HERMITE CONSTANT AND EXTREME FORMS FOR ALGEBRAIC NUMBER FIELDS

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1. *Introduction*

In this paper we consider a generalization to algebraic number fields of the classical Hermite constant γ_n. For this constant we extend the well-known Minkowski bound and study the notion of extreme forms in this setting.

Let us briefly recall the definition of Hermite's constant. Let (a_{ij}) be a positive definite symmetric real $n \times n$ matrix and let $f(x_1, \ldots, x_n) = \sum_{ij} a_{ij} x_i x_j$ be its associated quadratic form. Let us set

$$\mu(f) = \min\{f(x) \mid x \in \mathbb{Z}^n - \{0\}\}, \quad d(f) = \det(a_{ij}), \quad \gamma(f) = \mu(f)/d(f)^{1/n}.$$

Hermite's constant is defined to be

$$\gamma_n := \sup_f \{\gamma(f)\},$$

where f runs over all positive definite real quadratic forms. Hermite [**5**] proved that $\gamma_n \leqslant (4/3)^{n(n-1)/2}$ and later Minkowski improved this bound to $\gamma_n \leqslant 4\omega_n^{-2/n}$, where $\omega_n = \pi^{n/2}/\Gamma(\frac{1}{2}n+1)$ denotes the volume of the n-dimensional unit sphere. This constant has been widely studied and it is still an open problem to determine its exact value for $n \geqslant 9$. Following an idea suggested by R. Baeza, we extend the definition of Hermite's constant to arbitrary number fields and obtain upper bounds for it that extend Minkowski's bound for γ_n. Based on the works of Voronoi [**10**] and Oesterlé [**9**] we also generalize the notion of extreme forms. We establish the existence of such forms and give a partial characterization of them. We make use of Humbert's reduction theory [**6**, **7**] and some results from [**9**].

Let K be a number field with $[K\colon \mathbb{Q}] = m = r+2s$. Let $\{\sigma_1, \ldots, \sigma_r\}$ be the real embeddings of K and $\{\sigma_{r+1}, \ldots, \sigma_m\}$ be the complex embeddings with $\sigma_{r+i} = \bar{\sigma}_{r+s+i}$ for $1 \leqslant i \leqslant s$ where $^-$ denotes complex conjugation. We write $\mathcal{O}_K$ for the ring of algebraic integers of K. We consider tuples of forms $f = (f_{\sigma_i})$ for $1 \leqslant i \leqslant r+s$, where $f_{\sigma_i}(x) = \sum_{kj} a_{kj}^{(i)} x_k x_j$ is a positive definite real quadratic form in n variables for $1 \leqslant i \leqslant r$ and $f_{\sigma_i}(x) = \sum_{kj} a_{kj}^{(i)} x_k \bar{x}_j$ is a positive definite complex hermitian form in n variables for $r+1 \leqslant i \leqslant r+s$. We call such a system of $r+s$ forms, a *Humbert tuple*. For any Humbert tuple, let us define

$$\mu(f) = \min_{x \in \mathcal{O}_K^n - \{0\}} \left\{ \prod_{i=1}^{r} f_{\sigma_i}(\sigma_i(x)) \prod_{i=r+1}^{r+s} (f_{\sigma_i}(\sigma_i(x)))^2 \right\},$$

where for $x = (x_1, \ldots x_n) \in \mathcal{O}_K^n$ and $\sigma \in \{\sigma_1, \ldots, \sigma_{r+s}\}$ we write $\sigma(x) = (\sigma(x_1), \ldots, \sigma(x_n))$.

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We further define the determinant of a Humbert tuple $f=(f_{\sigma_i})$ to be

$$d(f)=\prod_{i=1}^{r}\det(f_{\sigma_i})\prod_{i=r+1}^{r+s}(\det(f_{\sigma_i}))^2. \tag{1}$$

Then we set $\gamma(f):=\mu(f)/d(f)^{1/n}$. Hermite–Humbert's constant is now defined to be $\gamma_{n,K}:=\sup_f\{\gamma(f)\}$, where f runs over all tuples of forms described above. According to Humbert's reduction theory, $\gamma_{n,K}<\infty$, but the upper bound remains unspecified in Humbert's work. In §2 we obtain, without using Humbert's work, an explicit upper bound for $\gamma_{n,K}$ that depends only on K and n.

REMARK 1. If K is a totally real number field, then $\gamma_{n,K}$ can also be defined in the following way. Let f be a positive definite quadratic form over K, that is, for each embedding σ of K into $\mathbb{R}$, and for $f(x)=\sum_{k,j}a_{kj}x_kx_j$ with $a_{kj}\in K$, we have $\sigma(f)=\sum_{kj}\sigma(a_{kj})x_kx_j$ is a positive definite real quadratic form. We set

$$\mu(f):=\min_{x\in\mathcal{O}_K^n-\{0\}}\{N_{K/\mathbb{Q}}(f(x))\},\quad d(f):=N_{K/\mathbb{Q}}(\det(f)),$$

where $N_{K/\mathbb{Q}}$ denotes the norm from K to $\mathbb{Q}$. Then put $\gamma(f):=\mu(f)/d(f)^{1/n}$. Since K is dense in $\mathbb{R}^{[K:\mathbb{Q}]}$, it follows that $\gamma_{n,K}$ can be defined by $\gamma_{n,K}:=\sup_f\{\gamma(f)\}$, where f runs over all positive definite quadratic forms defined over K.

In this paper we shall prove the following results.

THEOREM 1. *Let K be a number field of degree $[K:\mathbb{Q}]=m=r+2s$ with r real embeddings and $2s$ complex embeddings. Let D_K be the discriminant of K. Then*

$$\gamma_{n,K}\leqslant 4^{r+s}\omega_n^{-2r/n}\omega_{2n}^{-2s/n}|D_K|,$$

where ω_n denotes the volume of the n-dimensional unit sphere.

THEOREM 2. *There exists a Humbert tuple $f=(f_{\sigma_i})_{i=1}^{r+s}$ such that*

$$\gamma(f)=\gamma_{n,K}.$$

We call a tuple $f=(f_{\sigma_i})$ that satisfies the condition of Theorem 2 a *critical tuple*. The more general notion of extreme tuple will be studied in §3. We also introduce there the definitions of perfect and eutactic Humbert tuples. We prove that if a positive definite Humbert tuple is extreme, then it is perfect and eutactic. This result partially generalizes a theorem of Voronoi [**9**, **8**].

2. *Proof of Theorem* 1

We refer the reader to the statement of Theorem 1 given in the Introduction.

Proof of Theorem 1. Let σ_i for $1\leqslant i\leqslant r$ be the real embeddings of K and let σ_i for $r\leqslant i\leqslant r+2s$ be the complex embeddings of K, with $\sigma_{r+i}=\bar{\sigma}_{r+s+i}$ for $1\leqslant i\leqslant s$. Consider the Humbert tuple $f=(f_{\sigma_1},\dots,f_{\sigma_r},f_{\sigma_{r+1}},\dots,f_{\sigma_{r+s}})$ with the first r components being positive definite real n-dimensional quadratic forms and the remaining s components being positive definite complex hermitian forms of the same dimension. Suppose that each f_{σ_i} has a matrix $[f_{\sigma_i}]=A_i$. Then from (1),

$$d(f)=\prod_{i=1}^{r}\det(f_{\sigma_i})\prod_{i=r+1}^{r+s}(\det(f_{\sigma_i}))^2=\prod_{i=1}^{r}\det A_i\left(\prod_{i=r+1}^{r+s}\det A_i\right)^2.$$

For $1\leqslant i\leqslant r$ there is a matrix $B_i\in\mathbb{M}_{n\times n}(\mathbb{R})$ with $\det(B_i)=1$ such that $D_i=B_i^T A_i B_i:=A_i[B_i]$ is a diagonal matrix $D_i=\operatorname{diag}(d_1^{(i)},\dots,d_n^{(i)})$. Here B^T denotes the transpose of a matrix B (see [**4**]). Then for $x\in\mathcal{O}_K^n-\{0\}$ we have $A_i[\sigma_i(x)]=\|C_i B_i^{-1}\sigma_i(x)\|^2$, where $\|z\|^2=z^T z$ denotes the square of the Euclidean norm of the vector $z\in\mathbb{R}^n$, and C_i denotes the diagonal matrix $C_i=\operatorname{diag}(\sqrt{d_1^{(i)}},\dots,\sqrt{d_n^{(i)}})$. Similarly, for $r+1\leqslant i\leqslant r+s$, we diagonalize the matrices A_i and obtain $D_i=\bar{B}_i^T A_i B_i:=A_i[B_i]$, where $B_i\in\mathbb{M}_{n\times n}(\mathbb{C})$ and $C_i=\operatorname{diag}(c_1^{(i)},\dots,c_n^{(i)})$ such that $c_j^{(i)}\bar{c}_j^{(i)}=d_j^{(i)}$ for $1\leqslant j\leqslant n$. Thus $A_i[\sigma_i(x)]=\|C_i B_i^{-1}\sigma_i(x)\|^2$, where $\|z\|^2=\bar{z}^T z$.

Let us define the following domain in $\mathbb{R}^{nr}\times\mathbb{C}^{ns}$. For

$$l>0\quad\text{and}\quad y=(y_1,\dots,y_r,z_{r+1},\dots,z_{r+s})\in\mathbb{R}^{nr}\times\mathbb{C}^{ns},$$

let

$$V_l:=\{y\colon \|C_i B_i^{-1}y_i\|^2\leqslant l \text{ and } \|C_i B_i^{-1}z_i\|^2\leqslant l \text{ for } 1\leqslant i\leqslant r+s\}.$$

The volume of V_l, is given by

$$\operatorname{vol}(V_l)=\frac{\omega_n^r\, l^{rn/2}\,\omega_{2n}^s\, l^{sn}}{(\prod_{i=1}^{r}\det A_i)^{1/2}(\prod_{i=r+1}^{r+s}\det A_i)}=\frac{\omega_n^r\,\omega_{2n}^s\, l^{mn/2}}{d(f)^{1/2}}.$$

We impose conditions on l so that Minkowski's convex body theorem holds for V_l. Therefore we need

$$\operatorname{vol}(V_l)\geqslant 2^{mn}\operatorname{vol}(\mathcal{O}_K^n)=2^{mn}2^{-ns}|D_K|^{n/2}. \tag{2}$$

It will follow that for all $x=(x_1,\dots,x_{r+s})\in V_l$, the inequality

$$\frac{\prod_{i=1}^{r}f_{\sigma_i}(x_i)\prod_{i=r+1}^{r+s}(f_{\sigma_i}(x_i))^2}{d(f)^{1/n}}\leqslant C_{n,K}$$

holds for a certain constant $C_{n,K}$ which we will determine below.

In (2) it is enough to choose l so that the equality holds. Therefore it is enough to take l so that

$$\frac{\omega_n^r\,\omega_{2n}^s\, l^{mn/2}}{d(f)^{1/2}}=2^{mn}\operatorname{vol}(\mathcal{O}_K^n)=2^{mn}2^{-ns}|D_K|^{n/2}.$$

From now on we choose l so that

$$l=\left(\frac{2^{n(r+s)}}{\omega_n^r\,\omega_{2n}^s}|D_K|^{n/2}d(f)^{1/2}\right)^{2/mn}.$$

On the other hand, for $x\in V_l$ we have

$$\left[\prod_{i=1}^{r}f_{\sigma_i}(x_i)\prod_{i=r+1}^{r+s}(f_{\sigma_i}(x_i))^2\right]^n=\left(\prod_{i=1}^{r}\|C_i B_i^{-1}(x_i)\|^2\prod_{i=r+1}^{r+s}\|C_i B_i^{-1}(x_i)\|^4\right)^n\leqslant l^{rn}\,l^{2sn}$$
$$=\frac{2^{2nr}}{\omega_n^{2r}}\frac{2^{2ns}}{\omega_{2n}^{2s}}|D_K|^n\, d(f).$$

Therefore

$$\frac{\prod_{i=1}^{r} f_{\sigma_i}(x_i) \prod_{i=r+1}^{r+s} (f_{\sigma_i}(x_i))^2}{(d(f))^{1/n}} \leqslant \frac{2^{2r}}{\omega_n^{2r/n}} \frac{2^{2s}}{\omega_{2n}^{2s/n}} |D_K| = C_{n,K}.$$

By Minkowski's Convex Body Theorem, there exists $x \in \mathcal{O}_K^n - \{0\}$ so that $(\sigma_1(x), \ldots, \sigma_{r+s}(x)) \in V_t$ and thus we obtain $\gamma_{n,K} \leqslant 4^{r+s} \omega_n^{-2r/n} \omega_{2n}^{-2s/n} |D_K|$.

REMARK 2. It is clear that when $K = \mathbb{Q}$ we recover Minkowski's bound given in §1.

3. *Proof of Theorem* 2

In this section we keep the notation introduced above. Instead of working with Humbert tuples of forms we shall work with Humbert tuples of matrices of the form $A = (A_i)$ for $1 \leqslant i \leqslant r+s$, where the first r entries are positive definite symmetric real $n \times n$ matrices and the remaining s matrices are positive define $n \times n$ complex hermitian matrices. The definitions of μ, d and γ are extended in the obvious way for tuples of matrices. In the set of all tuples of matrices we introduce the following equivalence relation. Let $\{\sigma_i\}$ be the embeddings of K arranged as before, and let $A = (A_1, \ldots, A_{r+s})$, $B = (B_1, \ldots, B_{r+s})$. We say that $A = (A_i)$ is equivalent to $B = (B_i)$ if there exists $C \in \mathrm{GL}_n(\mathcal{O}_K)$ such that if we put $C_i = \sigma_i(C)$, then $A_i = C_i^T B_i C_i$ for $1 \leqslant i \leqslant r$, and $A_i = \bar{C}_i^T B_i C_i$ for $r+1 \leqslant i \leqslant r+s$. We denote by $\mathscr{H}$ the set of classes of tuples $[A]$.

Since $[K: \mathbb{Q}] = m = r + 2s$, we have an inclusion

$$\mathrm{GL}_n(K) \to (\mathrm{GL}_n(\mathbb{R}))^r \times (\mathrm{GL}_n(\mathbb{C}))^s.$$

Therefore $\mathrm{GL}_n(\mathcal{O}_K)$ can be identified with a subgroup $\Gamma \subseteq (\mathrm{GL}_n(\mathbb{R}))^r \times (\mathrm{GL}_n(\mathbb{C}))^s$. We introduce the map

$$\psi : (\mathrm{GL}_n(\mathbb{R}))^r \times (\mathrm{GL}_n(\mathbb{C}))^s / \Gamma \longrightarrow \mathscr{H}$$

defined by $\psi((A_1, \ldots, A_r, \ldots, A_{r+s})) = [(A_1^T A_1, \ldots, A_r^T A_r, \ldots, \bar{A}_{r+s}^T A_{r+s})]$. It is easily seen that ψ is a surjection and we use this fact to endow $\mathscr{H}$ with the quotient topology. We also define the continuous map

$$\phi : \mathscr{H} \longrightarrow (0, \infty),$$

where $[A] \mapsto \mu(A)/d(A)^{1/n}$. Then

$$\chi = \phi \circ \psi : (\mathrm{GL}_n(\mathbb{R}))^r \times (\mathrm{GL}_n(\mathbb{C}))^s / \Gamma \longrightarrow (0, \infty)$$

is also continuous.

Theorem 2 amounts to proving that the map χ attains its maximum.

Before we continue with the proof of Theorem 2, we need to introduce more definitions and notation. Let $\mathscr{L}^{r,s} = \{(x_1, \ldots, x_r; x_{r+1}, \ldots, x_{r+s}) \in \mathbb{R}^r \times \mathbb{C}^s\}$ be the set of $(r+s)$-tuples, where the first r components are real and the remaining s components are complex. Let

$$\mathscr{L}_*^{r,s} = \{\mathrm{x} \in \mathscr{L}^{r,s} : x_i \neq 0 \text{ for } 1 \leqslant i \leqslant r+s\}.$$

Thus we have a map $l\colon \mathscr{L}_*^{r,s} \to \mathbb{R}^{r+s}$ which is defined by $l(x_1, \ldots, x_{r+s}) = (\log|x_1|, \ldots, \log|x_r|, \log|x_{r+1}|^2, \ldots, \log|x_{r+s}|^2)$. If $\varepsilon_1, \ldots, \varepsilon_{r+s-1}$ is a set of independent units of K, then if we define

$$l^* = (\underbrace{1, \ldots, 1}_{r}, \underbrace{2, \ldots, 2}_{s}),$$

the set $\{l^*, l(\varepsilon_1), \ldots, l(\varepsilon_{r+s-1}))\}$ is an $\mathbb{R}$-basis for $\mathbb{R}^{r+s}$ (later we shall take this system of units as powers of fundamental units). Hence for any $x \in \mathscr{L}^{r,s}$ we have a unique representation for $l(x) \in \mathbb{R}^{r+s}$ as

$$l(x) = \xi l^* + \xi_1 l(\varepsilon_1^2) + \cdots + \xi_{r+s-1}(\varepsilon_{r+s-1}^2) \quad \text{with } \xi, \xi_1, \ldots, \xi_{r+s-1} \in \mathbb{R}.$$

We identify $\mathscr{L}^{r,s}$ with $\mathbb{R}^m$ as an $m = (r+2s)$-dimensional real linear space. In $\mathbb{R}^m$ we fix the following cone

$$X = X(\varepsilon_1^2, \ldots, \varepsilon_{r+s-1}^2) = \{x \in \mathscr{L}_*^{r,s} : 0 \leqslant \xi < 1, 0 \leqslant \arg x_1 < 2\pi/k\},$$

where k is the order of the group of roots of 1 contained in K. If the field K is not totally imaginary, then the condition $0 \leqslant \arg x_1 < 2\pi/k$ just means that $x_1 > 0$. Our next result is a generalized version of [**2**, Proposition 3.4].

Proposition 1. *Let $\{\sigma_j\}$ for $1 \leqslant j \leqslant r+s$ be the embeddings of K into $\mathbb{R}$ and $\mathbb{C}$, where the complex embeddings have been arranged as before. For $1 \leqslant i \leqslant r+s-1$, let $d_i = \min\{(|\sigma_j(\varepsilon_i)|, 1 \leqslant j \leqslant r+s\}$ and let $x = (x_1, \ldots, x_{r+s}) \in X$ with $x_i > 0$ for $1 \leqslant i \leqslant r$ and $x_i \neq 0$ for $r+1 \leqslant i \leqslant r+s$. If*

$$N(x) := x_1 \cdots x_r |x_{r+1}|^2 \cdots |x_{r+s}|^2 > (d_1 \cdots d_{r+s-1})^{-m},$$

then $x_i > 1$ for $1 \leqslant i \leqslant r$ and $|x_i| > 1$ for $r+1 \leqslant i \leqslant r+s$.

In order to prove Theorem 2 it is enough to prove the following result.

Proposition 2. *For each real number $v > 0$, there is a compact set $B \subseteq (\mathrm{GL}_n(\mathbb{R}))^r \times (\mathrm{GL}_n(\mathbb{C}))^s/\Gamma$ such that $[v, \infty) \cap \operatorname{Im}\chi \subseteq \chi(B)$.*

Proof. From Humbert's reduction theory [**7**, pp. 295–297] it follows that in each class $[A] \in \mathscr{H}$ of positive definite forms, there exists a representative $A = (A_1, \ldots, A_r, A_{r+1}, \ldots, A_{r+s})$ with $A_i = \bar{F}_i^T D_i F_i = D_i[F_i]$ for $1 \leqslant i \leqslant r+2$, where

$$F_i = \begin{pmatrix} 1 & \beta_{12} & \cdots & \beta_{1n} \\ 0 & 1 & \cdots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is a unipotent upper triangular matrix with $|\beta_{ij}| \leqslant C_1$ and the diagonal matrix $D_i = \operatorname{diag}(t_1^{(i)}, \ldots, t_n^{(i)})$ with $t_j^{(i)} \in \mathbb{R}^+$ is such that $t_j^{(i)}/t_{j+1}^{(i)} \leqslant C_2$ for $1 \leqslant j \leqslant r-1$, where C_1 and C_2 (and all the C_i below) are constants depending only on the field K and on n. Hence by the correspondence determined by ψ, in each class

$$[G] \in \left(\prod_{1}^{r} \mathrm{GL}_n(\mathbb{R}) \times \prod_{r+1}^{r+s} \mathrm{GL}_n(\mathbb{C}) \right) \Big/ \Gamma$$

there is a representative $G = (G_1, \ldots, G_{r+s})$ with each $G_i = \tilde{D}_i B_i$, where B_i is unipotent upper triangular with entries bounded in absolute value by constants depending only on K and n, and $\tilde{D}_i = \operatorname{diag}(q_1^{(i)}, \ldots, q_n^{(i)})$ with $|q_j^{(i)}/q_{j+1}^{(i)}| \leqslant C_3$ for $1 \leqslant i \leqslant r+s$, $1 \leqslant j \leqslant n$. Here the equality $(q_j^{(i)})^2 = t_j^{(i)}$ holds for $1 \leqslant j \leqslant n$, $1 \leqslant i \leqslant r$ and the equality $q_j^{(i)}\,\overline{q}_j^{(i)} = t_j^{(i)}$ holds for $1 \leqslant j \leqslant n$, $r+1 \leqslant i \leqslant r+s$.

For $\alpha = (\alpha_1, \ldots, \alpha_{r+s}) \in \mathbb{R}^r \times \mathbb{C}^2$ with $\alpha_i \neq 0$ it follows that $\chi([\alpha G]) = \chi([G])$, where $[\alpha G] = [(\alpha_1 G_1, \ldots, \alpha_{r+s} G_{r+s})]$. By changing the class of G to the class of αG, we may assume that $|\det G_i| = 1$ and $G_i = \tilde{D}_i B_i$ as before.

It is clear that in order to prove the proposition it is enough to consider $0 < v \leqslant v_1$ for a fixed v_1 that we shall choose as follows.

With all notation as before, let $0 < v_1 < 1$ be such that $1/v_1 > (d_1 \cdots d_{r+s-1})^{-m}$, where the d_i are defined as in Proposition 1 and the cone X is determined by a system of squares of fundamental units of K.

For $v \leqslant v_1$, let $[G] \in \chi^{-1}[v, \infty)$ be given by $[G] = [(G_1, \ldots, G_{r+s})]$ with $|\det G_i| = 1$ and $G_i = \tilde{D}_i B_i$ as before. Letting $e_1 = (1, 0, \ldots, 0)$, we get

$$\chi([G]) = \min_{x \in \mathcal{O}_K^n - \{0\}} \left\{ \prod_1^r \|\tilde{D}_i B_i \sigma_i(x)\|^2 \prod_{r+1}^{r+s} \|\tilde{D}_i B_i \sigma_i(x)\|^4 \right\}$$
$$\leqslant \prod_1^r \|\tilde{D}_i B_i \sigma_i(e_1)\|^2 \prod_{r+1}^{r+s} \|\tilde{D}_i B_i \sigma_i(e_1)\|^4 \leqslant t_1^{(1)} \cdots t_1^{(r)} \, |t_{(1)}^{(r+1)}|^2 \cdots |t_1^{(r+s)}|^2.$$

Consider the element $\omega = (\omega_1, \ldots, \omega_{r+s}) \in \mathbb{R}^{r+s}$ defined by $\omega_i = |q_1^{(i)}|^2$ for $i > 1$ and $\omega_1 = |q_1^{(1)}|^2/v^2$. Then

$$N(\omega) = (1/v^2)\left((t_1^{(1)} \cdots t_1^{(r)})\right)\left((t_1^{(r+1)} \cdots t_1^{(r+s)})^2\right) \geqslant \chi([G])/v^2 \geqslant v/v^2 = 1/v \geqslant 1/v_1.$$

By [**2**, Lemma 3.2], there exists a unit $\eta \in \mathcal{O}_K^*$ such that if $(\eta_1^2, \ldots, \eta_{r+s}^2) := (\sigma_1(\eta)^2, \ldots, \sigma_{r+s}(\eta)^2)$ then $\tilde{\omega} = (\eta_1^2 \omega_1, \eta_2^2 \omega_2, \eta_3^2 \omega_3, \ldots, \eta_{r+s}^2 \omega_{r+s}) \in X$. Since $N(\omega) = N(\tilde{\omega})$, we may apply Proposition 1 to get

$$(\eta_1 q_1^{(1)})^2/v^2 > 1, (\eta_2 q_1^{(2)})^2 > 1, \ldots, (\eta_r q_1^{(r)})^2 > 1, \ldots, |\eta_{r+s} q_1^{(r+s)}|^2 > 1. \tag{3}$$

Consider in the class $[G]$ of G, the representative $\hat{G}$ defined by

$$\hat{G} = (\eta_1 G_1, \ldots, \eta_{r+s} G_{r+s}) := (\hat{G}_1, \ldots, \hat{G}_{r+s}).$$

Then $\chi[\hat{G}] = \chi[G]$ and $\hat{G}$ has a decomposition $\hat{G}_i = \hat{D}_i B_i$, where $\hat{D}_i = \operatorname{diag}(\tilde{q}_1^{(i)}, \ldots, \tilde{q}_n^{(i)}) = \operatorname{diag}(\eta_i q_1^{(i)}, \ldots, \eta_i q_n^{(i)})$ for $1 \leqslant i \leqslant r+s$ and the $\tilde{q}_j^{(i)}$ satisfy $|\tilde{q}_j^{(i)}/\tilde{q}_{(j+1)}^{(i)}| \leqslant C_3$ for all $j = 1, \ldots, r+s-1$ and all $i = 1, \ldots, n$. By equation (3),

$$(\tilde{q}_1^{(1)})^2/v^2 > 1, \quad (\tilde{q}_1^{(2)})^2 > 1, \quad (\tilde{q}_1^{(3)})^2 > 1, \ldots, |(\tilde{q}_1^{(r+s)})|^2 > 1.$$

On the other hand, if we put $C = C_3^{(r+s)(n-1)n/2}$, then, for $i = 1, \ldots, r+s$, we have

$$1 = \left| \prod_{j=1}^{r+s} \tilde{q}_1^{(j)} \cdots \tilde{q}_n^{(j)} \right| \geqslant \left| \prod_{j=1}^{r+s} (\tilde{q}_1^{(j)})^n \right| \Big/ C \geqslant \begin{cases} q_1^{(1)^n}/C & \text{for } i = 1, \\ v^{n/2}(q_1^{(i)^n}/C) & \text{for } i = 2, \ldots, r+s. \end{cases}$$

Thus $C^{1/n} \geqslant \tilde{q}_1^{(ij)} \geqslant v^{1/2}$ for $i = 1$ and $C^{1/n}/v^{1/2} \geqslant |\tilde{q}_1^{(i)}| \geqslant 1$ for $i = 2, \ldots, r+s$.

The fact that the $\tilde{q}_1^{(i)}$ are bounded from below for all $1 \leqslant i \leqslant r+s$ together with the inequalities $|\tilde{q}_j^{(i)}/\tilde{q}_{(j+1)}^{(i)}| \leqslant C_3$ and $1 = |\prod_{i=1}^{r+s} \tilde{q}_1^{(i)} \cdots \tilde{q}_n^{(i)}|$ allow us to obtain an upper bound for all $\tilde{q}_j^{(i)}$ with $2 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant r+s$. Hence a compact subset of $(\mathrm{GL}_n(\mathbb{R}))^r \times (\mathrm{GL}_n(\mathbb{C}))^s$ surjects onto $\operatorname{Im}(\chi) \cap [v, \infty)$.

We have thus proved Theorem 2 (see the Introduction).

We shall call a Humbert tuple *extreme* if it is a maximum of the map χ. Those extreme tuples which are an absolute maximum of χ will be called *critical* tuples. In particular Theorem 2 asserts that there exist critical Humbert tuples.

4. *Extreme and perfect forms*

In this section we assume as before, that K is any number field with $[K\colon \mathbb{Q}] = m = r+2s$ and we keep all the notation already introduced. We shall generalize the classical notions of eutactic and perfect forms (see [**8**]) and we shall partially extend Voronoi's Theorem, namely by showing that extreme forms are perfect and eutactic [**10**].

Let $f = (f_i)$ for $1 \leqslant i \leqslant r+s$ be a positive definite Humbert tuple where each f_i for $1 \leqslant i \leqslant r+s$ is a positive definite quadratic or hermitian form of dimension n. For a vector $u \in \mathcal{O}_K^n$ we denote its class modulo units of $\mathcal{O}_K$ by $[u]$, that is, $[u] = \{\xi u\colon \xi \in \mathcal{O}_K^*\}$. We say that the vector $u = (u_1, \ldots, u_n) \in \mathcal{O}_K^n$ is a *minimal vector* for f if it satisfies $\mu(f) = \prod_{i=1}^{r+1} f_i(\sigma_i(u)) \prod_{i=r+1}^{r+s} (f_i(\sigma_i(u)))^2$.

Lemma 1. *A positive definite tuple f has only finitely many classes of minimal vectors.*

Proof. We keep all the notation introduced in §3. Let $y = (y_1, \ldots, y_{r+s})$ with $y_i \in \mathbb{R}^n$ for $1 \leqslant i \leqslant r$ and $y_i \in \mathbb{C}^n$ for $r+1 \leqslant i \leqslant r+s$. Let $f = (f_1, \ldots, f_{r+s})$ be a positive definite Humbert tuple. For any positive real number T, the set

$$B_T = \{y = (y_1, \ldots, y_{r+s})\colon f_i(y_i) \leqslant T \text{ for } 1 \leqslant i \leqslant r+s\}$$

is compact. Hence there are only finitely many $y \in \mathcal{O}_K^n$ such that $f_i(\sigma_i(y)) \leqslant T$ for $1 \leqslant i \leqslant r+s$.

For each $y \in \mathcal{O}_K^n$, there exists $\varepsilon \in \mathcal{O}_K^*$ such that $f_i(\sigma_i(\varepsilon y)) = \sigma_i(\varepsilon)^2 f_i(\sigma_i(y)) \in X$, where $X = X(\varepsilon_1^2, \ldots, \varepsilon_{r+s-1}^2)$ is defined as in §3. Therefore it is enough to prove that the set

$$X_\mu = \left\{ y \in \mathcal{O}_K^n\colon f_i(\sigma_i(y)) \in X \text{ and } \prod_{i=1}^{r+1} f_i(\sigma_i(y)) \prod_{i=r+1}^{r+s} (f_i(\sigma_i(y))^2 \leqslant \mu \right\}$$

is finite. But this easily follows from the considerations above and the fact that for a fixed positive real number T the set $\{\omega \in X\colon |N(\omega)| \leqslant T\}$ is bounded in $\mathbb{R}^r \times \mathbb{C}^s$, where $N(\omega)$ is defined as in Proposition 1.

Let $\{[u_j]\}_{j=1}^l$ be the set of classes of all minimal vectors of f. For j satisfying $1 \leqslant j \leqslant l$, put $f_i(\sigma_i(u_j)) = \lambda_{i,j}$.

Definition 1. Let f be a positive definite tuple with $\{[u_j]\}_{j=1}^l$ being its set of classes of minimal vectors. Then f is *perfect* if there exists a system of representatives of $\{[u_j]\}_{j=1}^l$, namely $\{u_1, \ldots, u_l\}$, such that f is uniquely determined by the equations $f_i(\sigma_i(u_j)) = \lambda_{i,j}$ for $1 \leqslant i \leqslant r+s$, $1 \leqslant j \leqslant 1$.

Remark 3. From the above definition it follows that a perfect tuple has at least $\frac{1}{2}n(n+1)$ distinct classes of minimal vectors.

Definition 2. Let $f(x_1, \ldots, x_n) = \sum_{ij} a_{ij} x_i x_j$, (respectively, $f(x_1, \ldots, x_n) = \sum_{ij} a_{ij} x_i \bar{x}_j$) be a positive definite real quadratic (respectively, complex hermitian) form. We define the *dual form* of f to be the form $f^*(x_1, \dots, x_n) = \sum_{ij} A_{ij} x_i x_j$, (respectively, $f^*(x_1, \dots, x_n) = \sum_{ij} A_{ij} x_i \bar{x}_j$), where (A_{ij}) is the adjoint of the matrix (a_{ij}). Notice that $f(x_1, \dots, x_n)$ is positive definite (hermitian) if and only if $f^*(x_1, \dots, x_n)$ is positive definite (hermitian).

DEFINITION 3. With the same notation as in the previous definition, let $f^* = (f_1^*, \dots, f_r^*)$ be the dual tuple of f. We say that f is *eutactic* if there exists $(\xi_1^{(i)}, \dots, \xi_l^{(i)}) \in \mathbb{R}^l$ for $1 \leqslant i \leqslant r+s$ with $\xi_j^{(i)} > 0$ for $1 \leqslant j \leqslant 1$ such that

$$f_i^*(x) = \sum_{k=1}^{l} \xi_k^{(i)} (\sigma_i(u_k) \cdot x)^2 \qquad \text{for } 1 \leqslant i \leqslant r,$$

$$f_i^*(x) = \sum_{k=1}^{l} \xi_k^{(i)} (\sigma_i(u_k) \cdot x) \overline{(\sigma_i(u_k) \cdot x)} \qquad \text{for } r+1 \leqslant i \leqslant r+s,$$

where $\sigma_i(u_k) \cdot x$ denotes the standard dot product of the vectors $\sigma_i(u_k)$ and $x = (x_1, \dots, x_n)$.

REMARK 4. We say that a class of Humbert tuples $[f]$ as defined in Section 2, is a perfect (eutactic) class if it contains a representative which is a perfect (eutactic) tuple.

The following lemma is a generalization of well-known classical result.

LEMMA 2. *Let f be a Humbert tuple, and let $\{[u_j]\}_{j=1}^l$ be the set of classes of its minimal vectors. Then there exists a neighbourhood V of f such that the set of classes of minimal vectors of any element in V is contained in $\{[u_j]\}_{j=1}^l$.*

Proof. Since for $\lambda = (\lambda_1, \dots, \lambda_r)$ with $\lambda_i > 0$, the minimal vectors of f and of $\lambda f = (\lambda_1 f_1, \dots, \lambda_r f_r)$ are the same, we assume that all the tuples to be considered, have determinant 1.

Assume that the lemma is not true. Then for each $\varepsilon > 0$, there exists a tuple $f_\varepsilon = (f_{1,\varepsilon}, \dots, f_{r+s,\varepsilon})$ and a minimal vector $v_\varepsilon \in \mathcal{O}_K^n$ of f_ε, such that v_ε is not a minimal vector of f. We may assume, as before, that $(f_{i,\varepsilon}(\sigma_i(v_\varepsilon))) \in X$ for $1 \leqslant i \leqslant r+s$. Since v_ε is a minimal vector for f_ε it follows that

$$\prod_{i=1}^{r} f_{i,\varepsilon}(\sigma_i(v_\varepsilon)) \prod_{i=r+1}^{r+s} (f_{i,\varepsilon}(\sigma_i(v_\varepsilon)))^2 \leqslant C_{n,K}(d(f_\varepsilon))^{1/n} = C_{n,K}(d(f))^{1/n} = C_{n,K},$$

where $C_{n,K}$ is a constant depending only on n and on the field K. Since we have assumed that $(f_{i,\varepsilon}(\sigma_i(v_\varepsilon))) \in X$ we conclude that there exists a constant $\tilde{D}_{n,f,K}$ such that $|f_{i,\varepsilon}(\sigma_i(v_\varepsilon))| < \tilde{D}_{n,f,K}$. Hence for all $1 \leqslant i \leqslant r+s$, we have

$$\|\sigma_i(v_\varepsilon)\| \leqslant D_{n,K,f},$$

where $D_{n,K,f}$ depends only on n, K and on the form f. Since $v_\varepsilon \in \mathcal{O}_K^n$, we conclude that there are only finitely many classes of such vectors v_ε. Passing to a subset of the ε we may assume that $[v_\varepsilon] = v$ for any ε small enough. By continuity, v is a minimal vector of f which is a contradiction. This proves the lemma.

We now state the analogue of a result due to Voronoi concerning a characterization of extreme positive definite tuples. The proof of this result is obtained by generalizing that in the classical case.

PROPOSITION 3. *Let f be an extreme Humbert tuple. Then f is perfect and eutactic.*

We prove only the case when the field K is a totally real number field. The general case follows in the same way after making the obvious changes. From now on we assume that the field K is totally real of degree r. In order to prove Proposition 3 we need the following lemma.

LEMMA 3. *Let $f = (f_1, \dots, f_r)$ and $g = (g_1, \dots, g_r)$ be two Humbert tuples. Let $F_{t,i} = (1-t)f_i + tg_i$ for $0 \leqslant t \leqslant 1$ and $1 \leqslant i \leqslant r$. Then*

$$\frac{d^2 \log(\prod_{i=1}^{r} \det F_{t,i})}{dt^2} < 0.$$

The proof of Lemma 3 follows from the classical case (see, for instance, [**8**, Lemma 1, Chapter 6, Section 39]), as $d^2 \log(\prod \det F_{t,i})/dt^2 = \sum d^2(\log \det F_{t,i})/dt^2$.

Proof of Proposition 3. We first prove that an extreme tuple is perfect. Let $f = (f_1, \dots, f_r)$ be an extreme tuple which is not perfect. Then there exists $g = (g_1, \dots, g_r)$, with $f \neq g$ such that g has the same classes of minimal vectors as f, say $\{[u_j]\}_{j=1}^{l}$, and there is set of representatives such that $g_i(\sigma_i(u_j)) = f_i(\sigma_i(u_j)) = \lambda_{i,j}$, for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant l$. Then the values of these forms coincide for any system of representatives of minimal vectors. Put

$$f_k = \sum_{ij} s_{ij}^{(k)} x_i x_j, \quad g_k = \sum_{ij} w_{ij}^{(k)} x_i x_j,$$

where $1 \leqslant i, j \leqslant n$ and $1 \leqslant k \leqslant r$. Assume that $w_{ij}^{(k)} \neq s_{ij}^{(k)}$ for some k and some (i,j). Consider the expression

$$F_{k,t} = tg_k + (1-t)f_k = f_k + t(g_k - f_k), \quad \text{where } t \in \mathbb{R}.$$

Let $F_t = (F_{1,t}, \dots, F_{n,t})$, then $F_0 = f$. Since the tuple f consists of positive definite forms, there is an interval $(-t_0^{(k)}, t_0^{(k)}) \subseteq \mathbb{R}$, where $F_{k,t}$ is positive definite for every $t \in (-t_0^{(k)}, t_0^{(k)})$ and $1 \leqslant k \leqslant r$. Choose $t_0 = \inf_{1 \leqslant k \leqslant r}\{t_0^{(k)}\}$. Then

$$d(F_t) = \prod_{k=1}^{r} \det F_{k,t} > 0 \quad \text{for } t \in (-t_0, t_0).$$

Since f is extreme, there exists $t_1 > 0$ such that

$$\mu(F_t)\, d(F_t)^{-1/n} \leqslant \mu(f)\, d(f)^{-1/n} = \mu(F_0)\, d(F_0)^{-1/n}$$

for all $t \in (-t_1, t_1)$. We may assume that $t_1 < t_0$. Therefore by Lemma 2, there exists a class of minimal vectors $[u_0] = [(u_{0,1}, \dots, u_{0,n})]$ such that

$$d(F_t)^{-1/n} \prod_{k=1}^{r} \sum_{i,j} (s_{i,j}^{(k)} + t(w_{ij}^{(k)} - s_{ij}^{(k)}))(\sigma_k(u_{0,i})\,\sigma_k(u_{0,j}))$$
$$\leqslant d(f)^{-1/n} \prod_{k=1}^{r} \sum_{i,j} s_{i,j}^{(k)}(\sigma_k(u_{0,i})\,\sigma_k(u_{0,j}))$$

for $t \in (-t_1, t_1)$. By assumption $\sum_{i,j} t(w_{ij}^{(k)} - s_{ij}^{(k)})(\sigma_k(u_{0,i})\,\sigma_k(u_{0,j})) = 0$. Hence

$$d(F_t)^{-1/n} \prod_{k=1}^{r} (\sum_{i,j} s_{i,j}^{(k)}\,\sigma_k(u_{0,i})\,\sigma_k(u_{0,j})) \leqslant d(f)^{-1/n} \prod_{k=1}^{r} \sum_{i,j} s_{i,j}^{(k)}(\sigma_k(u_{0,i})\,\sigma_k(u_{0,j})).$$

Therefore $d(f) \leqslant d(F_t)$ for $t \in (-t_1, t_1)$. Thus the map $D: \mathbb{R} \to \mathbb{R}$ defined by $D(t) = d(F_t)$ has a local minimum at 0. Lemma 3 then leads to a contradiction and the tuples $f = (f_1, \ldots, f_r)$ and $g = (g_1, \ldots, g_r)$ must be the same, that is $f_k = g_k$ for $1 \leqslant k \leqslant r$.

We now show that an extreme tuple is eutactic. Because f is eutactic if and only if for $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r$ with $\lambda_k > 0$ for $1 \leqslant k \leqslant r$, the tuple $\lambda f = (\lambda_1 f_1, \ldots, \lambda_r f_r)$ is eutactic, hence we may assume that $\det(f_k) = 1$ for $1 \leqslant k \leqslant r$.

For any fixed $(t_{ij}^{(k)}) \not\equiv (0)$ with $(t_{ij}^{(k)}) = (t_{ji}^{(k)})$, consider a linear half-space

$$\Psi_k = \{(\xi_{ij}^{(k)}) \in \mathbb{R}^{n(n+1)/2} \colon \sum_{i,j=1}^{n} t_{ij}^{(k)} \xi_{ij}^{(k)} \geqslant 0 \text{ and } \xi_{ij}^{(k)} = \xi_{ji}^{(k)}\}.$$

Suppose that Ψ_{k_0} contains the elements $\sigma_{k_0}(u_{is}) \sigma_{k_0}(u_{js})$, that is,

$$\sum_{i,j} t_{ij}^{k} \sigma_{k_0}(u_{is}) \sigma_{k_0}(u_{js}) \geqslant 0 \quad \text{for } 1 \leqslant s \leqslant l.$$

Here u_{is} denotes the ith coordinate of the sth minimal vector. Consider the forms

$$F_{\rho,k} = f_k \qquad \text{for } k \neq k_0,$$

$$F_{\rho,k_0} = \sum_{i,j} (s_{ij}^{(k_0)} + \rho t_{ij}^{(k_0)}) x_i x_j.$$

Then the tuple $F_\rho = (f_1, \ldots, f_{k-1}, F_{\rho,k_0}, \ldots, f_r)$ is positive definite for ρ small enough. Since f is extreme, we may also assume that for such a ρ the inequality

$$\mu(F_\rho)\, d(F_\rho)^{-1/n} \leqslant \mu(f)\, d(f)^{-1/n}$$

holds. By Lemma 2, there exists $u_0 \in [u_0] = [(u_{1,0}, \ldots, u_{n,0})] \in \mathcal{O}_K^n$ such that

$$d(F_\rho)^{-1/n} \left(\prod_{k \neq k_0} \left(\sum_{i,j} s_{ij}^{(k)} \sigma_k(u_{i0}) \sigma_k(u_{j0}) \right) \right) \left(\sum_{i,j} (s_{ij}^{(k_0)} + \rho t_{ij}^{(k_0)}) \sigma_{k_0}(u_{i0})(u_{j0}) \right)$$
$$\leqslant d(f)^{-1/n} \prod_{k} \left(\sum_{i,j} s_{ij}^{(k)} \sigma_k(u_0) \sigma_k(u_{j0}) \right).$$

By assumption $\sum_{i,j} t_{ij}^{(k_0)} \sigma_{k_0}(u_{i0})(u_{j0}) \geqslant 0$. Hence for small enough positive ρ we have

$$d(F_\rho)^{-1/n} \left(\prod_{k \neq k_0} \left(\sum_{i,j} s_{ij}^{(k)} \sigma_k(u_{i0}) \sigma_k(u_{j0}) \right) \right) \left(\sum_{i,j} (s_{ij}^{(k_0)} + \rho t_{ij}^{(k_0)}) \sigma_{k_0}(u_{i0})(u_{j0}) \right)$$
$$\geqslant d(F_\rho)^{-1/n} \left(\prod_{k} \left(\sum_{i,j} s_{ij}^{(k)} \sigma_k(u_{i0}) \sigma_k(u_{j0}) \right) \right).$$

After cancelling the contribution from the $k \neq k_0$, we find that $\det f_{k_0} \leqslant \det(F_{\rho,k_0})$. By Lemma 3

$$\left. \frac{d(\det(F_{\rho,k_0}))}{d\rho} \right|_{\rho=0} > 0.$$

But

$$\left. \frac{d(d(F_{\rho,k_0}))}{d\rho} \right|_{\rho=0} = \left. \frac{d(\det(F_{\rho,k_0}))}{d\rho} \right|_{\rho=0} = \sum_{i,j} t_{ij}^{(k_0)} \frac{\partial(\det f_{k_0})}{\partial s_{ij}^*} = \sum_{i,j} t_{ij}^{(k_0)} (s_{ij}^{(k_0)})^*.$$

Hence, the point in $\mathbb{R}^{n(n+1)/2}$ that represents the form $f_{k_0}^*$ lies in the interior of any linear half-space Ψ_{k_0} that contains those points representing the forms $(\sigma_i(u_s) \cdot x)^2$ for

$1 \leqslant s \leqslant l$. In particular it lies in the interior of the convex linear hull in $\mathbb{R}^{n(n+1)/2}$ determined by the forms $(\sigma_i(u_s)\cdot x)^2$ for $1 \leqslant s \leqslant l$. This proves that the tuple f is eutactic.

We finally give an example and some concluding remarks. We shall consider the following form $g(x,y) = x^2+y^2+xy$ which is known to be extreme over $\mathbb{Z}$ (in fact it is critical over $\mathbb{Z}$, that is, it realizes the Hermite constant γ_2). Let $f = (f_1, f_2)$ be the Humbert tuple over the real quadratic number field $K = \mathbb{Q}(\sqrt{d})$ defined by $f_1 = f_2 = x^2+y^2+xy$. Since $\mu(f) = \min\{(u^2+v^2+uv)(\bar{u}^2+\bar{v}^2+\overline{uv}) : (0,0) \neq (u,v) \in \mathcal{O}_K\}$, we conclude that $\mu(f) = 1$.

We want to determine the minimal vectors of f over K, that is, all $(u,v) \in \mathcal{O}_K^2$ with $N_{K/\mathbb{Q}}(u^2+v^2+uv) = 1$. This equation can be written as

$$N_{K/\mathbb{Q}}(u+\tfrac{1}{2}v)^2 + \tfrac{9}{16}N_{K/\mathbb{Q}}(v)^2 + \tfrac{3}{4}[v^2(\bar{u}^2+\tfrac{1}{2}\bar{v})^2 + \bar{v}^2(u+\tfrac{1}{2}v)^2] = 1. \qquad (4)$$

Notice that by symmetry we may interchange the roles of u and v in the last equation.

If $N_{K/\mathbb{Q}}(v) \geqslant 2$, then $\frac{9}{16}N_{K/\mathbb{Q}}(v) > 1$ and we get a contradiction with (4). Hence we must have $N_{K/\mathbb{Q}}(v)$, $N_{K/\mathbb{Q}}(u) \in \{0, 1, -1\}$.

If $u = 0$, then $N_{K/\mathbb{Q}}(v)^2 = 1$ and v is then a unit. Hence the class of (u,v) is $[(u,v)] = [(0,1)]$. Similarly if $v = 0$ we obtain $[(u,v)] = [(1,0)]$. Assume now that $N_{K/\mathbb{Q}}(v)^2 = N_{K/\mathbb{Q}}(u)^2 = 1$. Since we are considering classes of minimal vectors, we may put $u = 1$. Then $[(u,v)] = [(1,\varepsilon)]$ with $\varepsilon \in \mathcal{O}_K^*$. Equation (4) becomes

$$N_{K/\mathbb{Q}}(\varepsilon)\, N_{K/\mathbb{Q}}(1+\varepsilon) + \mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon) + \mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon^2) = 0.$$

Let $N_{K/\mathbb{Q}}(\varepsilon) = 1$. Since $\varepsilon^2 - \mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon)\,\varepsilon + N_{K/\mathbb{Q}}(\varepsilon) = 0$, we have $\varepsilon^2 - \mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon)\,\varepsilon = -1$. Hence $\mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon^2) - \mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon)^2 = -2$. Inserting this in the above equation we get $N_{K/\mathbb{Q}}(1+\varepsilon) + \mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon) - 2 + \mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon)^2 = 0$. Since $N_{K/\mathbb{Q}}(1+\varepsilon) = 2 + \mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon)$, we get $2\mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon) = -\mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon)^2$. Now K is totally real, so we must have $\mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon) = -2$. But $\mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon) = -2$ and $N_{K/\mathbb{Q}}(\varepsilon) = 1$ imply that $\varepsilon = -1$. Hence $[(1,\varepsilon)] = [(1,-1)]$.

Let $N_{K/\mathbb{Q}}(\varepsilon) = -1$. Then $\varepsilon^2 - \mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon)\,\varepsilon = 1$ and $\mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon^2) = 2 + \mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon)^2$. Our equation in this case is $-\mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon) + \mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon) + 2 + \mathrm{Tr}_{K/\mathbb{Q}}(\varepsilon)^2 = 0$ and this is not possible.

We have then proved that the only classes of minimal vectors of the tuple f over the field K are $[(1,0)]$, $[(0,1)]$, $[(1,-1)]$. Inserting these values in the equations defining a perfect tuple (see Definition 1) we easily see that f is a perfect tuple over K. The relations defining an eutactic tuple are also satisfied by f and we think that f could be an extreme tuple (or even a critical tuple) over K. In general we do not expect f to be a critical tuple for any quadratic extension K, since according to [**1**] the invariant $\gamma_{n,K}$ is bounded from below by a function that depends linearly on $|D_K|^{1/2}$, where D_K is the discriminant of K.

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References

1. R. Baeza, 'The volume of the space of Humbert reduced forms', preprint.
2. R. Baeza and M. I. Icaza, 'Decomposition of positive definite integral quadratic forms as a sum of positive definite forms', Proceedings of Symposia in Pure Mathematics 58.2 (American Mathematical Society, Providence, 1995).
3. Z. Borevich and I. R. Shafarevich, *Number theory* (Academic Press, New York, 1967).

4. F. R. GANTMACHER, *The theory of matrices*, vols. 1 and 2 (Chelsea, New York, 1959).
5. C. HERMITE, *Oeuvres* (Gauthiers-Villars, Paris, 1905).
6. P. HUMBERT, 'Théorie de la reduction des formes quadratiques définies positives dans un corps algebrique K fini', *Comment. Math. Helv.* 12 (1939–1940) 263–306.
7. P. HUMBERT, 'Reduction des formes quadratiques dans un corps algebrique fini', *Comment. Math. Helv.* 23 (1949) 50–63.
8. C. G. LEKKERKERKER, *Geometry of numbers*, Bibliotheca Mathematica (Wolters-Noordhof, Groningen; North-Holland, Amsterdam, 1969).
9. J. OESTERLÉ, 'Empilements de sphères', *Séminaire Bourbaki*, vol. 1989–90, exp. 727, *Astérisque* 189–190 (1990).
10. G. VORONOI, Sur quelques propriétés des formes quadratiques positives parfaites', *J. Reine Angew. Math.* 133 (1908) 97–178.

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