JORDAN MEASURE AND RIEMANN INTEGRATION.[1]

By Orrin Frink Jr.

In this paper it is shown that the Riemann integral can be defined in terms of Jordan measure in about the same way that the Lebesgue integral is defined in terms of Lebesgue measure. This result is not surprising, but it does not seem to be in the literature. There is a paper on the subject by J. Ridder, "Over de Integraldefinities van Riemann en Lebesgue", Christiann Huygens, vol. 4, (1925-6) pp. 346-350, but the results stated in this paper are incorrect. Ridder attempts to show that a necessary and sufficient condition that a bounded function defined over a closed interval be integrable Riemann is that the function be "Jordan measurable", that is, for every pair of numbers k and l, the set of points x of I for which $k \leqq f(x) < l$, and the set for which $k < f(x) \leqq l$, are measurable Jordan. But it is easily seen that not even all continuous functions are "measurable Jordan" in this sense. For example, let C be a closed set which is not measurable Jordan (such as a nowhere dense closed set of positive Lebesgue measure) contained in the closed interval I. Let $f(x)$ be the distance of a point x of I from the set C. Then $f(x)$ is continuous, but the set of points of I for which $-1 < f(x) \leqq 0$ is the set C, which is not measurable Jordan. The situation is more complicated than Ridder supposed, as can be seen from the theorems below, where several necessary and sufficient conditions for Riemann integrability in terms of Jordan measure are derived. Some consequences of these results, believed to be new, are also given; for example, it is shown that a bounded function integrable Riemann can be uniformly approximated by functions which take on only a finite number of values, at sets measurable Jordan.

We shall confine our attention to bounded functions defined over a closed interval (not necessarily 1-dimensional). This is sufficiently general, since a function defined over only a subset of the interval may be considered to have the value zero elsewhere. Improper integrals will not be considered. The following notation will be used. If A is any bounded set of points, then by $\bar{A}$, the *closure* of A, we shall mean the set A plus its limit points, and by $\underline{A}$, the *interior* of A, we shall mean the set of all interior points of A. Also $b(A) = \bar{A} - \underline{A}$ is the *boundary* of A, and $f(A) = A - \underline{A}$ is the *frontier* of A. Then the set $\bar{A} - A$ is the frontier of the complement of A.

[1] Received January 23, 1932 and, in revised form, March 29, 1932.

If A is measurable Lebesgue we shall denote its measure by mA. The word *measure*, unless otherwise qualified, will always mean Lebesgue measure. The *outer content* of A is defined to be $m\bar{A}$, and the *inner content* of A is defined to be $m\underline{A}$. Since the sets $\bar{A}$ and $\underline{A}$ are closed and open respectively, they are measurable. If the outer and inner contents of A are equal, their common value is cA, the *content* or *Jordan measure* of A, and A is said to be *measurable Jordan*. It follows immediately that a set is measurable Jordan if and only if its boundary is of measure zero. If $mf(A) = 0$, we say, following Carathéodory, that A is *Jordan measurable from within*, or JMI, and if $\bar{A}-A$, the frontier of the complement of A, is of measure zero, we say A is *Jordan measurable from without*, or JMO.

With the aid of some simple point-set theory the following consequences of these definitions can be established. The logical sum of a countable number of sets JMI is JMI. The logical product of a countable number of sets JMO is JMO. The complement of a set JMI is JMO, and conversely. A set is measurable Jordan if and only if it is both JMI and JMO. A closed interval is measurable Jordan. If A and B are measurable Jordan, then their logical sum $A+B$, their logical product AB, and their logical difference $A-B$, are all measurable Jordan.

It is assumed that the reader is familiar with the ordinary theory of Riemann and Lebesgue integration, and in particular with the theorem that a function defined and bounded on an interval is integrable Riemann if and only if its points of discontinuity form a set of measure zero. Given a function $f(x)$ defined over a closed interval I, we shall denote by G_k, E_k, L_k, GE_k, and LE_k the sets of points of I for which $f(x) > k$, $= k$, $< k$, $\geqq k$, or $\leqq k$ respectively.

THEOREM 1. *If $f(x)$ is defined, bounded, and integrable Riemann on the closed interval I, then for all except a countable number of values of k the set of points G_k for which $f(x) > k$ is measurable Jordan.*

Proof. Since $f(x)$ is integrable Lebesgue, the set of points E_k for which $f(x) = k$ is measurable Lebesgue, and for all except a countable number of values of k is of measure zero. For, no two distinct sets E_k have a point in common, and all are contained in I, hence only a countable number of them can have positive measure.

Suppose now that k is so chosen that $mE_k = 0$, and consider the set G_k of points where $f(x) > k$. What we wish to prove is that G_k is measurable Jordan, that is, that its boundary is of measure zero. Suppose p is any point of the boundary of G_k. Three cases arise. I. If $f(p) > k$, then p is both a member of and a boundary point of G_k, and is therefore a limit point of points where $f(x) \leqq k$. Hence p must be a point of discontinuity of $f(x)$. II. If $f(p) = k$, then p is a point of E_k. III. If

$f(p) < k$, then since p is a boundary point but not a member of G_k, it must be a limit point of G_k, that is of points where $f(x) > k$. Then it is a point of discontinuity of $f(x)$. Hence in any case a boundary point of G_k is a member of either E_k, which is of measure zero, or of the set D of discontinuities of $f(x)$, which is also of measure zero since $f(x)$ is integrable Riemann. Since it is a subset of a set of measure zero, the boundary of G_k is of measure zero, hence G_k is measurable Jordan.

THEOREM 2. *If $f(x)$ is defined, bounded, and integrable Riemann on the closed interval I, then for all except a countable number of values of k, and in fact whenever $mE_k = 0$, the sets G_k, E_k, L_k, GE_k, and LE_k are all measurable Jordan.*

Proof. This follows from Theorem 1, from the fact that $-f(x)$ is integrable Riemann, and from the fact that the difference of two sets measurable Jordan is measurable Jordan. In fact, L_k is the set of points where $-f(x) > -k$, $GE_k = I - L_k$, $LE_k = I - G_k$, $E_k = GE_k - G_k$.

THEOREM 3. *If $f(x)$ is defined, bounded, and integrable Riemann on the closed interval I, then for every k the sets G_k and L_k are each the sum of a countable number of sets measurable Jordan, and hence are JMI, and the sets GE_k, LE_k, and E_k are each the logical product of a countable number of sets measurable Jordan and hence are JMO.*

Proof. For any number k we can find a sequence of numbers $k_1, k_2, \cdots, k_n, \cdots$ approaching k from above and such that for every n we have $mE_{k_n} = 0$, and a sequence of numbers $l_1, l_2, \cdots, l_n, \cdots$ approaching k from below and such that for every n we have $mE_{l_n} = 0$, since all except a countable number of numbers have this latter property. Then $G_k = \sum_{n=1}^{\infty} G_{k_n}$, since in the first place every x which is a member of G_k is a member of every G_{k_n} for which $f(x) > k_n$, and in the second place every point of G_{k_n} is also a point of G_k. Similar reasoning shows us that

$$L_k = \sum_{n=1}^{\infty} L_{l_n}, \qquad GE_k = \prod_{n=1}^{\infty} G_{l_n} = I - L_k, \qquad LE_k = \prod_{n=1}^{\infty} L_{k_n} = I - G_k,$$

$$E_k = GE_k \cdot LE_k = \prod_{n=1}^{\infty} G_{l_n} \cdot \prod_{n=1}^{\infty} L_{k_n}.$$

But by Theorem 2 the sets G_{k_n}, L_{k_n}, G_{l_n}, L_{l_n} are all measurable Jordan. This completes the proof of Theorem 3.

It should be noted that a set can be *JMI* without being the sum of a countable number of sets measurable Jordan. An example is any set of the second category of measure zero. For, every set of measure zero is *JMI*; but since a set of content zero is nowhere dense, the logical sum of a countable number of sets of content zero is necessarily of the first category. Similarly, the complement with respect to I of a set which is of measure zero and of the second category is an example of a set which is JMO without being the logical product of a countable number of sets measurable Jordan.

It follows from Theorem 2 that if $mE_k = mE_l = 0$, the set of points of I for which $k \leqq f(x) < l$ is measurable Jordan. We are now in a position to define the Riemann integral in terms of Jordan measure in the same way in which the Lebesgue integral is defined in terms of Lebesgue measure.

Let $f(x)$ be a function defined, bounded, and integrable Riemann on the closed interval I, and let m be a number less than the greatest lower bound of $f(x)$ and M be a number greater than the least upper bound of $f(x)$. Consider a partition of the interval from m to M, that is, a finite number of values $y_0, \cdots, y_n$ such that $m = y_0 < y_1 < y_2 < \cdots < y_k < \cdots < y_n = M$, and subject to the condition K that $mE_{y_k} = 0$ for every k. Let δ, the norm of the partition, be the maximum of $y_k - y_{k-1}$ for all values of k. Let A_k be the set of points of I for which $y_{k-1} \leqq f(x) < y_k$. Then it follows from Theorem 2 that A_k is measurable Jordan. Let cA_k denote the Jordan measure of A_k, which of course is equal to its Lebesgue measure. Then we can define the Riemann integral of $f(x)$ over the interval I by the relation

$$\int_I f(x)\,dx = \lim_{\delta \to 0} \sum_{k=1}^{n} y_k\, cA_k.$$

This limit exists in the sense that there is a number L such that for every $\epsilon > 0$ there exists an $\eta > 0$ such that for every partition $y_0, \cdots, y_n$ subject to condition K, and whose norm δ is less than η, we have $\left| L - \sum_{k=1}^{n} y_k\, cA_k \right| < \epsilon$. The existence of this limit L is proved in the same way as for the Lebesgue integral, and of course it is equal to the Lebesgue integral. That there exist partitions of arbitrarily small norm subject to condition K follows from the fact that $mE_{y_k} = 0$ for all except a countable number of values of y_k.

Having obtained some necessary conditions for the Riemann integrability of bounded functions in terms of Jordan measure, we now look for sufficient conditions of the same type. As we shall prove, the properties stated in the conclusions of Theorems 2 and 3 are each sufficient for Riemann integrability. We shall first show that a condition apparently weaker than either of these is also sufficient.

THEOREM 4. *If the function $f(x)$ is defined and bounded on the closed interval I, and its bounds are m and M, and if for a set of values of k everywhere dense on the interval from m to M the sets G_k and L_k of points* *of I for which* $f(x) > k$ *and* $f(x) < k$ *respectively are JMI, then* $f(x)$ *is integrable Riemann.*

Proof. It is sufficient to prove that the set D of discontinuities of $f(x)$ is of measure zero. Note that from the hypothesis of our theorem it follows that for *every* k, G_k and L_k are *JMI*. For, as before, we can find a sequence of values k_n approaching k from above such that every G_{k_n} is *JMI*, and a sequence of values l_n approaching k from below such that every L_{l_n} is *JMI*. Then $G_k = \sum_{n=1}^{\infty} G_{k_n}$ and $L_k = \sum_{n=1}^{\infty} L_{l_n}$. Hence both G_k and L_k are *JMI*, since each is the sum of a countable number of sets *JMI*.

Now consider the set O_n of those discontinuities of $f(x)$ at which the oscillation of $f(x)$ is $> 1/n$, suppose p is any point of O_n, and let $f(p) = k$. If r and s are rational numbers such that $r < k < s$ and $(s - r) < 1/5n$, then p is a point of G_r and also of L_s. But, since the oscillation of $f(x)$ is $> 1/n$ at p, p is a limit point either of points where $f(x) < r$ or of points where $f(x) > s$. Then p is a frontier point of either G_r or L_s, since it is either a point of G_r and a limit point of points not in G_r, or a point of L_s and a limit point of points not in L_s. Since the sets G_r and L_s are *JMI*, their frontiers $f(G_r)$ and $f(L_s)$ are of measure zero. All points of O_n, then, are included in the set $\sum_{i=1}^{\infty} f(G_{r_i}) + \sum_{i=1}^{\infty} f(L_{r_i})$ where the sequence $r_1, r_2, \cdots, r_i, \cdots$ is some enumeration of the rational numbers. Since this set is of measure zero, so also is O_n. But the set of discontinuities of $f(x)$ is $D = \sum_{n=1}^{\infty} O_n$. Hence D is of measure zero and $f(x)$ is integrable Riemann. We have used throughout the fact that a countable number of sets of measure zero has as its logical sum a set of measure zero.

By combining Theorems 2, 3, and 4, a large number of different necessary and sufficient conditions for the Riemann integrability of bounded functions can be obtained. Some of these are found in

THEOREM 5. *Each of the following ten properties is by itself a necessary and sufficient condition for the Riemann integrability of a function* $f(x)$, *defined and bounded on the closed interval I, and whose bounds are m and M*:

a) *For all except a countable number of values of k the sets* G_k *and* L_k *are measurable Jordan.*

b) *For all except a countable number of values of k the sets* GE_k *and* LE_k *are measurable Jordan.*

c) *For a set of values of k everywhere dense on the interval from m to M the sets* G_k *and* L_k *are measurable Jordan.*

d) *For a set of values of k everywhere dense on the interval from m to M the sets GE_k and LE_k are measurable Jordan.*
e) *For all values of k the sets G_k and L_k are each the logical sum of a countable number of sets measurable Jordan.*
f) *For all values of k the sets GE_k and LE_k are each the logical product of a countable number of sets measurable Jordan.*
g) *For all values of k the sets G_k and L_k are JMI.*
h) *For all values of k the sets GE_k and LE_k are JMO.*
i) *For a set of values of k everywhere dense on the interval from m to M the sets G_k and L_k are JMI.*
j) *For a set of values of k everywhere dense on the interval from m to M the sets GE_k and LE_k are JMO.*

It should be noted that if for a set of values of k everywhere dense on the interval from m to M the sets GE_k and LE_k are JMO, then for the same values of k the sets $G_k = I - LE_k$ and $L_k = I - GE_k$ are JMI, since the complement with respect to I of a set JMO is JMI, and hence $f(x)$ is integrable Riemann by Theorem 4. This is the only fact needed in Theorem 5 not already covered by Theorems 2, 3, and 4.

THEOREM 6. *If $f(x)$ and $g(x)$ are defined, bounded, and integrable Riemann on the closed interval I, then the set of points of I where $f(x) > g(x)$, and the set of points of I where $f(x) \neq g(x)$ are each the logical sum of a countable number of sets measurable Jordan, and the set of points where $f(x) = g(x)$, and the set of points where $f(x) \geqq g(x)$ are each the logical product of a countable number of sets measurable Jordan.*

Proof. This follows from the fact that if $f(x)$ and $g(x)$ are integrable Riemann, so is their difference $d(x) = f(x) - g(x)$. The four sets of points in question are the points of I at which $d(x) > 0$, $\neq 0$, $= 0$, $\geqq 0$, respectively. The conclusion then follows from Theorem 3.

THEOREM 7. *If $f(x)$ and $g(x)$ are defined, bounded, and integrable Riemann on the closed interval I, then the necessary and sufficient condition that $\int_I |f(x) - g(x)|\, dx = 0$ is that the set of points U at which $f(x) \neq g(x)$ be the logical sum of a countable number of sets of content zero.*

Proof. Since the Riemann and Lebesgue integrals are equal when both exist, it follows from the corresponding theorem for Lebesgue integrals that the necessary and sufficient condition for the vanishing of the integral in question is that $mU = 0$. But since $f(x)$ and $g(x)$ are integrable Riemann, it follows from Theorem 6 that U is the sum of a countable number of sets measurable Jordan. Since it is also of measure zero, it must be the logical sum of a countable number of sets of content zero.

It should be noted that Theorem 7, and in fact a somewhat sharper result, can also be deduced from a theorem to be found in the first edition of Hobson's "Theory of Functions of a Real Variable", Cambridge 1907, page 347. This theorem states that a necessary and sufficient condition that $\int_I f(x)\,dx$ exist in the sense of Riemann and be zero for every subinterval of I is that for every $k > 0$ the set of points of I for which $|f(x)| \geqq k$ is of content zero. From this it follows that under the hypothesis of Theorem 7 a necessary and sufficient condition that $\int_I |f(x) - g(x)|\,dx = 0$ is that for every $k > 0$, the set of points of I for which $|f(x) - g(x)| \geqq k$ is of content zero. This implies Theorem 7.

THEOREM 8. *If $f(x)$ is defined, bounded, and integrable Riemann on the closed interval I, and U is any subset of I which is the sum of a countable number of sets of content zero, then there exists a function $g(x)$, also defined, bounded and integrable Riemann over I, such that U is the set of points at which $f(x) \neq g(x)$.*

Proof. We define $g(x)$ to be $f(x) + u(x)$, where $u(x)$ is integrable Riemann and differs from zero only at points of U, as follows. We are given that $U = \sum_{n=1}^{\infty} C_n$, where each C_n is of content zero. If we let $D_1 = C_1$, $D_2 = C_2 - C_1$, $\cdots$, $D_n = C_n - (C_1 + C_2 + \cdots + C_{n-1})$, then $U = \sum_{n=1}^{\infty} D_n$, where each D_n is of content zero, and no two distinct sets D_n have a point in common. This follows from the fact that the logical sum and difference of two sets of content zero are sets of content zero. Now let $u(x)$ be the function which for all values of n is equal to $1/n$ for each x which is a member of D_n, and is equal to zero for each x which is a member of I but not of U. Then $u(x)$ is integrable Riemann by Theorem 5a), since for all values of k except zero the sets G_k and L_k for the function $u(x)$ are measurable Jordan. Hence the function $g(x)$ as defined above is bounded and integrable Riemann and differs from $f(x)$ only at points of U, which was to be proved.

Theorems 7 and 8 are important in connection with those function spaces for which the "distance" of two functions is defined in terms of the Riemann integral of some power of their absolute difference, and in other connections. Now every set which is the logical sum of a countable number of sets of content zero is both of the first category and of measure zero. Is the converse true? It is not, as the following rather complicated example will show. It seems to be even more difficult to construct a similar example for linear sets.

Let a Jordan arc AB in the plane be of superficial (two-dimensional) measure unity, and let it be in one to one continuous correspondence with a straight line segment OX of unit length in such a way that, if S is any subset of OX and S' is its image in AB, and if one of the sets S, S' is measurable, then the other is also, and the linear measure of S is equal to the superficial measure of S'. It is well known that such a correspondence can exist. Now let S be a subset of OX of linear measure zero, whose complement with respect to OX is of the first category, and let S' be the image of S in AB. Of course S', being a subset of an arc, is nowhere dense in the plane and hence of the first category, and also it is of measure zero. Nevertheless, S' is not the sum of a countable number of sets of (two-dimensional) content zero. For, if it were, it would be a subset of an F_σ of measure zero, and in fact, of an F_σ of measure zero contained in AB, since AB is a closed set. Call the image of this F_σ on OX the set K. Now we have that K is also an F_σ of measure zero, and is thus of the first category. But K contains S, which is of the second category, and this involves a contradiction.

THEOREM 9. *If the function $f(x)$, defined over the closed interval I, takes on only a finite number of values $y_1, y_2, \cdots, y_n$, at the sets E_1, $E_2, \cdots, E_n$ respectively, then the necessary and sufficient condition that $f(x)$ be integrable Riemann is that each of the sets E_i be measurable Jordan, and the integral in this case is equal to $\sum_{i=1}^{n} y_i \, cE_i$.*

Proof. In the first place if each set E_i is measurable Jordan, then by Theorem 5 $f(x)$ is integrable Riemann, since it is bounded and the sets G_k and L_k are measurable Jordan for all values of k. Conversely, if $f(x)$ is integrable Riemann, then the set D of its points of discontinuity is of measure zero. Now if p is a boundary point of one of the sets E_i, then it is at the same time a point of some set E_r and a limit point of points not in E_r, which implies that it is a point of discontinuity of $f(x)$. Hence the boundaries of the sets E_i are all contained in the set D, which is of measure zero. But then these boundaries are of measure zero, and hence the sets E_i are measurable Jordan, which was to be proved. That the integral in this case is equal to $\sum_{i=1}^{n} y_i \, cE_i$ follows from our discussion after Theorem 3, where a definition of the Riemann integral was given similar to that of the Lebesgue integral.

THEOREM 10. *The necessary and sufficient condition that a function $f(x)$, defined and bounded on the closed interval I, be integrable Riemann is that $f(x)$ be the limit of a uniformly convergent sequence of functions each of which assumes only a finite number of functional values over sets measurable Jordan.*

Proof. If $f(x)$ is the limit of such a sequence, then by Theorem 9 each function of the sequence is integrable Riemann. Then by a well known theorem, $f(x)$, being the limit of a uniformly convergent sequence of functions integrable Riemann, is itself integrable Riemann.

Suppose now on the other hand $f(x)$ is given to be integrable Riemann. We wish to prove that for every integer n there exists a function $g_n(x)$, taking on only a finite number of values over sets measurable Jordan, and such that $|f(x)-g_n(x)|<1/n$. Suppose m is a number less than the lower bound of $f(x)$, and M is a number greater than the upper bound of $f(x)$. In view of Theorem 2 we can select a finite number of values $m=y_0<y_1<\cdots<y_i<\cdots<y_r=M$ on the interval from m to M such that $y_i-y_{i-1}<1/2n$ and such that the set of points A_i for which $y_{i-1}\leqq f(x)<y_i$ is measurable Jordan for all values of i from 1 to r. Define $g_n(x)$ to be the function which for all values of i is equal to y_i for every x which is a member of A_i. Then $|f(x)-g_n(x)|<1/n$, and $g_n(x)$ takes on only a finite number of values y_i at sets A_i measurable Jordan, which was to be proved.

Theorem 10 gives us another method of defining the Riemann integral in terms of Jordan measure, which however is not essentially different from the method previously described. It should be remarked that the theorems of this paper are not restricted to functions defined over a linear interval; the proofs hold equally well for functions defined over a closed interval I of any number of dimensions.