Although the preceding proof requires no linear algebra, one can compare it to the proof of [1], which uses eigenvalues. Thinking about it gives another simple proof, provided one knows a little linear algebra in characteristic $p$.

**Second Proof of the Friendship Theorem.** We again reduce to the case in which the graph is $k$-regular, i.e., each vertex has exactly $k$ adjacent vertices and the total number of vertices is $n = k(k - 1) + 1$, with $k \geq 3$. Let $G = \{v_1, \ldots, v_n\}$. We let $A$ be the adjacency matrix of $G$, whose $(i, j)$ entry is 1 if $v_i$ and $v_j$ are adjacent, and 0 otherwise. The matrix $A$ has zeroes on the diagonal, so the trace of $A$ is 0. We let $B$ be the $n$ by $n$ matrix having a 1 in every entry. The trace of $B$ is $n$.

By assumption and the fact that $G$ is $k$-regular, $A^2 = (k - 1)I + B$, and $AB = kB$, where $I$ is the identity matrix of size $n$ by $n$. We now pass to the field $\mathbb{Z}_p$, where $p$ is a prime dividing $k - 1$. We continue to call the matrices $A$ and $B$, though we now think of them with entries in $\mathbb{Z}_p$. Observe that both $n$ and $k$ are now equal to 1. Hence $A^2 = B$, and furthermore $AB = kB = B$. It follows that for all $l \geq 2$, $A^l = B$. Let $\text{tr} \ C$ denote the trace of a square matrix $C$. In characteristic $p$, $\text{tr} \ A^p = (\text{tr} \ A)^p$. We reach a contradiction: $1 = n = \text{tr} \ B = \text{tr} \ A^p = (\text{tr} \ A)^p = 0$. □

The relation between the first proof and the second is simply that the trace of $A^p$ counts the closed walks of length $p$ in the graph. The relationship between the second proof and the usual proof is clear: in characteristic 0, one computes the eigenvalues of $A^2$ and then proves that $A$ could not have trace 0. The second proof takes advantage of the fact that $(a + b)^p = a^p + b^p$ in characteristic $p$. This allows us to avoid the actual computation of the eigenvalues, to push the calculation of the trace out to the $p$th power.

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**A Short Proof of Lebesgue’s Density Theorem**

Claude-Alain Faure

Lebesgue’s one-dimensional density theorem [1] says that almost all points of an arbitrary set $E \subseteq \mathbb{R}$ are points of density for $E$. We recall that a point $x \in \mathbb{R}$ is a point of density for $E$ if one has
\[ d_+(E, x) := \liminf_{y \to x} \frac{m^*(E \cap (x, y))}{y - x} = 1, \]

and
\[ d_-(E, x) := \liminf_{y \to x} \frac{m^*(E \cap (y, x))}{x - y} = 1, \]

where \( m^*(A) \) denotes the Lebesgue outer measure of the set \( A \subseteq \mathbb{R} \). For a survey of various proofs of this theorem, see [2], where a new constructive proof is given by the authors.

A short proof of the theorem is in [6]. Our proof does not use measurable functions, but only the usual properties of the outer measure. Furthermore, it is valid for non-measurable sets \( E \subseteq \mathbb{R} \). We use the following properties of the Lebesgue outer measure \( m^* \):

(P1) \( A \subseteq B \Rightarrow m^*(A) \leq m^*(B) \),

(P2) \( m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^*(A_n) \),

(P3) if \( U \subseteq \mathbb{R} \) is an open set whose components are the intervals \((c_n, d_n)\), then \( m^*(U) = \sum_n (d_n - c_n) \),

(P4) \( m^*(A) = \inf\{m^*(U)/A \subseteq U \text{ and } U \text{ is open}\}\),

(P5) \( m^*(A \cap (a, c)) = m^*(A \cap (a, b)) + m^*(A \cap (b, c)) \).

Properties (P3) and (P4) can be taken as definition of the outer measure \( m^* \). The following lemma is a slight modification of Riesz’s Rising Sun Lemma [3]:

**Lemma.** Let \( G : [a, b] \to \mathbb{R} \) be a continuous function and let \( U \subseteq (a, b) \) be an open set. Then the set

\[ U_G := \{x \in U/\text{there exists } y > x \text{ with } (x, y) \subseteq U \text{ and } G(x) > G(y)\} \]

is also open. Moreover, if \((c, d)\) is a component of \( U_G \), then \( G(c) \geq G(d) \).

**Proof.** Trivially, the set \( U_G \) is open. Now let \((c, d)\) be any component of \( U_G \). We show that \( G(x) \geq G(d) \) for all \( x \in (c, d) \). Let \( s := \max\{r \in [x, d]/G(x) \geq G(r)\} \), and suppose that \( s < d \). Thus \( G(x) < G(d) \) and \( s \in U_G \). There exists some \( t > s \) with \((s, t) \subseteq U \) and \( G(s) > G(t) \). If \( t \leq d \), then \( G(x) \geq G(s) > G(t) \) contradicts the maximality of \( s \). And if \( t > d \), then \( G(d) > G(x) \geq G(s) > G(t) \) implies that \( d \in U_G \), a contradiction. Therefore \( s = d \) and hence \( G(x) \geq G(d) \). \( \square \)

**Theorem.** The set \( A := \{x \in E/d_+(E, x) < 1\} \) has outer measure zero.

**Proof.** It is enough to verify that \( A_n := \{x \in E \cap (-n, n)/d_+(E, x) < n/(n + 1)\} \) has outer measure zero. We consider the map \( G : [-n, n] \to \mathbb{R} \) defined by

\[ G(x) = m^*(E \cap (-n, x)) - \frac{n}{n + 1}x. \]

For \( x < y \) one has \( G(y) - G(x) = m^*(E \cap (x, y)) - n(y - x)/(n + 1) \) by property (P5). Since \( 0 \leq m^*(E \cap (x, y)) \leq y - x \), it follows that the map \( G \) is continuous.

Now let \( \varepsilon > 0 \). By (P4) there exists an open set \( U \subseteq (-n, n) \) such that \( A_n \subseteq U \) and \( m^*(U) < m^*(A_n) + \varepsilon \). Since \( d_+(E, x) < n/(n + 1) \) for every \( x \in A_n \), one de-
duces that $A_n \subseteq U_G$. Let $(c_k, d_k)$ denote the components of $U_G$. By the lemma one has $G(c_k) \geq G(d_k)$ for each $k$ and hence $m^*(E \cap (c_k, d_k)) \leq n(d_k - c_k)/(n + 1)$. By (P2), (P1), and (P3) one thus obtains

$$m^*(A_n) \leq \sum_k m^*(A_n \cap (c_k, d_k)) \leq \sum_k \frac{n}{n + 1} (d_k - c_k) = \frac{n}{n + 1} m^*(U_G).$$

Therefore $m^*(A_n) < n(m^*(A_n) + \varepsilon)/(n + 1)$, which implies that $m^*(A_n) < ne$. The assertion follows because $\varepsilon$ is arbitrary.

By symmetry, the set $B := \{x \in E / d_-(E, x) < 1\}$ also has outer measure zero. Hence $d_+(E, x) = d_-(E, x) = 1$ for almost all $x \in E$, and the proof of Lebesgue’s theorem is complete.

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A Simple Proof of $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

and Related Identities

Josef Hofbauer

1. A PROOF FOR

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}. \quad (1)$$

Repeated application of the identity

$$\frac{1}{\sin^2 x} = \frac{1}{4 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}} = \frac{1}{4} \left[ \frac{1}{\sin^2 \frac{x}{2}} + \frac{1}{\cos^2 \frac{x}{2}} \right] = \frac{1}{4} \left[ \frac{1}{\sin^2 \frac{x}{2}} + \frac{1}{\sin^2 \frac{\pi + x}{2}} \right] \quad (2)$$

yields