real number $k > 0$, there exists a $\gamma \in \mathcal{O}_K$ such that $N(\xi - \gamma) < k$, then there exists a $\zeta = \sum_{j=0}^{n-1} a_j\alpha^j \in K$ with the following properties:*

1. *$\zeta + \mathcal{O}_K = \xi_j$ for some $\xi_j \in \mathrm{Orb}(\overline{\xi})$;*
2. *$|a_i| < \mu_i$ $(0 \le i < n)$ for some constants $\mu_i > 0$ depending only on K;*
3. *$N(\zeta) < k$.*

Since the number of $\zeta \in K$ satisfying 1. and 2. is finite, this theorem allows us to compute $M(\xi, K)$.

3.3. **Weighted Norms.** In light of Weinberger's result we are interested in functions f that might serve as Euclidean functions on number fields K with unit rank ≥ 1 and class number 1. Of course, if R is Euclidean we can always take $f = f_{\min}$; but this function is not very useful if we want to prove that R is Euclidean because $f_{\min}$ is rather hard to compute. Lenstra [120] proposed to look at "weighted norms" instead: first we define a multiplicative function $\phi : I_K \to \mathbb{R}$, where I_K denotes the group of fractional ideals of $\mathcal{O}_K$, by giving its values on the prime ideals; to this end choose a prime ideal $\mathfrak{p}$, a real number $c > 1$, and define $\phi(\mathfrak{p}) = c$, $\phi(\mathfrak{q}) = N(\mathfrak{q}) := (R : \mathfrak{q})$ for every prime ideal $\mathfrak{q} \ne \mathfrak{p}$. Then extend ϕ multiplicatively to all ideals of $\mathcal{O}_K$ and put $\phi(0) = 0$ and $\phi(\alpha) = \phi(\alpha\mathcal{O}_K)$ for elements $\alpha \in K^\times$. Then $\phi = \phi_{\mathfrak{p},c}$ is a well defined multiplicative function with the property (3), and

$$w(\mathfrak{p}) = \{c > 0 : \phi_{\mathfrak{p},c} \text{ is a Euclidean function on } \mathcal{O}_K\}$$

is called the *Euclidean window* of the weighted norm ϕ; see [30].

Proposition 3.7. *The Euclidean window* $w(\mathfrak{p})$ *of a weighted norm* f *is a (possibly empty) interval contained in* $(1, \infty)$.

Using an incredibly simple idea, Clark [39] succeeded in proving that $f = f_{\mathfrak{p},c}$ is a Euclidean function in the quadratic number field $\mathbb{Q}(\sqrt{69}\,)$ for $\mathfrak{p} = (23, \sqrt{69}) = (\frac{23+\sqrt{69}}{2})$ and every $c > 25$. This was done as follows: first one observes that M_1 is isolated and that $M_2 < 1$. For every $\xi \in \overline{K} \setminus C_1$ we can find $\eta \in \mathcal{O}_K$ such that $|N(\xi - \eta)| < 1$, where N denotes the usual norm; if the numerator of $\xi - \eta$ is not divisible by $\mathfrak{p}$, we will also have $f(\xi - \eta) < 1$. In order to take care of the points $\xi - \eta$ with numerator divisible by $\mathfrak{p}$, we show that for every $\xi \in \overline{K} \setminus C_1$ we can find $\eta_1, \eta_2 \in \mathcal{O}_K$ such that $|N(\xi - \eta_j)| < 1$ for $j = 1, 2$ and $\eta_1 - \eta_2 \not\equiv 0 \bmod \mathfrak{p}$. Unfortunately, this method does not seem to work for other quadratic number fields; there are, however, numerous examples in degree 3 (cf. Sect. 5)

Building on work of Gupta, M. Murty & V. Murty [87] on the Euclidean algorithm for S-integers, Clark & M. Murty [42] devised a method for proving number fields to be Euclidean with respect to functions different from the norm; this method applies to totally real Galois extensions of degree ≥ 3 with an additional property. In his thesis, Clark [38] verified this condition for the 165 totally real quartic number fields with class number 1 and discriminant less than 10^6 as well as the cyclic cubic number fields with discriminant less than $5 \cdot 10^5$ and class number 1. See Mandavid [109] for a detailed exposition.

Harper & Murty [91] proved that if K is a finite Galois extension of $\mathbb{Q}$ with unit rank > 3, then $\mathcal{O}_K$ is Euclidean if and only if it is a principal ideal domain; if K is abelian, unit rank ≥ 3 is sufficient.

3.4. **Euclidean Minima for k-stage Algorithms.** Cooke & Weinberger have made some very interesting observations concerning the k-stage Euclidean algorithm in number fields: define continued fractions $[\gamma_1, \gamma_2, \ldots, \gamma_k]$ of length k (with coefficients $\gamma_j \in \mathcal{O}_K$) by

$$[\gamma_1, \gamma_2, \ldots, \gamma_k] = \gamma_1 + \cfrac{1}{\gamma_2 + \cfrac{1}{\gamma_3 + \cdots + \cfrac{1}{\gamma_k}}}$$

Let $CF_k(K)$ be the set of all continued fractions of length $\leq k$ with coefficients in $\mathcal{O}_K$. Then for all $\alpha, \beta \in \mathcal{O}_K$ there exists a k-stage division chain of length $k \leq n$ starting from (α, β) such that $|N(\rho_k)| < |N(\beta)|$ if and only if we can find $\gamma \in CF_k(K)$ with $|N(\alpha/\beta - \gamma)| < 1$.

The k-stage Euclidean minimum of K is the real number

$$M^k(K) = \inf\ \{\kappa : \text{ for all } \xi \in K \text{ there is a } \gamma \in CF_k(K) : |N(\xi - \gamma)| < \kappa\}$$

and the inhomogeneous minimum of K is defined by replacing K by $\overline{K}$.

Let us define sets $B_k = \{\xi \in K : |N(\xi - \gamma)| \geq 1 \text{ for all } \gamma \in CF_k(K)\}$ for $k \geq 1$; obviously we have $B_1 \supseteq B_2 \supseteq \ldots \supseteq B_\infty = \bigcap B_k$; if $B_\infty = B_k$ for some $k \in \mathbb{N}$ we say that K has *Euclidean depth* k and write $\mathrm{Ed}(K) = k$.

Theorem 3.8. *Assume that GRH holds. Then* $\mathrm{Ed}(K) \leq 4$ *for every number field* K *with unit rank* ≥ 1, *and* $\mathrm{Ed}(K) \leq 2$ *if* K *has at least one real embedding.*

The inequalities $\mathrm{Ed}(K) \le 5$ (resp. $\mathrm{Ed}(K) \le 3$) are due to Cooke & Weinberger [51]; it can be shown, however, that these inequalities are strict (cf. the remarks of Lenstra in [51], as well as [116]).

Using results of Vaserstein, Cooke [50] was able to show that B_∞ is discrete. By defining a suitable equivalence relation on the points in B_∞, Cooke could show that the number of equivalence classes equals $h-1$, where h is the class number of K.

In his paper [41], Clark could remove the assumption of the validity of GRH from Thm. 3.8 for a certain class of real normal fields.

For methods of computing the greatest common divisor in algebraic number fields which are not norm-Euclidean, see Kaltofen & Rolletschek [112] and F. George [81] for quadratic fields, and H. Cohen [44] in general.

4. Quadratic Number Fields

4.1. **Complex Quadratic Number Fields.** If K is an imaginary quadratic number field, i.e. $K = \mathbb{Q}(\sqrt{-m}\,)$, m a square free integer, the situation is completely clear (we write $D(-m)$ for the ring of integers in $\mathbb{Q}(\sqrt{-m}\,)$):

Proposition 4.1. *The rings $D(-m)$ are Euclidean if and only if $m = 1,2,3,7,11$, and in these cases the norm is a Euclidean function.*

In order to prove Prop. 4.1 we have to show:

a) $D(-m)$ is norm-Euclidean for $m = 1,2,3,7,11$;
b) if f is a Euclidean function on $D(-m)$, then $m = 1,2,3,7,11$.

The first proofs of a) are due to Gauss ($m = 1,3$ [79, 80]), Dirichlet [68], Cauchy ($m = 1,3$ [26]), Wantzel [193], Traub ($m = 1,2$ [185]), and Dedekind [65], who also noticed that these values of m are the only ones for which K is norm-Euclidean. Proofs for this fact have later been given by Birkhoff [13] and Schatunowsky [174]. In 1948, Motzkin [147] gave the first proof of b); this result has been rediscovered several times, for example by Dubois & Steger [69] or Chadid [25].

Wantzel [193] and Traub [185] were the first to show that $M(f) = 1$ for $R = \mathbb{Z}[\sqrt{-3}\,]$, where f is the norm, although the following proposition can easily be deduced from a result of Dirichlet [68]:

Proposition 4.2. *The Euclidean minimum $M(R,N)$ of $R = D(-m)$ with respect to the norm is given by*

$$\frac{|m|+1}{4}, \text{ if } R = \mathbb{Z}\left[\sqrt{-m}\,\right], \text{ and}$$
$$\frac{(|m|+1)^2}{16m}, \text{ if } R = \mathbb{Z}\left[\frac{1+\sqrt{-m}}{2}\right]$$

This implies the inequalities $\frac{|d|}{16} < M(K) \le \frac{|d|+4}{16}$ for imaginary quadratic fields K with discriminant d.

Concerning k-stage Euclidean rings, we have the result of P. Cohn [46]:

Proposition 4.3. *$D(-m)$ is k-stage Euclidean if and only if it is Euclidean.*

The Dedekind-Hasse-test 2.4 with $f = N$ has often been applied to show that $D(-19)$ is a principal ideal domain; cf. Wilson [199], Campoli [21], Feyzioglu [75]. The results of Prop. 4.2 can be used to improve the Minkowski bounds of quadratic extensions of imaginary quadratic Euclidean fields ([118], as well as [142]).

4.2. **Real Quadratic Number Fields.** As above, let $D(m)$ denote the ring of integers of $\mathbb{Q}(\sqrt{m}\,)$, where m is assumed to be squarefree. The real quadratic number fields that are norm-Euclidean are known:

Theorem 4.4. *The rings D(m) are norm-Euclidean if and only if*

$$m = 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73.$$

The if-part of Thm. 4.4 can be proved easily; it is the "only if" that causes the difficulties. The following table shows the evolution of the proof:

1848 Wantzel [194] shows that $\mathbb{Q}(\sqrt{m}\,)$ is norm-Euclidean for $m = 2, 3, 5$, and claims that this holds also for $m = 6, 7, 13, 17$.

1927 Dickson [66] shows that $\mathbb{Q}(\sqrt{m}\,)$ is norm-Euclidean if $m = 2, 3, 5, 13$ and asserts that these are the only such values.

1932 Perron [162] exhibits Dickson's error by showing that $\mathbb{Q}(\sqrt{m}\,)$ is norm-Euclidean for $m = 6, 7, 11, 17, 21, 29$; moreover he asks if every real quadratic number field with class number 1 is norm-Euclidean. In a letter to Perron (see [162]), Schur shows that $\mathbb{Q}(\sqrt{47}\,)$ is not norm-Euclidean.

1934 Oppenheim [160] finds a clever method to prove that $\mathbb{Q}(\sqrt{m}\,)$ is norm-Euclidean for $m = 2, 3, 5, 6, 7, 11, 17, 21, 29, 33, 37, 41$, and shows that $\mathbb{Q}(\sqrt{m}\,)$ is not norm-Euclidean for $m = 23, 31, 53$.

1935 Fox [77] and Berg [11] show that if $\mathbb{Q}(\sqrt{m}\,)$ is norm-Euclidean and $m \equiv 2, 3 \bmod 4$, then $m = 2, 3, 6, 7, 11, 19$, and Berg is able to prove that $\mathbb{Q}(\sqrt{19}\,)$ is indeed norm-Euclidean. Hofreiter [98, 99] shows that $\mathbb{Q}(\sqrt{57}\,)$ is norm-Euclidean; moreover he proves that $\mathbb{Q}(\sqrt{21}\,)$ is the only norm-Euclidean field among the $\mathbb{Q}(\sqrt{m}\,)$ with $m \equiv 21 \bmod 24$.

1936 Behrbohm & Redei [10] find all norm-Euclidean $\mathbb{Q}(\sqrt{m}\,)$ with $m \equiv 5 \bmod 24$.

1938 Schuster [177] treats the case $m \equiv 9 \bmod 24$. Erdös & Ko [74] show that there are only finitely many norm-Euclidean D(m) with $m \equiv 1 \bmod 8$ prime, and Heilbronn [93] extends this to composite values of m.

1940 A. Brauer [19] shows $m \le 109$ for all norm-Euclidean $\mathbb{Q}(\sqrt{m}\,)$ with $m \equiv 13 \bmod 24$.

1942 Rédei [168] finds all norm-Euclidean $\mathbb{Q}(\sqrt{m}\,)$, $m \equiv 17 \bmod 24$, and shows that $D(73)$ is norm-Euclidean. Moreover he shows that $D(m)$ is not norm-Euclidean for $m = 61, 89, 109, 113, 137$. This leaves only the $m \equiv 1 \bmod 24$ undecided. Rédei also claims that $D(97)$ is norm-Euclidean.

1944 Hua [100, 101] shows $m < e^{250}$, if $\mathbb{Q}(\sqrt{m}\,)$ is norm-Euclidean and $m \equiv 1 \bmod 4$ is prime.

1945 Hua & Shih [102] gives another proof that $D(61)$ is not norm-Euclidean.

1947 Inkeri [106] shows that the only norm-Euclidean fields with $\operatorname{disc} K < 5000$ are the known ones.

1948 Davenport ([61], published 1951) proves that $\operatorname{disc} K < 2^{14} = 16384$ for every norm-Euclidean real quadratic number field.

1949 Chatland [33] shows that there are no norm-Euclidean fields whose discriminants lie between 601 and 16 384.

1950 Chatland & Davenport [34], unaware of the results of Inkeri, show that there are no norm-Euclidean fields with $193 \le \operatorname{disc} K \le 601$.

1952 Barnes & Swinnerton-Dyer [5] discover that $D(97)$ is not norm-Euclidean.

We know the following bounds for Euclidean minima of quadratic fields:

Theorem 4.5. *For real quadratic fields K with $d = \operatorname{disc} K$, we have*

$$\frac{\sqrt{d}}{16+6\sqrt{6}} \;\le\; M(K) \;\le\; \frac{1}{4}\sqrt{d}$$

The upper bound, due to Minkowski (see Cassels [24]), is easily seen to be best possible:

Proposition 4.6. *Let n be an odd integer, put $m = n^2+1$ and $R = \mathbb{Z}[\sqrt{m}]$; then the Euclidean minimum of R is $M = \frac{n}{2}$, and this minimum is attained exactly at the points $\xi \equiv \frac{1}{2}\sqrt{m} \bmod R$.*

Since m is squarefree for an infinite number of n, and $R = D(m)$ in this case, the upper bound in Thm. 4.5 is in fact best possible. Heinhold [96], Barnes & Swinnerton-Dyer [5, 6, 7], and Varnavides [191] have given results of this kind for a lot of other orders in real quadratic fields.

The lower bound $D \ge \frac{1}{128}$ for the "Davenport constant" $D = \sup\ M(K)/\sqrt{d}$ is due to Davenport himself (cf. [57]). It was improved to $D \ge \frac{1}{51}$ by Cassels [23], and to the result given in Thm. 4.5 by Ennola [73].

The minima $M_i(\overline{K}), i \ge 1$, have been investigated thoroughly for many quadratic number fields; we cite a few examples that show some of the phenomena that can occur (cf. Davenport [53, 54, 55] for more examples):

Proposition 4.7. *Let $K = \mathbb{Q}(\sqrt{5}\,)$; then $\omega = \frac{1}{2}(1+\sqrt{5}\,)$ is the fundamental unit of K, and we have $M(K) = \frac{1}{4}$. There is an infinite sequence of isolated minima $M_i(K)$ given by*

$$M_{i+1}(\overline{K}) = \frac{F_{6i-2}+F_{6i-4}}{4(F_{6i-1}+F_{6i-3}-2)}$$

for all $i \ge 1$, where F_i is the i-th number in the Fibonacci sequence defined by $F_0 = F_1 = 1, F_{n+1} = F_n + F_{n-1}$. The sequence of minima begins with $M_1 = \frac{1}{4}, M_2 = \frac{1}{5}, M_3 = \frac{19}{121}, \ldots$, and we have $M_\infty(\overline{K}) = \lim M_i(\overline{K}) = \frac{1}{4\omega}$.

The sets $C_i(\overline{K}) = \{x \in \overline{K} : M(x) = M_i(\overline{K})\}$, where the minima are attained, are $C_1 = \{(0,\frac{1}{2}),(\frac{1}{2},0)\}, C_2 = \{(0,\pm\frac{1}{5}),(0,\pm\frac{2}{5})\}$, and, generally

$$C_k = \Big\{\xi \in K : \xi \equiv \frac{\omega^{6i-3}+1}{2(\omega^{6i-2}-1)}\varepsilon \bmod \mathcal{O}_K,\ \varepsilon \text{ a unit}\Big\}.$$

As Varnavides [188] has shown, $\mathbb{Q}(\sqrt{2})$ has similar properties; in general, however, the results are much more complicated (Inkeri [108]):

Proposition 4.8. *Let $K = \mathbb{Q}(\sqrt{13}\,)$; then $M_1(\overline{K}) = \frac{1}{3}, M_2(\overline{K}) = \frac{4}{13}$, and*

$$C_1 = \left\{ \left(\pm\frac{1}{6},\frac{1}{6}\right),\ \left(\pm\frac{1}{6}-\frac{1}{6}\eta^k, \pm\frac{1}{6}+\frac{1}{6\sqrt{13}}\eta^k\right) \right\},$$

where $k \in \mathbb{N}$ and $\eta = \frac{1}{2}\left(-3+\sqrt{13}\right)$, and

$$C_2 = \left\{ \left(0,\pm\frac{2}{13}\right), \left(0,\pm\frac{3}{13}\right) \right\}.$$

The minimum $M_1(\overline{K})$ is not attained.

Barnes & Swinnerton-Dyer [5] have generalized Prop. 4.8 to all fields $\mathbb{Q}(\sqrt{m})$ with $m = (2n+1)^2+4, n \ge 1$. They also computed an infinite sequence of minima for $m = 13$ and noticed that this sequence continues even beyond the limit $M_\infty = \lim_{k\to\infty} M_k$. On the other hand we know (cf. Godwin [83]):

Proposition 4.9. *Let $K = \mathbb{Q}(\sqrt{23})$; then the first minimum $M_1(\overline{K}) = \frac{77}{46}$ is attained and isolated, whereas $M_2(\overline{K}) = \frac{1}{46}\big(20\sqrt{23} - 31\big)$ is not isolated.*

It is easy to see that the points

$$x_k = \frac{1}{2} + \left(\frac{1}{2} - \frac{2}{23} + \frac{2}{23}\varepsilon^{-k}\right)\sqrt{23}, \quad k \in \mathbb{N}_0,$$

have an attained minimum $\mu = \frac{1}{46}\big(20\sqrt{23} - 31\big)$. Moreover, it is obvious that the x_k converge to $x = \frac{1}{2} + \big(\frac{1}{2} - \frac{2}{23}\big)\sqrt{23}$, and that $M(x) = M_1(\overline{K}) = \frac{77}{46}$. Godwin has shown that x is (up to conjugation and translation mod $\mathcal{O}_K$) the only point such that $M(x) > \mu$, and that each x_k is the limit of a series $x_{k,i}$ of (rational) points in K such that $\mu = \lim\, M(x_{k,i})$; since the $M(x_{k,i})$ are rational and M is not, $M_2(K)$ is not isolated. It seems likely that the x_k generate C_2, which would imply that M_2 is attained.

The same thing happens for $\mathbb{Q}(\sqrt{69})$ (see [30]):

Proposition 4.10. *In $K = \mathbb{Q}(\sqrt{69})$, we have*

$$\begin{aligned} M_1 &= \tfrac{25}{23}, & C_1 &= \big\{\pm\tfrac{4}{23}\sqrt{69}\big\}, \\ M_2 &= \tfrac{1}{1058}\big(3795 - 345\sqrt{69}\big), & C_2 &= \big\{(\pm P_k, \pm P_k')\big\}, \end{aligned}$$

where

$$P_k = \frac{1}{2}\varepsilon^{-k} + \left(\frac{4}{23} + \frac{1}{2\sqrt{69}}\varepsilon^{-k}\right)\sqrt{69}, \quad P_k' = \frac{1}{2}\varepsilon^{-k} - \left(\frac{4}{23} + \frac{1}{2\sqrt{69}}\varepsilon^{-k}\right)\sqrt{69}.$$

Here $\varepsilon = \frac{1}{2}\big(25 + 3\sqrt{69}\big)$ is the fundamental unit in $\mathbb{Q}(\sqrt{69})$, and the points P_k, P_k' have the limits $\pm\frac{4}{23}\sqrt{69}$ in C_1. The minimum $M_1(K) = M_1(\overline{K})$ is isolated, but $M_2(K) = M_2(\overline{K})$ is not.[2] In fact, the series of points $P_n = -\frac{3}{2} - \frac{15}{46}\sqrt{69} + \frac{1}{\varepsilon^n - 1}$ have minima that converge to $M_2(K)$ from below.

The first example of a quadratic number field with an infinite set C_2 such that C_1 is the set of accumulation points of C_2 is also due to Godwin [82]: in $\mathbb{Q}(\sqrt{73})$, $M_2(\overline{K})$ is isolated, and C_2 consists of irrational points converging to rational points of C_1; in particular, $M_2(K) < M_2(\overline{K})$!

The Euclidean and inhomogeneous minima $M_i(K)$ of real quadratic fields K may or may not have the following properties:

- A_i : $M_i(\overline{K})$ is attained;
- F_i : $C_i(\overline{K})$ is finite;
- E_i : $M_i(\overline{K}) = M_i(K)$;
- I_i : $M_i(\overline{K})$ is isolated;
- AP_i : I_i holds, and $C_i(\overline{K})$ is the set of accumulation points of $C_{i+1}(\overline{K})$;

we know that $F_i \Rightarrow A_i$, and that $F_2 \Rightarrow \neg AP_i$. Moreover, E_1 is true, and we conjecture that I_1 always holds.

The following combinations are known to occur:

[2] In the original version of this manuscript I falsely claimed that $M_2(K) < M_2(\overline{K})$.

m	A_1	F_1	AP_1	A_2	F_2	I_2	AP_2	E_2
5	x	x	*no*	x	x	x	*no*	x
7	x	x	*no*	x	x	x	x?	x
13	*no*	*no*	*no*	x	x	x	*no*	x
23	x	x	x	x?	*no*	*no*	$-$	x
69	x	x	x	x	*no*	x	?	*no*

Here "x" means, that D(m) has the corresponding property, while "x?" denotes a conjecture. This leaves, of course, a lot of questions unanswered:

- is there a $D(m)$ such that F_2 and AP_1 are simultaneously false?
- is there a $D(m)$ such that F_2 holds, but I_2 does not?
- etc.

It should be remarked that in $K = \mathbb{Q}(\sqrt{69}\,)$, the weighted norm $f_{\mathfrak{p},c}$ (with $\mathfrak{p} = (23, \sqrt{69}\,))$ and large enough c ($c \geq 49$ is sufficient) has an irrational Euclidean minimum $M_1(\mathcal{O}_K, f_{\mathfrak{p},c}) = \frac{1}{23}(-600 + 75\sqrt{69}\,) = 0.9998604...$ (see [30]).

The known examples of 2-stage norm-Euclidean rings $D(m)$ are

$$m = \begin{array}{rrrrrrrrrrrr} 14, & 22, & 23, & 31, & 38, & 43, & 46, & 47, & 53, & 59, & 61, & 62, \\ 67, & 69, & 71, & 77, & 89, & 93, & 97, & 101, & 109, & 113, & 129, & 133, \\ 137, & 149, & 157, & 161, & 173, & 177, & 181, & 193, & 197, & 201, & 213, & 253. \end{array}$$

The following observation concerning Euclidean windows can be proved easily using ideals of small norm:

Proposition 4.11. *Let $K = \mathbb{Q}(\sqrt{14}\,)$ and define a weighted norm f in K by $f(\mathfrak{p}) = c$, where $\mathfrak{p} = (2, \sqrt{14})$ is the unique prime ideal above (2). If f is a Euclidean function on $D(14)$, then necessarily $5 < c^2 < 7$, i.e., $w(\mathfrak{p}) \subseteq (\sqrt{5}, \sqrt{7})$.*

This shows again that $D(14)$ is not norm-Euclidean, because the absolute value of the norm coincides with $f_{\mathfrak{p},2}$, and $c = 2$ lies outside the Euclidean window of $\mathfrak{p}$. It is tempting to try the value $c = \sqrt{6}$; Nagata [151, 153] conjectured that this value makes $f_{\mathfrak{p},c}$ into a Euclidean function on $\mathbb{Z}[\sqrt{14}]$ and did some computations which support this conjecture. Bedocchi [8] has studied a function that – although not even being multiplicative – does not differ much from $f_{\mathfrak{p},\sqrt{6}}$. So far it has not been possible to prove that the Euclidean window of $\mathfrak{p}$ is non-empty using the method of Clark that succeeds for $D(69)$; even a modification of this idea due to R. Schroeppel and G. Niklasch does not seem to work (see also Hainke's thesis [88]). Cardon [22] shows that $\mathbb{Z}[\sqrt{14}, \frac{1}{2}]$ is Euclidean with respect to the absolute value of the S-norm, and Harper [89] showed that $\mathbb{Z}[\sqrt{14}, \frac{1}{p}]$ is Euclidean for each prime $p \in \mathbb{N}$. In his thesis [89], Harper succeeded in proving that $\mathbb{Z}[\sqrt{14}\,]$ *is* actually Euclidean.

Euclidean minima of real quadratic number fields have been computed by Heinhold [96], Davenport [53, 54, 55], Varnavides [187, 188, 189, 191], Bambah [3, 4], Inkeri [108], Barnes & Swinnerton-Dyer [5, 6, 7], Godwin [83], Bedocchi [9], and Lemmermeyer [116]; the table at the end of our survey gives the minima for all $m \leq 102$.

5. Cubic Number Fields

5.1. **Complex Cubic Number Fields.** It was Davenport [58] who first could prove that there are only finitely many norm-Euclidean complex cubic number fields; the best lower bound for $M(\overline{K})$ so far has been obtained by Cassels [23] (also, cf. van der Linden [138, 139, 140, 141], who notes that this bound does not seem to be best possible):

Proposition 5.1. *If K is a complex cubic number field with $d = |\operatorname{disc} K|$, then*

$$\frac{\sqrt{d}}{420} \le M(\overline{K}) \le \frac{d^{2/3}}{16\sqrt[3]{2}}.$$

In particular, $d < 170\,520$ if K is norm-Euclidean.

The upper bound is due to Swinnerton-Dyer (for fields with $|d| \le 1236$, Prop. 5.1 has been proved by direct computation), who also showed that the exponent $2/3$ and the constant $16\sqrt[3]{2}$ cannot be improved. Note that we cannot define a Davenport constant since we do not know if the exponent $1/2$ in the lower bound is best possible or not; it seems that no one has yet dared to conjecture that this exponent can be improved to $2/3$.

Already in 1848 Wantzel [194] claimed that the cubic field with discriminant -23 is norm-Euclidean. The next result concerning the Euclidean algorithm in complex cubic fields was obtained more than a hundred years later by Prasad [165], who showed $M(K) = \frac{1}{5}$ for the cubic field with $\operatorname{disc} K = -23$. In 1967, Godwin [85] showed that the fields with $-23 \ge \operatorname{disc} K \ge -152$ are norm-Euclidean, and E. Taylor [183, 184] found all norm-Euclidean fields with $0 > \operatorname{disc} K > -680$. The pure cubic number fields wich are norm-Euclidean were determined by Cioffari [37]: there are only three, namely $\mathbb{Q}(\sqrt[3]{m})$ with $m = 2, 3, 10$. See the tables at the end of this survey for known results on Euclidean minima of cubic fields.

In the tables below, let E denote the number of fields in a given interval which are norm-Euclidean; the number of those which are not norm-Euclidean will be denoted by N.

TABLE 1.

$\operatorname{disc} K$	E	N	Σ
$0 < d \le 200$	18	1	19
$200 < d \le 400$	15	9	24
$400 < d \le 600$	16	10	26
$600 < d \le 800$	7	20	27
$800 < d \le 1000$	2	29	31
$1000 < d \le 1200$	0	29	29
$1200 < d \le 1400$	0	35	35
$1400 < d \le 1600$	0	27	27
Σ	58	160	218

It is surprising that all cubic fields with $0 > \operatorname{disc} K > -500$ have an attained Euclidean minimum $M_1(\overline{K})$ with finite $C_1(K)$; this has to be seen in contrast to the situation for quadratic fields, where already $\mathbb{Q}(\sqrt{13}\,)$ and $\mathbb{Q}(\sqrt{29}\,)$ have infinite $C_1(K)$ and minima $M_1(\overline{K})$ which are not attained.

As in the quadratic case it is possible to compute the Euclidean minima of an infinite sequence of fields:

Proposition 5.2. *Let K be the number field defined by the real root α of $f(x) = x^3+2ax-1$ (where $a \geq 1$) and let $R = \mathbb{Z}[\alpha]$. Then $M(K) = M(\overline{K}) = \frac{1}{2}(a^2-a+1)$, and this minimum is attained exactly at $\xi \equiv \frac{1}{2}(1+\alpha+\alpha^2) \bmod R$.*

This result is due to Swinnerton-Dyer [182] for sufficiently large $a \geq 1$; Lemmermeyer [116] observed that it is valid for all $a \geq 1$. This sequence incidentally shows that the upper bound in Prop. 5.1 is best possible. Similar results for sufficiently large a are known for other families of cubic number fields (cf. Swinnerton-Dyer [182]).

Computer calculations have led to the following

Conjecture. There are exactly 58 norm-Euclidean complex cubic fields, and their discriminants are -23, -31, -44, -59, -76, -83, -87, -104, -107, -108, -116, -135, -139, -140, -152, -172, -175, -200, -204, -211, -212, -216, -231, -239, -243, -244, -247, -255, -268, -300, -324, -356, -379, -411, -419, -424, -431, -440, -451, -460, -472, -484, -492, -499, -503, -515, -516, -519, -543, -628, -652, -687, -696, -728, -744, -771, -815, -876.

The idea of Clark [39] has been used to show that the complex cubic fields with discriminants $-199, -327$, -351 and -367 are Euclidean with respect to weighted norms.

Let K be the field generated by a root α of the polynomial x^3+3x^2+6x+1, and let $f = f_{\mathfrak{p},c}$ be the weighted norm for the prime ideal $\mathfrak{p} = (11, \alpha-1)$. The Euclidean minimum $M_1(K)$ of $\mathcal{O}_K$ with respect to f is not known for all values $c \in w(\mathfrak{p})$, but it can be shown that $M_f(K) = \frac{187}{189}$ for all $c \geq \frac{189}{8}$. This minimum is attained $\bmod \mathcal{O}_K$ at the points

$$P = \pm\frac{1}{21}(10+6\alpha+6\alpha^2), \pm\frac{1}{21}(12+3\alpha+10\alpha^2), \pm\frac{1}{21}(15+16\alpha+9\alpha^2).$$

On the other hand, $M_f(K) = \frac{11}{c}$ for all real c in the interval $[11, \frac{189}{17})$, and this minimum is attained at the points $P = \pm(5+2\alpha+6\alpha^2)/11 \bmod \mathcal{O}_K$.

5.2. **Totally Real Cubic Number Fields.** Remak [169] proved Minkowski's conjecture for the cubic case, i.e.

Proposition 5.3. $M(\overline{K}) \leq \frac{1}{8}\sqrt{\operatorname{disc} K}$ *for totally real cubic fields.*

This implies in particular that the cubic number field with $\operatorname{disc} K = 49$ is norm-Euclidean. Some minima $M(K)$ have been computed by Davenport [56] ($\operatorname{disc} K = 49, 81$), Clarke [43] ($\operatorname{disc} K = 148$), Samet [171, 172] (for an infinite class of fields whose discriminants are "big enough"), Smith [179], and Lemmermeyer [116]. Clark [40] independently has shown some fields to be norm-Euclidean.

5.2.1. *Cyclic Fields.* Heilbronn [94] proved that the number of norm-Euclidean cyclic cubic fields is finite, but could give no bound for the discriminants of such fields. Smith [178] examined the cyclic cubic fields with discriminant $< 10^8$ and found that

- the fields with conductors $\mathfrak{f} = 7, 9, 13, 19, 31, 37, 43, 61, 67$ are Euclidean with respect to the norm;
- the fields with conductors $\mathfrak{f} = 73, 79, 97, 139, 151$, and $163 < \mathfrak{f} < 10^4$ are not norm-Euclidean.

Since fields with class number 1 have conductors which are prime powers, this left only the fields with conductors $\mathfrak{f} = 103, 109, 127, 157$ undecided; these were shown to be Euclidean by Godwin & Smith [86]. In the meantime, Lemmermeyer [116] had found that there are no norm-Euclidean fields with conductors $10^4 < \mathfrak{f} < 5 \cdot 10^5$.

5.2.2. *Non-cyclic Totally Real Fields.* Heilbronn [94] has conjectured that there are infinitely many norm-Euclidean fields of this type. The numerical results obtained so far are in favour of Heilbronn's conjecture, and in fact most of the fields with discriminants $\operatorname{disc} K < 10^4$ are norm-Euclidean. The following table gives the number E of totally real cubic fields (cyclic and non-cyclic) that are known to be norm-Euclidean; since the proportion of non-Euclidean fields seems to be growing, it is tempting to conjecture that the norm-Euclidean cubic fields have density 0.

TABLE 2.

$\operatorname{disc} K$	E	N	Σ
$0 < d \le 1000$	26	1	27
$1000 < d \le 2000$	29	5	34
$2000 < d \le 3000$	31	4	35
$3000 < d \le 4000$	36	6	42
$4000 < d \le 5000$	28	7	35
$5000 < d \le 6000$	35	7	42
$6000 < d \le 7000$	30	8	38
$7000 < d \le 8000$	37	10	47
$8000 < d \le 9000$	30	11	41
$9000 < d \le 10000$	29	10	39
$10000 < d \le 11000$	34	9	43
$11000 < d \le 12000$	37	16	53
$12000 < d \le 13000$	31	6	37
Σ	382	94	476

Explicit information on the real cubic fields with $\operatorname{disc} K < 13{,}000$ is given at the end of this article. There you can also find a table with cubic fields that have been shown to be Euclidean with respect to a weighted norm ([40],[28],[30]).

6. Quartic Number Fields

6.1. **Totally Complex Quartic Fields.** Davenport [59, 60] and Cassels [23] proved that the number of norm-Euclidean totally complex quartic fields is finite and gave a bound for the discriminants of such fields; his computation of the bound, however, was shown to contain a mistake by van der Linden [140].

Proposition 6.1. *If K is a totally complex quartic field and $d = \operatorname{disc} K$, then $M(K) > C \cdot \sqrt{d}$ for some constant $C > 0$. The best known C gives* $\operatorname{disc} K < 230\,202\,117$ *for Euclidean fields.*

There exist slightly better bounds for quadratic extensions of imaginary quadratic fields given by van der Linden ([140], 10.2), who used them to find all totally complex cyclic quartic fields that are norm-Euclidean:

Proposition 6.2. *The only norm-Euclidean totally complex cyclic quartic fields are $\mathbb{Q}(\zeta_5)$ and the quartic subfield of $\mathbb{Q}(\zeta_{13})$, where ζ_m denotes a primitive m-th root of unity.*

Let $D(m,n)$ denote the ring of integers in $\mathbb{Q}(\sqrt{m}, \sqrt{n}\,)$; the norm-Euclidean rings $D(-m,n)$, $m > 0$, have been determined by Lemmermeyer [116]:

Theorem 6.3. *The following list of norm-Euclidean rings $D(-m,n)$ with $m > 0$ is complete:*

$$\begin{array}{llllll} m=1, & n= & 2,3,5,7; & m=3, & n= & 2,5,-7,-11,17,-19; \\ m=2, & n= & -3,5; & m=7, & n= & 5. \end{array}$$

Eisenstein [72] established the Euclidean algorithm in $D(-1,2)$ and $D(-1,3)$ (these are the rings of integers in $\mathbb{Q}(\zeta_8)$ and $\mathbb{Q}(\zeta_{12})$, respectively); other proofs were given later by Masley [143, 144], and Lakein [114] showed that $D(-m,n)$ is norm-Euclidean for all the values (m,n) above except $(2,5)$, $(3,17)$, $(3,-19)$, and $(7,5)$. Sauvageot [173] showed that certain rings $D(m,n)$ are not norm-Euclidean, for example $D(-1,n)$ with $n \geq 15$. The proof of Thm. 6.3 in [119] is an extension (and correction) of her arguments; surprisingly, it is far less difficult than the proof of the corresponding result for real quadratic fields.

Proposition 6.4. *Suppose that $m > 0$ is no square and 4th-power free; then $\mathbb{Q}(\sqrt[4]{-m})$ is norm-Euclidean if and only if $m = 2,3,7,12$.*

This is largely due to Cioffari [37], who showed that if K is Euclidean then $m = 2,3,7,12,44,67$, or $2p^2$ for prime p; moreover he showed that $\mathbb{Q}(\sqrt[4]{-m})$ is norm-Euclidean for $m = 2,3,7$.

Apart from Prop. 6.1 – Prop. 6.4, there are only partial results on the Euclidean nature of complex quartic fields (cf. [116, 119])

Proposition 6.5. *Assume that K is a norm-Euclidean complex quartic field;*

i) *if K contains $k = \mathbb{Q}(\sqrt{2})$, then K is one of the fields $k(\sqrt{-1})$, $k(\sqrt{-3})$, $k\big(\sqrt{-5-2\sqrt{2}}\,\big)$;*
ii) *if K contains a real quadratic number field and 2 is totally ramified in K, then $K = \mathbb{Q}(\zeta_8) = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$;*
iii) *if K contains a real quadratic number field and 2 is the square of a prime ideal in K, then K is one of the fields $\mathbb{Q}(\zeta_{12})$, $\mathbb{Q}(\sqrt{-3}, \sqrt{2})$, $\mathbb{Q}(\sqrt{-3}, \sqrt{-2})$, $\mathbb{Q}(\sqrt{5}, \sqrt{-2})$;*
iv) *if $K = \mathbb{Q}(i, \sqrt{a+bi})$ with $i^2 = -1$ and $a + bi \equiv \pm 1 + 2i \bmod 4$, then $a + bi = \pm 1 + 2i, \pm 3 + 2i, \pm 5 + 2i, \pm 1 + 6i, \pm 7 + 2i$.*

All the fields given above are norm-Euclidean.

Best upper bounds on $M(\overline{K})$ seem to depend on the existence of a quadratic subfield of K; Davenport & Swinnerton-Dyer [62] found

Theorem 6.6. *Suppose that K is a totally complex quartic field which does not contain a real quadratic subfield. Then $M(\overline{K}) < C \cdot d^{3/4}$.*

They also claimed that the exponent 3/4 is best possible. For fields K that have real quadratic subfields, the best possible bound on $M(\overline{K})$ is $\frac{1}{32}\sqrt{d}$, as can be deduced from

Proposition 6.7. *Let $n \geq 1$ be odd, and put $m = n^2 + 1$; then the order $R = \mathbb{Z}[i, \sqrt{m}, \frac{1}{2}(\sqrt{m} + \sqrt{-m})]$ has Euclidean minimum $M = \frac{m}{4}$, and M is attained exactly at the points congruent to $\frac{1}{2}(1 + i + \sqrt{m}) \bmod \mathcal{O}_K$. If m is squarefree, we find $R = \mathcal{O}_K$, $\operatorname{disc} K = (8m)^2$, and $M(\overline{K}) = M(K) = \frac{1}{32}\sqrt{d}$.*

I do not know a family of totally complex quartic fields such that $M(K)$ is asymptotically equal to $C \cdot d^{3/4}$.

6.2. **Quartic Fields with Unit Rank** 2. Thanks to computations of R. Quême [166] we know quite a few examples of norm-Euclidean fields; on the other hand, negative results are quite rare:

Proposition 6.8. *There are only finitely many norm-Euclidean fields $\mathbb{Q}(\sqrt[4]{m}\,)$.*

Egami [70] proved (5.8) for all $m \neq 2p^2$ using estimates from analytic number theory; Lemmermeyer [116] gave an elementary proof using the technique of Behrbohm & Redei [10] and showed that in fact

$$m = 2, 3, 5, 7, 12, 13, 20, 28, 52, 61, 116, 436,$$

if $\mathbb{Q}(\sqrt[4]{m}\,)$ is norm-Euclidean. It should not be too hard to complete the classification of norm-Euclidean pure quartic fields. The fields with $m = 2$, 5, 12 and 20 are meanwhile known to be norm-Euclidean, and those with $m = 7$, 28, 52 and 436 are not. This leaves the open cases $m = 3$, 13, 61 and 116.

The following theorem is due to Davenport & Swinnerton-Dyer, who also claim that the exponent 2/3 is best possible:

Theorem 6.9. *$M(K) < C \cdot |d|^{2/3}$ for quartic fields with unit rank 2.*

Many quartic fields with mixed signature that are known to be Euclidean have been found by Lenstra [120, 124] using the method described in Sect. 9 below; in his dissertation, G. Kacerovsky [111] contributed the five quadratic extensions of $\mathbb{Q}(\sqrt{2})$ with smallest discriminants. Finally Quême (1997) used a computer to find lots of new Euclidean fields of this type.

6.3. **Totally Real Quartic Fields.** Almost no negative results are known; using the method of Heilbronn [95], Egami [71] has shown that there are some classes of cyclic fields which are not norm-Euclidean. A few more examples can be found in Clark's thesis [38], for example the bicyclic field $\mathbb{Q}(\sqrt{14}, \sqrt{22}\,)$.

The norm-Euclidean real quartic fields were found by Godwin [84], Kacerovsky [111], Cohn & Deutsch [45], Lemmermeyer [116], Niklasch & Quême [157], and R. Quême [166].

7. Quintic Number Fields

Most norm-Euclidean quintic fields before 1997 have been found with Lenstra's method (see Section 9); exceptions are the fields discovered by Godwin [84] and Schroeppel [175].

R. Quême has shown that the following quintic fields are Euclidean: the 92 fields with one real prime and discriminants $0 > \operatorname{disc} K \geq -37532$ except possibly $\operatorname{disc} K = -18463, -24671$; 146 fields with three real primes and discriminants $0 < \operatorname{disc} K \leq 17232$ except possible the field with $\operatorname{disc} K = 16129$; and the 25 totally real fields with $0 < \operatorname{disc} K \leq 161121$.

8. Cyclotomic Fields

It is known that the rings $\mathbb{Z}[\zeta_m]$ ($m \not\equiv 2 \bmod 4$) have class number 1 if and only if

$$\begin{aligned} m = 1,\ &3,\ 4,\ 5,\ 7,\ 8,\ 9,\ 11,\ 12,\ 13,\ 15,\ 16,\ 17,\ 19,\ 20,\\ &21,\ 24,\ 25,\ 27,\ 28,\ 32,\ 33,\ 35,\ 36,\ 40,\ 44,\ 45,\ 48,\ 60,\ 84; \end{aligned}$$

among these rings, the following are known to be norm-Euclidean:[3]

$$m = 1,\ 3,\ 4,\ 5,\ 7,\ 8,\ 9,\ 11,\ 12,\ 13,\ 15,\ 16,\ 20,\ 24,$$

and Lenstra [121] has shown that $K = \mathbb{Q}(\zeta_{32})$ is not norm-Euclidean; his proof actually shows that $M(K) \ge \frac{97}{64}$. There are only a few Euclidean minima known so far:

m	1	3	4	8	12
$M(K)$	$\frac12$	$\frac13$	$\frac12$	$\frac12$	$\frac14$

If we define $\Lambda(K) = \min\{N\mathfrak{a} : \mathfrak{a}$ is an integral ideal $\ne (0), \mathcal{O}_K\}$, then we have $M(K) = \Lambda(K)^{-1}$ in all these cases (of course we always have $M(K) \ge \Lambda(K)^{-1}$).

A masterful exposition of the interesting history of the Euclidean algorithm in cyclotomic fields can be found in Lenstra [128]; the names of the many mathematicians involved are displayed in the following table:

$(K : \mathbb{Q})$	m	
1	1	Euclid (ca. 300 B.C.)
2	3	Gauss, Wantzel (1847, 1848)
	4	Gauss (1832), Dirichlet (1844)
4	5	Kummer (1844), Cauchy [27], Ouspensky [161], Branchini [18], Chella [35], Landau [115], Lenstra (1975)
	8	Eisenstein (1850), Cauchy [27], Chella (1924), Lakein (1972), Masley (1975), Lenstra (1975)
	12	Eisenstein (1850), Cauchy [27], Chella (1924), Lakein (1972), Masley (1975), Lenstra (1975)
6	7	Kummer (1844), Cauchy [27], Chella (1924), Lenstra (1975)
	9	Cauchy [27], Chella (1924), Lenstra (1975)
8	15	Cauchy [27], Lenstra (1975)
	16	Ojala (1977)
	20	Lenstra (1975)
	24	Lenstra (1978)
10	11	Lenstra (1975)
12	13	McKenzie [110]

Kummer conjectured (in a letter to Kronecker) that the fields $\mathbb{Q}(\zeta_p)$, $p = 17$, 19 are also Euclidean, but this has not been verified so far. For more details on Euclid's algorithm in cyclotomic number fields, see Akhtar [2] and Philibert [163].

It was known for a long time that only a finite number of complex subfields of cyclotomic fields have class number 1, and recently they have been determined (K. Yamamura, The determination of the imaginary abelian number fields with class

[3] Thanks to Julien Houriet for notifying me of the fact that $m = 21$ somehow had crept into this list.

number one; Math. Comp. 62 (1994), 899–921); there are exactly 172 such fields. The Euclidean fields among them are known for $(K : \mathbb{Q}) = 2, 4$.

9. Exceptional Sequences

In 1974, Lenstra discovered that a modification of an idea originally due to Hurwitz [103, 104, 105] yields a new method to find Euclidean number fields K of high degree ($5 \le (K : \mathbb{Q}) \le 10$): he called a sequence $\omega_1, \omega_2, \ldots, \omega_m$ an exceptional sequence of length m in K if the differences $\omega_i - \omega_j, i \neq j$, are units in $\mathcal{O}_K$.

Let r (resp. $2s$) denote the number of real (resp. non-real) embeddings of K in $\mathbb{C}$, and let $d = |\operatorname{disc} K|$. Then Lenstra was able to prove

Theorem 9.1. *There exist constants $\alpha_{r,s} > 0$ with the following property: if K contains an exceptional sequence of length $m > \alpha_{r,s}\sqrt{d}$, then K is norm-Euclidean.*

Lenstra showed that the "Minkowski bounds"

$$\alpha_{r,s} = \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s, \quad \pi = 3.14159\ldots, \quad n = (k : \mathbb{Q}) = r + 2s,$$

were good enough to find many new Euclidean fields, and that, for most of the values r, s, the bounds of Rogers are even better. For totally real fields, the $\alpha_{r,s}$ given by Lenstra have been sharpened by Niklasch & Quême [157].

For a given number field K, the length of exceptional sequences is bounded: if $\omega_1, \omega_2, \ldots, \omega_m$ is an exceptional sequence of maximal length in K, then $\lambda(K) = m$ is called *Lenstra's constant*. If $\Lambda(K)$ denotes the minimal norm of an integral ideal $\neq (0), (1)$ in $\mathcal{O}_K$, then it is easily seen that $\lambda(K) \le \Lambda(K)$. Note the analogy $M(K) \ge \Lambda(K)^{-1}$; computations have confirmed that both inequalities tend to be equalities for fields with very small discriminants. Moreover, we know the values of $\lambda(K)$ and $\Lambda(K)$ for cyclotomic fields $K = \mathbb{Q}(\zeta_p)$ of prime conductor: the decomposition law for abelian extensions of $\mathbb{Q}$ shows that $\Lambda(K) = p$. Lenstra [120, 124] found that in fact $\lambda(K) = \Lambda(K)$:

Proposition 9.2. *Let p be prime, $\zeta = \zeta_p$ a primitive p-th root of unity, and $K = \mathbb{Q}(\zeta)$. Then the sequence*

$$\omega_j = \frac{\zeta^j - 1}{\zeta - 1}, 1 \le j \le p,$$

shows that $\lambda(K) = \Lambda(K) = p$.

The analogous question for the maximal real subfields $k = \mathbb{Q}(\zeta + \zeta^{-1})$ of $\mathbb{Q}(\zeta)$ is not yet completely settled: here $\Lambda(k) = p$ unless $p \ge 5$ is a Fermat prime ($p = 2^{2^n} + 1$), where $\Lambda(k) = p - 1$. Lenstra [124] could show that $\lambda(k) \ge \frac{p+1}{2}$, and Leutbecher & Niklasch [137] improved this to $\lambda(k) \ge p - 1$. For all $p \le 17$ it is known that $\lambda(k) = \Lambda(k)$, but the general case is still open.

Similar questions can be asked for ray class fields of prime conductor over imaginary quadratic number fields; Mestre [146] used elliptic curves to construct exceptional sequences for such fields, but it is not known how far from best possible his bounds are.

Exceptional sequences were studied by Lenstra [120, 124], Leutbecher & Martinet [135, 136] (these two articles contain several open problems), Leutbecher [132, 133], Niklasch [154], Leutbecher & Niklasch [137], and Niklasch & Quême [157].

Even sequences where many (not all) differences are units can be used to show that a given number field is norm-Euclidean; see e.g. Leutbecher & Niklasch [137] or Leutbecher [134].

Lenstra's theorem was generalized by Lemmermeyer [116]: call $\omega_1, \omega_2, \ldots, \omega_m$ a k-exceptional sequence of length m if $\omega_i - \omega_j$ is a nonzero element of the Motzkin set E_k for all $1 \le i < j \le m$. Then the following theorem gives a device to discover k-stage norm-Euclidean number fields:

Proposition 9.3. *If K contains a k-exceptional sequence of length $m \ge \alpha_{r,s}\sqrt{d}$, for the same constants $\alpha_{r,s}$ as in Thm. 9.1, then K is k-stage norm-Euclidean.*

As a corollary of Prop. 9.3, we conclude that every Euclidean number field is also k-stage norm-Euclidean for a suitable $k \ge 1$: choose any sequence $\omega_1, \omega_2, \ldots, \omega_m$ in $\mathcal{O}_K$ such that $m \ge \alpha_{r,s}\sqrt{d}$; since $R = \mathcal{O}_K$ is Euclidean, we have $R = E_\infty(R)$ by Motzkin. Therefore, the $\omega_i - \omega_j$, $1 \le i < j \le m$, are non-zero elements of E_k for some $k \ge 1$, and R is k-stage Euclidean with respect to the norm.

It is not known whether k-exceptional sequences are always finite for $k \ge 3$.

Another generalization of Thm. 9.1 is due to Blöhmer [14]; he considered sequences $\omega_1, \omega_2, \ldots, \omega_m$ in $\mathcal{O}_K$ such that the $N(\omega_i - \omega_j)$ are ± 1 or prime and showed that $\mathcal{O}_K$ is principal if $m \ge \alpha_{r,s}\sqrt{d}$.

10. Gauss's Measure Function

Let K be a number field; in order to prove that $|N_{K/\mathbb{Q}}|$ is a Euclidean function on $\mathcal{O}_K$ it is sufficient to find a function $F : K \longrightarrow \mathbb{R}$ such that

a) $|N_{K/\mathbb{Q}}(\alpha)| \le F(\alpha)$ for all $\alpha \in K$;
b) for all $\xi \in K$, there is a $\gamma \in \mathcal{O}_K$ such that $F(\xi - \gamma) < 1$.

Define $\mathcal{M}_K(\alpha) = \frac{1}{n}\sum |\sigma(\alpha)|^2$, where $n = (K : \mathbb{Q})$ is the degree of K, and where the sum is over all n embeddings $\sigma : K \longrightarrow \mathbb{R}$. Except for the factor $\frac{1}{n}$, this function was introduced by Gauss. It was then used by Lenstra [121], Ojala [158] and McKenzie [110] to find Euclidean cyclotomic fields. This function $\mathcal{M}$ has the following properties:

Proposition 10.1. *Let $K \subseteq L$ be number fields, and put $n = (K : \mathbb{Q})$. Then*

1. $|N_{K/\mathbb{Q}}(\alpha)| \le \mathcal{M}_K(\alpha)^{n/2}$;
2. $\mathcal{M}_L(\alpha) - \mathcal{M}_L(\alpha - \beta) = \mathcal{M}_K\big(\frac{1}{(L:K)}\mathrm{Tr}_{L/K}(\alpha)\big) - \mathcal{M}_K\big(\frac{1}{(L:K)}\mathrm{Tr}_{L/K}(\alpha) - \beta\big)$ *for all $\alpha \in L$, $\beta \in K$.*
3. *If $L = K(\zeta_m)$, then $(L : K)\mathcal{M}_L(\alpha) = \frac{1}{m}\sum_{j=1}^m \mathcal{M}_K\big(\mathrm{Tr}_{L/K}(\alpha\zeta_m^j)\big)$.*

These slight generalizations of results of Lenstra [121] can be found in [116]. If we put

$$F_K = \{\xi \in K : \mathcal{M}(\xi) \le \mathcal{M}(\xi - \gamma) \text{ for all } \gamma \in \mathcal{O}_K\}$$

and $c(K) = \sup\{\mathcal{M}_K(\xi) : \xi \in F_K\}$, then for every $\xi \in K$ there is a $\gamma \in \mathcal{O}_K$ such that $\mathcal{M}(\xi - \gamma) \le c(K)$. Thus K is norm-Euclidean if $c(K) < 1$; sometimes even $c(K) = 1$ is sufficient. Call $c' \in \mathbb{R}$ a usable bound if $c' \ge c(K)$, and if for all $\xi \in F_K$ such that $\mathcal{M}(\xi) = c'$ there exists a root of unity $\zeta \in \mathcal{O}_K$ and a $\gamma \in \mathcal{O}_K$ such that $\mathcal{M}(\xi - \gamma) = \mathcal{M}(\xi - \gamma - \zeta) = c'$. In particular, every $c' > c(K)$ is a usable bound.

Proposition 10.2. *If c' is a usable bound for K, then K is norm-Euclidean.*

The central result is

Theorem 10.3. *Let ζ_m be a primitive m^{th} root of unity, and $L = K(\zeta_m)$. Then $c(L) \le (L:K)c(K)$, and if c' is a usable bound, then so is $(L:K)c'$.*

It is an easy exercise to show that $c(\mathbb{Q}) = \frac{1}{4}$, and that $c(\mathbb{Q})$ is a usable bound. This implies at once that $\mathbb{Q}(\zeta_m)$ is norm-Euclidean for $m = 3, 4, 5, 8, 12$. Lenstra [121] determined the exact value of $c(K)$ for cyclotomic fields of prime conductor:

Proposition 10.4. *For $K = \mathbb{Q}(\zeta_p)$, p an odd prime, $c(K) = \frac{p+1}{12}$ is a usable bound.*

Thus $\mathbb{Q}(\zeta_m)$ is norm-Euclidean for $m = 7, 11$ (directly) and for $m = 9, 15, 20$ (by using the subfields $\mathbb{Q}(\zeta_3)$ and $\mathbb{Q}(\zeta_4)$).

Since $c(K) = M(K)$ for imaginary quadratic number fields, only $\mathbb{Q}(\sqrt{-3}\,)$ and $\mathbb{Q}(\sqrt{-4}\,)$ have $c(K) \le \frac{1}{2}$; by an elementary argument (related to Dirichlet's result 4.2) one can compute $c(K)$ for real quadratic fields:

Proposition 10.5. *Let $K = \mathbb{Q}(\sqrt{m}\,)$ be a real quadratic number field (m is assumed to be squarefree). Then*

$$c(K) = \begin{cases} \frac{1+m}{4}, & \text{if } m \equiv 2, 3 \bmod 4 \\ \frac{(1+m)^2}{16m}, & \text{if } m \equiv 1 \bmod 4 \end{cases}$$

Thus $c(K) = \frac{9}{20}$ for $K = \mathbb{Q}(\sqrt{5}\,)$, hence the fields $\mathbb{Q}(\sqrt{5}, \sqrt{-1}\,)$, $\mathbb{Q}(\sqrt{5}, \sqrt{-3}\,)$ and $\mathbb{Q}(\sqrt{5}\,)$ are norm-Euclidean.

We also know $c(K)$ for certain families of biquadratic number fields:

Proposition 10.6. *Let $m, n \in \mathbb{Z}$ be squarefree and put $K = \mathbb{Q}(\sqrt{m}, \sqrt{n}\,)$; then*

$$c(K) = \begin{cases} \frac{1+|m|}{4}\big(1 + \frac{1+|m|}{4|m|}\,|n|\big) & \text{if } \mathcal{O}_K = \mathbb{Z}[\sqrt{m}, \sqrt{n}, \frac{1}{2}(\sqrt{n} + \sqrt{mn})], \\ \frac{(1+|m|)^2}{16|m|}(1 + |n|) & \text{if } \mathcal{O}_K = \mathbb{Z}[\frac{1}{2}(1+\sqrt{m}), \sqrt{n}, \frac{1}{2}(\sqrt{n} + \sqrt{mn})], \end{cases}$$

and these bounds are usable.

This leaves the case $\mathcal{O}_K = \mathbb{Z}[\frac{1}{2}(1+\sqrt{m}), \frac{1}{2}(1+\sqrt{n}), \frac{1}{4}(1+\sqrt{m})(1+\sqrt{n})]$ open; it is easy to see that $c(K) \le \frac{(1+|m|)^2}{16|m|}\big(1 + \frac{|n|}{4}\big)$, and this implies that $\mathbb{Q}(\sqrt{-3}, \sqrt{5}\,)$ and $\mathbb{Q}(\sqrt{-3}, \sqrt{-7}\,)$ are norm-Euclidean.

Our last result on usable bounds is

Proposition 10.7. *Let $\mu = a + b\sqrt{-3}$ be a prime in $\mathbb{Z}[\zeta_3]$, where a is odd and $a + b \equiv 1 \bmod 4$. Put $p = a^2 + 3b^2$ and $K = \mathbb{Q}(\sqrt{a + b\sqrt{-3}}\,)$; then $c(K) \le \frac{1}{12}(4 + \sqrt{p}\,)$.*

The best possible bound is not known here, but this result is good enough to show that $\mu = -1 + 2\sqrt{-3}$, $3 + 2\sqrt{-3}$, 5, $-5 + 2\sqrt{-3}$, -7, $-3 + 4\sqrt{-3}$, $7 - 2\sqrt{-3}$ yield norm-Euclidean fields.

We conjecture that there are only finitely many number fields with bounded $c(K)$.

E. Bayer-Fluckiger recently introduced the concept of thin fields; thin fields are necessarily norm-Euclidean, but much more rare.

11. Number Fields of Degree ≥ 6

Most of the norm-Euclidean number fields of degree ≥ 6 have been found with Lenstra's method; exceptions are some cyclotomic fields (cf. Sect. 8), those found by R. Quême [166], and the field $\mathbb{Q}(\zeta_{32} + \zeta_{32}^{-1})$ that was shown to be norm-Euclidean by J.-P. Cerri [31, 32] in 1997. The discriminants and generating polynomials for the other fields can be found in the papers of Lenstra, Leutbecher, Martinet, Mestre, Niklasch, and Quême cited in Sect. 9. Recently, Julien Houriet (not yet published) found three norm-Euclidean fields of degree 10, 11 and 12 with $r + s = 6$ among a list of fields computed by Denis Simon.

We conclude our survey with the now traditional

Table 3. Table of all known norm-Euclidean number fields (November 1997)

$r+s$ \ n	1	2	3	4	5	6	7	8	9	10	11	12	Σ
1	1	5											6
2		16	58	118									192
3			382	681	92	28							1183
4				257	146	37	39	45					524
5					25	12	26	65	92	50			270
6						7	4	5	2	1	1	2	22
7							0	0	0	0	0	0	0
8								1	0	0	0	0	1
Σ	1	21	440	1056	263	84	70	115	94	51	1	2	2198

Similar tables can be found in Lenstra [124], Leutbecher [132], Leutbecher & Niklasch [137].

Moreover, the fields in Table 4 are known to be Euclidean with respect to a weighted norm (Clark [40], Niklasch [155], Cavallar & Lemmermeyer [30]):

As we have mentioned in Sect. 3, Clark has found a lot of totally real cubic and quartic number fields which are Euclidean with respect to functions different from the norm, for example the quartic field $\mathbb{Q}(\sqrt{14}, \sqrt{22}\,)$ (see Clark & Murty [42]).

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Table 4.

disc K	$M_1(K)$	$M_2(K)$	$N\mathfrak{p}$	w($\mathfrak{p}$)
-367	1	9/13	13	$(13, 279/8)$
-351	1	9/11	11	$(11, \infty)$
-327	101/99	< 0.9	11	$(101/9, \infty)$
-199	1	< 0.47	7	$(7, \infty)$
985	1	5/11	5	$(5, \infty)$
1345	7/5	< 0.4	5	$(7, \infty)$
1825	7/5	< 0.5	5	$(7, \infty)$
1929	1	3/7	7	$(7, \infty)$
1937	1	5/9	3	$(3, \infty)$
2777	5/3	17/19	3	$\varnothing$
2836	7/4	7/8	2	$(7, \infty)$
2857	8/5	< 0.5	5	$(8, \infty)$
3305	13/9	37/45	3	$(\sqrt{13}, 5)$
3889	13/7	1	7	$(13, \infty)$
4193	7/5	< 0.65	5	$(7, \infty)$
4345	7/5	11/13	5	$(7, \infty)$
4360	41/35	7/10	7	$(41/5, \infty)$
5089	17/11	7/11	11	$(17, \infty)$
5281	1	< 0.6	5	$(5, \infty)$
5297	21/11	23/33	11	$(21, \infty)$
5329	9/8	63/73	2^3	$(9, 73)$
5369	21/19	17/19	19	$(21, \infty)$
5521	23/7	8/7	7	$(23, \infty)$
7273	973/601	729/601	601	$(973, \infty)$
7465	1	< 0.8	5	$(5, \infty)$
7481	1	< 0.7	5	$(5, \infty)$

12. Tables

The following table gives the minima $M_1(K)$ for all real quadratic number fields $\mathbb{Q}(\sqrt{m}\,)$, $m \le 102$:

m	M_1	m	M_1	m	M_1
5	1/4	2	1/2	47	253/94
13	1/3	3	1/2	51	287/102
17	1/2	6	3/4	55	9/4
21	5/7	7	9/14	58	3/2
29	4/5	10	3/2	59	125/59
33	29/44	11	19/22	62	13/4
37	3/4	14	5/4	66	15/4
41	23/32	15	3/2	67	341/162
53	9/7	19	170/171	70	891/500
57	14/19	22	27/22	71	7393/3479
61	1611/1525	23	77/46	74	5/2
65	1	26	5/2	78	7/2
69	25/23	30	3/2	79	585/158
73	1541/2136	31	45/31	82	9/2
77	19/11	34	9/4	83	631/166
85	16/9	35	5/2	86	10030/5203
89	1004287/1000004	38	11/4	87	169/58
93	44/31	39	5/2	91	5/2
97	33679354/31404817	42	7/4	94	4708623/2143294
101	5/4	43	11829/6962	95	7/2
		46	79877/48668	102	19/4

To the best of my knowledge, there are no minima known for fields beyond this limit, except for some sequences of fields like $\mathbb{Q}(\sqrt{m}\,)$, $m = n^2 \pm r$, $r|4$ etc. (compare 3.3).

This is what we know about minima for the 2-stage algorithm:

m	$M^2(K)$	B_1	$B_2 = B_\infty$	Eucl. depth
6	1/4	$\varnothing$	$\varnothing$	1
10	1	$\{(0, \frac{1}{2})\}$	$\{(0, \frac{1}{2})\}$	1
14	1/4	$\{(\frac{1}{2}, \frac{1}{2})\}$	$\varnothing$	2
15	1	?	$\{(\frac{1}{2}, \frac{1}{2})\}$	2
26	1	?	$\{(0, \frac{1}{2})\}$	2
30	3/2	?	$\{(0, \frac{1}{2})\}$	2
34	1	?	$\{(\pm\frac{1}{3}, \pm\frac{1}{3})\}$	2
35	7/5	?	$\{(0, \pm\frac{2}{5}), (\frac{1}{2}, \frac{1}{2})\}$	2
39	5/2	?	$\{(\frac{1}{2}, \frac{1}{2})\}$	2
65	1	$\{(\frac{1}{4}, \pm\frac{1}{4})\}$	$\{(\frac{1}{4}, \pm\frac{1}{4})\}$	1
85	1	?	$\{(\pm\frac{1}{6}, \pm\frac{1}{6})\}$	2

Only the classes mod $\mathcal{O}_K$ of the sets B_1 and $B_2 = B_\infty$ are given.

The table below gives the known Euclidean minima for complex cubic number fields with $|\mathrm{disc}\, K| \le 971$:

d_K		$M_1(K)$	$M_2(K)$	d_K		$M_1(K)$	$M_2(K)$
-23	E	$1/5$	$\ge 1/7$	-116	E	$1/2$	
-31	E	$1/3$	$< 1/4$	-135	E	$3/5$	
-44	E	$1/2$	$1/4$	-139	E	$1/2$	
-59	E	$1/2$	$1/4$	-140	E	$1/2$	
-76	E	$1/2$	$1/3$	-152	E	$1/2$	
-83	E	$1/2$		-172	E	$3/4$	
-87	E	$1/3$		-175	E	$3/5$	
-104	E	$1/2$		-199	N	1	< 0.47
-107	E	$1/2$		-200	E	$1/2$	
-108	E	$1/2$	$1/3$	-204	E	$61/116$	
-211	E	$59/106$		-283	H	$3/2$	
-212	E	$5/8$		-300	E	$23/30$	
-216	E	$1/2$		-307	N	$9/8$	$3/4$
-231	E	$7/9$		-324	E	$23/36$	$7/11$
-239	E	$8/9$		-327	N	$101/99$	
-243	E	$11/18$		-331	H	$3/2$	
-244	E	$1/2$		-335	N	1	
-247	E	$5/7$		-339	N	$9/8$	1
-255	E	$13/15$		-351	N	1	$9/11$
-268	E	$13/22$	$\ge 6/11$	-356	E	$7/8$	
-364	N	$9/8$		-451	E	$41/48$	
-367	N	1	$9/13$	-459	N	$9/8$	
-379	E	$397/648$	$\ge 11/18$	-460	E	$43/50$	$23/30$
-411	E	$17/22$	$\ge 8/11$	-472	E	$46/61$	
-419	E	$4/5$		-484	E	$59/76$	
-424	E	$19/27$	$\ge 53/76$	-491	H	2	≥ 1
-431	E	$43/64$		-492	E	$25/32$	
-436	N	$79/78$		-499	E	$23/27$	
-439	N	$17/15$	≥ 1	-503	E	$\ge 307/544$	
-440	E	$737/1090$		-515	E	$4/5$	$\ge 11/14$
-516	E	$36/53$		-628	E	$625/664$	
-519	E	$44712/45747$		-643	H	$25/16$	
-524	N	$5/4$		-648	H	$5/4$	
-527	N	$13/7$		-652	E	$21/23$	
-543	E	$\ge 158664/170633$		-655	N	$40/23$	
-547	N	$9/8$		-671	N	$25/19$	
-563	H	2		-675	N	$9/8$	
-567	N	$25/17$	$\ge 19/17$	-676	H	$7/4$	
-588	H	$5/2$		-679	N	$9/8$	
-620	N	$13/8$	$5/4$	-680	N	$(*)$	

d_K		$M(K)$	d_K		$M(K)$
-687	E	$937/945$	-751	H	$25/9$
-695	N	$25/13$	-755	N	1
-696	E	$186/199$	-756	N	$306/293$
-707	N	$271/270$	-759	N	$11/8$
-716	N	$121/109$	-771	E	$223/252$
-728	E	(§)	-780	N	$499/498$
-731	H	2	-804	N	$\geq 2771/2568$
-743	N	1	-808	N	$\geq 2031/1964$
-744	E	$992/999$	-812	N	$44/31$
-748	N	$62/51$	-815	E	$24543/25325$
-823	N	$37/25$	-891	H	$7/2$
-835	N	$110353/106265$	-907	N	$\geq 113/108$
-839	N	$25/17$	-908	N	$227/91$
-843	N	$134/131$	-931	H	$7/2$
-856	N	$\geq 454951/428544$	-932	N	$68425/56788$
-863	N	$29/11$	-940	N	$407/358$
-867	N	$1115/1028$	-948	N	$\geq 2120/1959$
-876	E	$353/372$	-959	N	$19/7$
-883	N	$49/47$	-964	N	$\geq 132/127$
-888	N	$2715/2602$	-971	N	$829/778$
-972	N	$5/4$	-1036	N	$133/101$
-972	N	$179/162$	-1048	N	$617/488$
-980	H	$7/4$	-1055	N	$\geq 1483/1370$
-983	N	$31/11$	-1059	N	$2381/1854$
-984	N	$\geq 22367/21296$	-1067	N	$\geq 160/121$
-996	N	$\geq 6713/5646$	-1068	N	$\geq 1499/1350$
-999	N	$\geq 294557/272112$	-1075	N	$777/680$
-1004	N	$3167/2298$	-1080	N	$\geq 10253/1000$
-1007	N	$41/23$	-1083	H	$3/2$
-1011	N	$271/207$	-1087	N	$15/8$
-1096	N	$\geq 207/199$	-1176	H	$4/3$
-1099	H	$47/26$	-1187	N	$11/8$
-1107	H	2	-1188	N	$\geq 22319/14072$
-1108	N	$\geq 4995/4384$	-1191	N	$11/9$
-1135	N	$5115/4033$	-1192	H	$265/168$
-1144	N	$4867/3222$	-1196	N	$197/94$
-1147	N	$136/99$	-1203	N	$\geq 4775/4608$
-1164	N	$\geq 1064/918$	-1207	N	$13/9$
-1172	N	$572/443$	-1208	N	$845/656$
-1175	N	$37/13$	-1219	N	$\geq 709/622$

d_K		$M(K)$	d_K		$M(K)$
-1228	H	$7/2$	-1291	N	$196/139$
-1228	H	$9/2$	-1292	N	$98/53$
-1228	H	$9/2$	-1295	N	$11/7$
-1231	N	$15/8$	-1300	N	$1381/978$
-1235	N	$283/169$	-1315	N	$249/157$
-1236	N	$\geq 5017/4246$	-1316	N	$931/601$
-1255	H	$\geq 8/5$	-1319	N	$49/17$
-1259	N	$13/8$	-1323	H	$5/2$
-1267	N	$\geq 1503/1048$	-1327	N	$56/31$
-1272	N	$\geq 16648/15987$	-1336	N	$967/844$
-1347	N	$47441858/35095129$	-1399	H	$37/9$
-1351	N	$81/43$	-1407	N	$15/8$
-1355	N	$\geq 95/79$	-1419	N	$1903/1406$
-1356	H	$7/4$	-1420	N	$1193/561$
-1356	H	$9/4$	-1423	H	$25/7$
-1356	H	$5/3$	-1427	N	$41236/26029$
-1363	N	$892/663$	-1431	N	$119/59$
-1371	H	$9/2$	-1432	N	$\geq 46751/33530$
-1383	N	$227/131$	-1439	N	$\geq 51777/550016$
-1388	N	$10711/5780$	-1448	N	$9395/6268$
-1452	N	$3425/1947$	-1563	H	$9/2$
-1464	N	$\geq 98048/93807$	-1567	N	$311/171$
-1480	N	$\geq 5801/3930$	-1572	H	$7/4$
-1484	N	$\geq 14503/10874$	-1579	N	$1197/824$
-1491	N	$\geq 17053/12018$	-1580	N	$223/109$
-1512	N	$\geq 49952/32217$	-1583	N	$1049/337$
-1515	N	$\geq 24182/17025$	-1588	H	$345/172$
-1539	N	$15906827/11384640$	-1599	N	$13/8$
-1547	N	$250/149$	-1603	N	$812/513$
-1559	N	$\geq 150079/137093$	-1607	N	

In this table as well as in those below, E means that the corresponding field is Euclidean (more exactly: that $M(K) < 0.999$), N indicates that it is not norm-Euclidean although it has class number 1, and H that the field has class number > 1. Instead of upper bounds on $M(K)$ we have sometimes given lower bounds, especially in those cases where we conjecture them to be exact without being able to prove this. The table is ordered in the same way as those at Bordeaux (i.e. for fields with the same discriminants, such as -972 or -1228).
(*) The Euclidean minimum $M(K)$ for the field with $\operatorname{disc} K = -680$ is

$$M(K) = \frac{81956632}{81182612}.$$

It is attained at the points

$$P_1 = \frac{1}{828394}(152556 - 267595\alpha - 332013\alpha^2),$$
$$P_2 = \frac{1}{828394}(-273732 + 188225\alpha + 300357\alpha^2),$$
$$P_3 = \frac{1}{828394}(-374312 + 21305\alpha + 407143\alpha^2).$$

(§) The Euclidean minimum $M(K)$ for the field with $\operatorname{disc} K = -728$ is

$$M(K) = \frac{7483645229}{8158377554}.$$

Euclidean minima of totally real cubic number fields

d_K		$M(K)$	d_K		$M(K)$	d_K		$M(K)$
49	E	1/7	469	E	1/2	788	E	1/2
81	E	1/3	473	E	1/3	837	E	1/2
148	E	1/2	564	E	1/2	892	E	1/2
169	E	5/13	568	E	1/2	940	E	1/2
229	E	1/2	621	E	1/2	961	E	16/31
257	E	1/3	697	E	13/31	985	N	1
316	E	1/2	733	E	1/2	993	E	31/63
321	E	1/3	756	E	1/2	1016	E	1/2
361	E	8/19	761	E	1/3	1076	E	1/2
404	E	1/2	785	E	3/5	1101	E	1/2
1129	E	1/3	1425	E	13/15	1708	E	1/2
1229	E	16/29	1436	E	1/2	1765	E	13/20
1257	E	9/25	1489	E	29/43	1772	E	1/2
1300	E	7/10	1492	E	1/2	1825	N	7/5
1304	E	1/2	1509	E	1/2	1849	E	22/43
1345	N	7/5	1524	E	1/2	1901	E	1/2
1369	E	31/37	1556	E	3/4	1929	N	1
1373	E	1/2	1573	E	19/22	1937	N	1
1384	E	11/16	1593	E	< 0.36	1940	E	1/2
1396	E	1/2	1620	E	1/2	1944	E	1/2
1957	H	2	2241	E	3/5	2636	E	1/2
2021	E	1/2	2292	E	1/2	2673	E	64/81
2024	E	1/2	2296	E	1/2	2677	E	139/224
2057	E	9/11	2300	E	27/40	2700	E	83/120
2089	E	1/2	2349	E	11/18	2708	E	1/2
2101	E	1/2	2429	E	1/2	2713	E	< 0.5
2177	E	< 0.39	2505	E	5/9	2777	H	5/3
2213	E	1/2	2557	E	1/2	2804	E	1/2
2228	E	1/2	2589	E	9/16	2808	E	1/2
2233	E	56/121	2597	H	5/2	2836	N	7/4
2857	N	8/5	3137	E	< 0.59	3325	E	
2917	E	8/13	3144	E	1/2	3356	E	
2920	E	13/20	3173	E	< 0.59	3368	E	
2941	E	1/2	3221	E	1/2	3496	E	
2981	E	1/2	3229	E	1/2	3508	E	
2993	E	< 0.49	3252	E		3540	E	
3021	E	1/2	3261	E		3569	E	
3028	E	1/2	3281	E		3576	E	
3124	E	1/2	3305	N	13/9	3580	E	
3132	E	1/2	3316	E		3592	E	

d_K		$M(K)$	d_K		$M(K)$	d_K		$M(K)$
3596	E		3892	E		4104	E	< 0.55
3604	E		3941	E		4193	N	7/5
3624	E		3957	E		4212	H	7/2
3721	E	121/183	3969	H	7/3	4281	E	< 0.7
3732	E		3969	H	1	4312	N	11/4
3736	E		3973	E	1/2	4344	E	< 0.7
3753	E		3981	H	3/2	4345	N	7/5
3873	E		3988	N	19/8	4360	N	41/35
3877	E		4001	E	7/9	4364	E	
3889	N	13/7	4065	E	3/5	4409	E	
4481	E		4729	N	149/73	4860	E	
4489	E	53/67	4749	E		4892	E	
4493	E		4764	E	17/24	4933	E	
4596	E		4765	E		5073	E	
4597	E		4825	E		5081	E	
4628	E		4841	E		5089	N	17/11
4641	E		4844	E		5172	E	
4649	E		4852	E		5204	E	
4684	N	13/8	4853	E		5261	E	
4692	E	< 0.7	4857	E		5281	N	1
5297	N	21/11	5468	E		5629	E	
5300	E		5477	E		5637	E	
5325	E		5497	E		5684	N	9/2
5329	N	9/8	5521	N	23/7	5685	E	
5333	E		5529	E		5697	E	
5353	E		5556	E		5724	E	
5356	E		5613	E		5741	E	
5368	E		5620	E		5780	E	
5369	N	21/19	5621	E		5821	E	
5373	E		5624	E		5853	E	
5901	E		6153	E		6420	E	
5912	E		6184	E		6452	N	5/4
5925	E		6185	N	17/15	6453	E	
5940	E		6209	E		6508	E	
5980	E		6237	E		6549	E	
6053	E		6241	N	223/79	6556	E	
6088	E		6268	E		6557	E	
6092	E		6289	N	1	6584	E	
6108	E		6396	E		6588	E	
6133	E		6401	N	35/27	6601	E	

d_K		$M(K)$	d_K		$M(K)$	d_K		$M(K)$
6616	E		6901	E		7220	H	9/4
6637	E		6940	E		7224	E	
6669	E		6997	E		7244	E	
6681	E		7028	E		7249	E	
6685	E		7032	E		7273	N	973/601
6728	E		7053	H	2	7388	E	
6809	H	7/3	7057	E		7404	E	
6856	E		7084	E		7425	E	
6868	N	5/4	7117	E		7441	E	
6885	N	67/40	7148	E		7444	E	
7453	E		7601	E		7745	N	7/5
7464	E		7628	E		7753	E	
7465	N	1	7636	E		7796	E	
7473	E	< 0.89	7641	E		7816	E	
7481	N	1	7665	E	21/25	7825	E	
7528	N	17/14	7668	E		7873	N	29/13
7537	N	227/91	7673	E		7881	E	
7540	E		7700	E		7892	E	
7572	E		7709	E		7925	E	
7573	N	41/32	7721	E		7948	E	
8017	E		8281	H	9/7	8532	E	
8057	E		8285	E		8545	E	
8069	H	9/2	8289	E		8556	E	
8092	E		8308	N	67/50	8572	N	17/16
8113	N	13/7	8372	E		8597	E	4/5
8173	E		8373	E		8628	E	
8220	E		8396	E		8637	E	
8276	E		8468	H	5/3	8680	E	
8277	E		8472	E		8692	N	11/10
8281	H	23/16	8505	E		8713	E	
8745	E		8920	E		9217	N	17/11
8761	E		9044	E		9281	E	
8769	E		9045	E		9293	E	
8789	N	23/12	9073	N	7/5	9300	E	
8828	E		9076	E		9301	H	2
8829	N	3/2	9133	E		9325	N	13/8
8837	E		9149	E		9364	E	
8884	E		9153	E		9409	N	337/97
8905	N	8/5	9192	E		9413	E	
8909	E		9204	E		9428	E	

d_K		$M(K)$	d_K		$M(K)$	d_K		$M(K)$
9460	E		9812	E		10004	E	
9517	E		9813	E		10040	E	
9565	E	$\geq 4/5$	9833	E		10069	E	
9612	E		9836	E		10077	E	
9653	N	35/12	9869	E		10164	N	27/22
9676	E		9897	E		10172	E	
9745	N	67/23	9905	N	9/5	10200	E	
9749	E		9937	E		10216	N	7/4
9800	H	9/5	9980	E		10233	E	
9805	E		9996	H	4/3	10260	E	
10261	N	11/7	10540	E		10721	E	
10273	H	27/7	10552	E		10733	E	
10292	E	9/10	10561	N	11/7	10740	E	
10301	E		10580	E		10812	E	
10309	H	11/2	10609	E		10844	E	
10324	E		10636	E		10865	E	
10333	N	1	10641	E		10868	E	
10353	E		10661	E		10889	H	13/5
10457	N	27/25	10664	E		10904	E	
10484	E		10712	E		10929	E	
10941	E		11085	E		11316	E	
10949	E		11092	E		11321	E	
10997	E		11097	N	11/9	11324	H	3/2
11013	E		11109	E		11348	H	9/4
11020	E		11124	N	5/4	11380	E	
11028	E		11137	E		11401	N	167/151
11032	E		11188	N	5/4	11417	H	11/3
11045	E		11197	H	31/8	11421	N	49/36
11057	E		11289	E		11448	E	
11060	E		11293	E		11476	E	
11505	E		11697	E		11853	E	
11545	E		11705	N	213/193	11880	E	
11576	E		11757	E		11881	E	
11608	E		11772	E		11884	E	
11637	N	5/4	11777	N	27/17	11885	E	
11641	E		11789	E		11965	N	23/8
11656	N	11/8	11821	N	23/16	12001	E	
11665	E		11829	E		12065	N	1
11672	E		11848	E		12081	N	152/149
11688	E		11849	N	19/9	12092	E	

12140	E	< 0.85	12325	E		12657	E	< 0.9
12177	E		12333	E		12660	N	23/18
12188	E		12401	E	< 0.75	12664	E	
12197	N	3/2	12409	E	< 0.9	12685	E	
12216	E	< 0.95	12436	E		12700	E	37/40
12248	E		12441	E	781/837	12724	E	< 0.95
12269	E		12552	E	< 0.9	12744	E	< 0.8
12284	E		12577	N	49/19	12765	E	23/20
12309	E		12632	E		12788	E	
12317	N	25/22	12652	E		12821	E	< 0.97

Euclidean minima of totally complex quartic number fields

d_K		$M(K)$	d_K		$M(K)$	d_K		$M(K)$
117	E	$\geq 1/7$	333	E		592	E	
125	E	$\geq 1/5$	392	E		605	E	
144	E	$1/4$	400	E	$5/16$	656	E	$1/2$
189	E	$\geq 1/3$	432	E		657	E	
225	E	$1/4$	441	E	$4/9$	697	E	
229	E		512	E	$1/2$	761	E	
256	E	$1/2$	513	E		784	E	$1/2$
257	E		549	E		788	E	
272	E	$1/4$	576	E		832	E	
320	E	$1/2$	576	E		837	E	
873	E		1076	E		1229	E	
892	E		1088	E		1257	E	
981	E		1088	E		1264	E	
985	E		1089	E		1280	N	$5/4$
1008	E		1129	E		1372	E	
1008	E		1161	E		1384	E	
1016	E		1168	E		1396	E	
1025	E		1197	E		1413	E	
1040	E		1197	E		1421	E	
1040	E		1225	E	$9/16$	1424	E	
1436	N		1600	E	$11/16$	1825	E	
1489	E		1616	E		1856	E	
1492	E		1629	E		1872	H	
1509	E		1728	E		1929	E	
1521	H	1	1737	E		1936	N	$5/4$
1525	E		1765	E		1937	E	
1552	E		1805	N		1940	E	
1556	E		1808	E		1953	E	
1568	E		1809	E		1953	E	
1593	E		1813	E		2021	E	
2048	E		2169	E		2312	E	
2048	N		2192	E		2320	E	
2057	E		2197	E		2320		
2061	E		2213			2349		
2089			2256	E		2368	E	
2112	E		2272	E		2429	E	
2112			2292	E		2448	H	
2133	E		2296	E		2457	H	
2156			2304	H		2457	H	
2156	E		2304	H	$5/2$	2493		

d_K		$M(K)$	d_K		$M(K)$	d_K		$M(K)$
2560	N	$5/4$	2709			2889	H	
2560	N	$5/4$	2725	E		2917		
2597			2736			2920		
2597	E		2736	E		2925	H	
2601	E	$13/16$	2744			2960		
2624			2781	E		2960		
2673	E		2817			2981		
2677			2836	E		3024	E	
2704	N		2873			3024	H	
2709			2880	H		3025	H	
3028			3221	E		3429		
3033	E		3229			3528	H	
3072			3249	E	$\geq 7/9$	3573		
3072			3261			3600	H	
3088	N		3305			3600	H	
3136	H	$9/8$	3316	E		3600	H	
3136			3328			3624		
3136			3357			3625	H	
3141	E		3368	E		3636	H	
3173	E		3392	E	$\geq 50/53$	3648		
3648			3773			4001		
3681			3789			4032		
3700	H		3856			4032		
3725	N		3877			4077		
3728	N		3889			4112		
3732			3897	H		4113		
3753			3904	N		4212		
3753			3973			4221		
3757	E		3988			4221		
3760			3993			4225	H	

Euclidean minima of quartic number fields of mixed signature

d_K		$M(K)$	d_K		$M(K)$	d_K		$M(K)$
−275	E		−688	E		−1192	E	
−283	E		−731	E		−1255	E	
−331	E		−751	E		−1323	E	
−400	E		−775	E		−1328	E	
−448	E		−848	E		−1371	E	
−475	E		−976	E		−1375	E	
−491	E		−1024	E		−1399	E	
−507	E		−1099	E		−1423	E	
−563	E		−1107	E		−1424	E	
−643	E		−1156	E		−1456	E	
−1472	E		−1823	E		−2000	E	
−1472	E		−1856	E		−2048	E	
−1475	E		−1879	E		−2051	E	
−1588	E		−1927	E		−2068	E	
−1600	E		−1931	E		−2092	E	
−1728	E		−1963	E		−2096	E	
−1732	E		−1968	E		−2116	E	
−1775	E		−1975	E		−2151	E	
−1791	E		−1984	E		−2183	E	
−1792	E		−1984	E		−2191	E	
−2219	E		−2480	E		−2764	E	
−2243	E		−2488	E		−2767	E	
−2284	E		−2563	E		−2787	E	
−2312	E		−2608	E		−2816	E	
−2319	E		−2619	E		−2824	E	
−2327	E		−2687	E		−2843	E	
−2375	E		−2696	E		−2859	E	
−2412	E		−2704	E		−2911	E	
−2443	E		−2736	E		−2943	E	
−2475	E		−2763	E		−3008	E	
−3052	E		−3284	E		−3475	E	
−3119	E		−3303	E		−3504	E	
−3163	E		−3312	E		−3544	E	
−3175	E		−3312	E		−3559	E	
−3188	E		−3376	E		−3571	E	
−3216	E		−3407	E		−3600	E	
−3223	E		−3411	E		−3632	E	
−3267	E		−3424	E		−3723	E	
−3271	E		−3431	E		−3747	E	
−3275	E		−3436	E		−3751	E	

d_K		$M(K)$	d_K		$M(K)$	d_K		$M(K)$
−3775	E		−3951	E		−4152	E	
−3776	E		−3967	E		−4192	E	
−3776	E		−3984	E		−4204	E	
−3816	E		−4027	E		−4275	E	
−3875	E		−4027	E		−4287	E	
−3887	E		−4032	E		−4319	E	
−3888	E		−4063	E		−4384	E	
−3891	E		−4103	E		−4400	E	
−3899	E		−4107	E		−4423	E	
−3919	E		−4108	E		−4432	E	
−4475	E		−4615	E		−4775	E	
−4491	E		−4648	E		−4780	E	
−4492	E		−4652	E		−4799	E	
−4503	E		−4663	E		−4832	E	
−4544	E		−4671	E		−4864	E	
−4564	N		−4675	E		−4907	E	
−4568	E		−4703	E		−4944	E	
−4595	E		−4744	E		−4975	E	
−4608	E		−4748	E		−4979	E	
−4608	E		−4752	E		−4999	E	
−5036	E		−5348	E		−5552	E	
−5056	E		−5371	E		−5568	E	
−5184	E		−5424	E		−5591	E	
−5224	E		−5431	E		−5595	E	
−5231	E		−5432	E		−5616	E	
−5243	E		−5448	E		−5616	E	
−5260	E		−5476	E		−5636	E	
−5275	E		−5488	N	$\geq 9/7$	−5644	E	
−5323	E		−5491	E		−5675	E	
−5343	E		−5548	E		−5732	N	
−5748	E		−5987	E		−6331	E	
−5755	E		−6043			−6336	E	
−5792	E		−6064	E		−6336	E	
−5816	E		−6071	E		−6343	E	
−5867	E		−6075	E		−6371	E	
−5887	E		−6079	E		−6387	E	
−5888	E		−6091	E		−6399	E	
−5932	E		−6199	E		−6411	E	
−5963	E		−6275	E		−6444	E	
−5975	E		−6283	E		−6480		

d_K		$M(K)$	d_K		$M(K)$	d_K		$M(K)$
−6484	E		−6656	E		−6775	E	
−6507	E		−6664	E		−6791	E	
−6571	E		−6687	E		−6800	E	
−6571	E		−6691	E		−6848	E	
−6571	E		−6700	E		−6848	H	
−6591	E		−6724	E		−6863	E	
−6592	E		−6739	E		−6880	E	
−6603	E		−6763	E		−6883	E	
−6604	E		−6768	E		−6883	E	
−6611	E		−6768	E		−6883		
−6896	E		−6987	E		−7344	E	
−6912			−7087			−7351	E	
−6912	E		−7088	E		−7375	E	
−6924	E		−7155	E		−7407	E	
−6928	E		−7199	E		−7412	E	
−6928	E		−7259	E		−7463	E	
−6939	E		−7267	E		−7472	E	
−6967	E		−7331	E		−7492	E	
−6975	E		−7335	E		−7528	E	
−6976	E		−7344	E		−7532	E	
−7571	E		−7732	E		−7948	E	
−7600	E		−7744	E		−7952	E	
−7616	E		−7771	E		−7971	E	
−7616	E		−7775	E		−7975	H	
−7652	E		−7779	E		−7975	H	
−7668	E		−7803	E		−7988	E	
−7692	E		−7864	E		−8000	E	
−7699	E		−7912	E		−8048	E	
−7703	E		−7936	E		−8108	E	
−7715	E		−7947	E		−8112	E	
−8123			−8207	E		−8492		
−8127	E		−8208	E		−8571	E	
−8128	E		−8236	E		−8579	E	
−8131	E		−8248	E		−8587	E	
−8152	E		−8256	E		−8591	E	
−8172			−8275	E		−8619	E	
−8180	E		−8287	E		−8619	E	
−8183			−8303	E		−8624	E	
−8196	E		−8375	H		−8640	E	
−8203	E		−8392	E		−8640	E	

d_K		$M(K)$	d_K		$M(K)$	d_K		$M(K)$
−8640	E		−8752	E		−8912	E	
−8667			−8752	E		−8960	E	
−8672	E		−8752	E		−8972	E	
−8676	E		−8763	E		−8975	E	
−8684	E		−8787	E		−9004	E	
−8707	E		−8856	E		−9008	E	
−8712	E		−8867	E		−9008	E	
−8712	E		−8875	E		−9012	E	
−8724	E		−8896			−9015	E	
−8739	E		−8896	E		−9019	E	
−9028	E		−9187	E		−9408	E	
−9036	E		−9216			−9408	E	
−9059	E		−9247			−9423	E	
−9071	E		−9248			−9452	E	
−9099	E		−9251	E		−9463	E	
−9127			−9260	E		−9475	E	
−9136	E		−9356	E		−9484	E	
−9136	E		−9364			−9488	E	
−9155	E		−9384	E		−9491	E	
−9163			−9395	E		−9519	E	
−9527	E		−9728	E		−9896	E	
−9531	E		−9747	E		−9899	E	
−9583	E		−9748	E		−9972		
−9612	E		−9751	E		−10048	E	
−9663	E		−9783	E		−10059	E	
−9664	E		−9823	E		−10064	E	
−9667	E		−9843	E		−10079		
−9680	E		−9875	E		−10091		
−9687	E		−9887	E		−10120	E	
−9704	E		−9888	E		−10152	E	
−10156	E		−10288	E		−10476	E	
−10160	E		−10288	E		−10531		
−10163	E		−10296	E		−10559	E	
−10187			−10339	E		−10611	E	
−10192	E		−10348	E		−10640	E	
−10224	E		−10355			−10688		
−10224	E		−10367	E		−10691	E	
−10247	E		−10404	E		−10704	E	
−10252	E		−10475	H		−10719	E	
−10287	E		−10475	E		−10720	E	

d_K		$M(K)$	d_K		$M(K)$	d_K		$M(K)$
−10732	E		−10832	E		−11003	E	
−10735	E		−10859			−11043	E	
−10751	E		−10895	E		−11052	E	
−10771	E		−10912	E		−11112	E	
−10775	E		−10951	E		−11127	E	
−10796	E		−10960	E		−11155	E	
−10800	H		−10975	H		−11163	E	
−10800	E		−10975	E		−11200	E	
−10816	E		−11003	E		−11200	H	
−10816	E		−11003	E		−11252	E	
−11275	E		−11440	E		−11627		
−11275	E		−11448	E		−11675		
−11279	E		−11500	H		−11731	E	
−11280			−11552	E		−11812	E	
−11300	E		−11568			−11823	E	
−11403			−11588	E		−11843	E	
−11404	E		−11596	E		−11884	E	
−11407	E		−11600	H		−11907		
−11408	E		−11600	H		−11943	E	
−11419	E		−11607	E		−11944	E	
−11948	E							

Euclidean minima of real quartic number fields

d_K	$M(K)$	d_K	$M(K)$	d_K	$M(K)$
725	E	2777	E	5125	E
1125	E	3600	E	5225	E
1600	E	3981	E	5725	E
1957	E	4205	E	5744	E
2000	E	4225	E	6125	E
2048	E	4352	E	6224	E
2225	E	4400	E	6809	E
2304	E	4525	E	7053	E
2525	E	4752	E	7056	E
2624	E	4913	E	7168	E
7225	E	8525	E	10025	E
7232	E	8725	E	10273	E
7488	E	8768	E	10304	E
7537	E	8789	E	10309	E
7600	E	8957	E	10512	E
7625	E	9225	E	10816	E
8000	E	9248	E	10889	E
8069	E	9301	E	11025	E
8112	E	9792	E	11197	E
8468	E	9909	E	11324	E
11344	E	13068	E	14013	E
11348	E	13448	E	14197	E
11525	E	13525	E	14272	E
11661	E	13625	E	14336	E
12197	E	13676	E	14400	E
12357	E	13725		14656	E
12400	E	13768	E	14725	E
12544	E	13824	E	15125	
12725	E	13888	E	15188	E
13025	E	13968	E	15317	E
15529	E	17069	E	18496	E
15952	E	17417	E	18625	E
16225	E	17424	E	18688	E
16317	E	17428	E	18736	E
16357	E	17600	E	19025	E
16400		17609	E	19225	E
16448	E	17725		19429	E
16448	E	17989	E	19525	E
16609	E	18097	E	19600	E
16997	E	18432		19664	E

d_K	$M(K)$	d_K	$M(K)$	d_K	$M(K)$
19773	E	21208	E	22221	E
19796	E	21308	E	22545	E
19821	E	21312	E	22592	E
20032	E	21469	E	22676	E
20225	E	21568	E	22784	E
20308	E	21725	E	22896	E
20808	E	21737	E	23252	E
21025	H	21801	E	23297	E
21056	E	21964	E	23301	E
21200	E	22000		23377	E
23525		24525	E	25808	E
23552	E	24749	E	25857	E
23600	E	24832	E	25893	E
23665	E	24917	E	25961	E
23724	E	25088	E	26032	E
24197	E	25225	E	26125	E
24336		25488	E	26176	E
24400	E	25492	E	26224	E
24417	E	25525	E	26225	
24437	E	25717	E	26525	E
26541	E	27792	E	29248	E
26569	E	28025	E	29268	E
26825	E	28224	E	29813	E
26873	E	28224	E	29952	E
27004	E	28400		30056	E
27225	E	28473	E	30056	E
27329	E	28669	E	30125	
27472	E	28677	E	30273	E
27648	E	28749	E	30400	E
27725		29237	E	30512	E
30544	E	31808	E	33344	E
30725	E	32081	E	33424	E
30776	E	32225		33428	E
30972	E	32368	E	33452	E
30976	E	32448	E	33489	E
31225	E	32625		33525	E
31288	E	32737	E	33625	
31532	E	32821	E	33709	E
31600	E	32832	E	33725	
31744	E	33097	E	33813	E

d_K		$M(K)$	d_K		$M(K)$	d_K		$M(K)$
33844	E		35152			36416	E	
34025	E		35225			36517	E	
34196	E		35312	E		36677	E	
34225	E		35392	E		36761	E	
34704	E		35401	E		36928	E	
34816			35537	E		37108	E	
34868	E		35537	E		37229	E	
35013	E		35537	E		37349	E	
35125	E		35856	E		37485	E	
35136	E		36025	E		37485	E	
37489	E		39528	E				
37525								
37773	E							
37885	E							
37952								
38000								
38225								
38720	E							
38725								
38864								

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