A MANIFOLD EMBEDDING THEOREM

PETE L. CLARK

1. INTRODUCTION TO MANIFOLDS

A manifold is a second countable Hausdorff topological space $X$ such that for all $p \in X$, there is an open neighborhood $U_p$ of $p$ which is homeomorphic to $\mathbb{R}^{n(p)}$ for some positive integer $n(p)$. An $n$-manifold is a second countable Hausdorff topological space such that for all $p \in X$, there is an open neighborhood $U_p$ of $p$ which is homeomorphic to $\mathbb{R}^n$.

**Exercise 1.1.**

a) Show: a countable coproduct $\bigsqcup_{i=1}^{\infty} M_i$ of manifolds is a manifold.

b) Let $d > 1$. Show: $\mathbb{R} \bigsqcup \mathbb{R}^d$ is not a $d$-manifold for any $d \in \mathbb{Z}^+$.

Recall that a topological space is weakly locally connected if every point admits a connected neighborhood and is locally connected if every point admits a neighborhood base consisting of connected neighborhoods. Also recall that weakly locally connected is indeed weaker than being connected, even for very nice spaces: there is a compact, connected (hence weakly locally connected!) subset of $\mathbb{R}^2$ which is not locally connected, the topologist’s sine curve.

**Exercise 1.2.**

a) Show: manifolds are weakly locally connected.

b) Show: for a topological space, the following are equivalent:

(i) $X$ is weakly locally connected.

(ii) $X$ is (homeomorphic to) the coproduct of its connected components.

c) Show: manifolds are locally connected.

Let $M$ be a manifold. Then $M$ is locally connected and second countable, so is the coproduct of its connected components, which form a countable set. It is often the case that the study of manifolds reduces easily to the case of connected manifolds.

It is natural to suspect that a connected manifold must be an $n$-manifold for some positive integer $n$. And in fact it is true, but annoyingly difficult to prove. In particular, if this holds then for all $1 \leq m < n$ we must have that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$. This is easy to show when $m = 1$; for $m \geq 2$ it is most naturally approached using the methods of algebraic topology.

**Exercise 1.3.**

a) Let $m < n \in \mathbb{Z}^+$. Show: if $\mathbb{R}^m \cong \mathbb{R}^n$ then $S^m \cong S^n$.

b) (Exercise for a future course) Show that the $m$th homotopy of group of $S^m$ is nontrivial and the $m$th homotopy group of $S^n$ is trivial, so $S^m \not\cong S^n$.

c) (Exercise for a future course) Show that for a positive integer $d$, the $d$th homology group of $S^m$ is nontrivial iff $d = m$. Deduce $S^m \not\cong S^n$.

We can get what we want using the following result of L.E.J. Brouwer.
Theorem 1. (Invariance of Domain) Let $U \subset \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}^n$ be a continuous injection. Then $f$ is an open map.

Exercise 1.4.
a) Use Invariance of Domain to show that if $\mathbb{R}^m \cong \mathbb{R}^n$ then $m = n$.
b) Use Invariance of Domain to show that if a point $p$ admits an open neighborhood $U_p \cong \mathbb{R}^m$ and an open neighborhood $V_p \cong \mathbb{R}^n$ then $m = n$. Thus there is a well-defined function $\dim : M \to \mathbb{Z}$, the dimension at $p$.
c) Show: the function $\dim : M \to \mathbb{Z}$ is locally constant.
d) Every connected manifold is an $m$-manifold for a unique $m \in \mathbb{Z}$.

Exercise 1.5. Use Invariance of Domain to show: if $M$ is a manifold, $p$ is a point of $M$ and $N$ is a neighborhood of $M$ which is homeomorphic to $\mathbb{R}^N$, then in fact $N$ is an open neighborhood of $p$.

Exercise 1.6. For $N \in \mathbb{Z}$, let $\frac{1}{2}\mathbb{R}^N = \{(x_1, \ldots, x_N) \mid x_N \geq 0\}$. We call $\frac{1}{2}\mathbb{R}^N$ closed Euclidean half-space. A second countable Hausdorff space is a manifold with boundary if every point $p \in M$ there is $N \in \mathbb{Z}$ and an open neighborhood $U$ of $p$ which is homeomorphic either to $\mathbb{R}^N$ or to $\frac{1}{2}\mathbb{R}^N$.
a) Show that the Mobius band is a manifold with boundary.
b) Use Invariance of Domain to show that no point in a manifold with boundary has both a neighborhood homeomorphic to $\mathbb{R}^N$ and a neighborhood homeomorphic to $\frac{1}{2}\mathbb{R}^N$. Therefore there is a well-defined set of points of the second subset, called the boundary $\partial M$ of $M$.
c) Show that $\partial M$ is closed and $M^\circ = M \setminus \partial M$ is a manifold.

2. Finite Partitions of Unity

An open covering $\mathcal{U} = \{U_i\}$ of a topological space $X$ is locally finite if for all $p \in X$, there is a neighborhood $N_p$ such that $\{i \in \mathcal{U} \mid U_i \cap N_p \neq \emptyset\}$ is finite. Certainly any finite cover is locally finite.

For a function $f : X \to \mathbb{R}$, the support of $f$ is

$$\text{supp } f = \overline{f^{-1}(\mathbb{R} \setminus \{0\})}.$$ 

Thus $p$ does not lie in the support of $f$ iff there is a neighborhood $N_p$ of $p$ on which $f$ is identically 0.

Let $X$ be a topological space. A family of functions $\mathcal{F} = \{f : X \to [0, 1]\}$ is a partition of unity if:

- (PU1) $\forall x \in X$, $\exists$ a neighborhood $U_x$ of $x$ with $\{f \in \mathcal{F} \mid \text{supp } f \cap U_x \neq \emptyset\}$ finite;
- (PU2) For all $x \in X$, $\sum_{f \in \mathcal{F}} f(x) = 1$.

Notice that because of (PU1), the sum in (PU2) amounts to a finite sum.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of $X$. A partition of unity $\mathcal{F} = \{f_i : X \to [0, 1]\}_{i \in I}$ is subordinate to the covering if $\text{supp } f_i \subset U_i$ for all $i \in I$.

Theorem 2. (Existence of Finite Partitions of Unity) Let $X$ be quasi-normal, and let $\mathcal{U} = \{U_i\}_{i=1}^n$ be a finite open cover of $X$. Then there is a partition of unity $\{f_i : X \to [0, 1]\}_{i=1}^n$ which is subordinate to $\mathcal{U}$. 
Step 1: We show there are open subsets \( V_1, \ldots, V_n \) of \( X \) with \( X = \bigcup_{i=1}^{n} V_i \) and \( \overline{V}_i \subset U_i \) for all \( 1 \leq i \leq n \). Let \( A_1 = X \setminus \bigcup_{i=2}^{n} U_i \). Then \( A_1 \) is closed, and since \( \bigcup_{i=1}^{n} U_i = X \), we have \( A_1 \subset U_1 \). By quasi-normality, there is an open subset \( V_1 \) with \( A_1 \subset V_1 \subset \overline{V}_1 \subset U_1 \), and thus \( \{V_1, U_2, \ldots, U_n\} \) covers \( X \). Let \( 2 \leq k \leq n \). Having constructed open subsets \( V_1, \ldots, V_{k-1} \) such that \( V_i \subset U_i \) for all \( 1 \leq i \leq k-1 \) and such that \( \{V_1, \ldots, V_{k-1}, U_k, U_{k+1}, \ldots, U_n\} \) covers \( X \), let

\[
A_k = X \setminus \left( \bigcup_{i=1}^{k-1} V_i \cup \bigcup_{j=k+1}^{n} U_j \right).
\]

Then \( A_k \) is closed in \( X \) and \( A_k \subset U_k \), so by quasi-normality there is an open subset \( V_k \) with \( A_k \subset V_k \subset \overline{V}_k \subset U_k \), and thus \( \{V_1, \ldots, V_k, U_{k+1}, \ldots, U_n\} \) covers \( X \) and \( V_k \subset U_i \) for all \( 1 \leq i \leq k \). We are done by induction: take \( k = n \).

Step 2: Apply Step 1 to the finite open covering \( \{U_i\}_{i=1}^{n} \) of \( X \) to get a finite open covering \( \{V_i\}_{i=1}^{n} \) of \( X \) with \( \overline{V}_i \subset V_i \) for all \( i \). Then apply Step 1 again (!) to get a finite open covering \( \{W_i\}_{i=1}^{n} \) of \( X \) with \( \overline{W}_i \subset V_i \) for all \( i \). By the Tietze Extension Theorem, for all \( 1 \leq i \leq n \) there is a continuous function \( g_i : X \rightarrow [0, 1] \) with \( g_i|_{\overline{W}_i} \equiv 1 \) and \( g_i|_{X \setminus V_i} \equiv 0 \). Thus for all \( 1 \leq i \leq n \) we have

\[
\text{supp } g_i \subset V_i \subset U_i.
\]

Define

\[
g : X \rightarrow [0, 1], \quad g(x) = \sum_{i=1}^{n} g_i(x).
\]

Because \( X = \bigcup_{i=1}^{n} W_i \) we have \( g(x) > 0 \) for all \( x \in X \). For \( 1 \leq i \leq n \), put

\[
f_i : X \rightarrow [0, 1], \quad f_i(x) = \frac{g_i(x)}{g(x)}.
\]

Then \( \{f_i : X \rightarrow [0, 1]\}_{i=1}^{n} \) is a partition of unity subordinate to \( \{U_i\}_{i=1}^{n} \).

\( \square \)

**Remark.** We gave the definition of partitions of unity subordinate to a locally finite cover, but the Existence Theorem above concerned only finite covers. That is sufficient for our application, but elsewhere in the subject the general case becomes very important. I want to remark in passing that a Hausdorff space admits partitions of unity subordinate to any given locally finite cover if it is paracompact. The latter is probably the most important general topological property that we do not have time to cover in this course. Sorry!

### 3. An Embedding Theorem For Compact Manifolds

**Exercise 3.1.** Let \( M \) be a manifold, and let \( N \in \mathbb{Z}^+ \). Suppose every connected component of \( M \) can be embedded in \( \mathbb{R}^N \). Show: \( M \) can be embedded in \( \mathbb{R}^N \).

**Theorem 3.** (Manifold Embedding Theorem) Let \( M \) be a compact manifold. Then there is a continuous embedding \( \iota : M \hookrightarrow \mathbb{R}^{2n+1} \).

**Proof.** By compactness, \( M \) admits a finite covering \( U \) by open sets \( U_1, \ldots, U_n \) such that each \( U_i \) is homeomorphic to \( \mathbb{R}^{m(i)} \). Let \( m = \max_{i=1}^{n} m(i) \). Then each \( U_i \) can be embedded in \( \mathbb{R}^m \); choose such an embedding \( \iota_i : U_i \rightarrow \mathbb{R}^m \). Since \( M \) is compact,
it is normal, so by Theorem 2 there is a partition of unity \( \left\{ f_i : X \to [0, 1] \right\}_{i=1}^n \) subordinate to \( U \). Let \( A_i = \text{supp} f_i \). For all \( 1 \leq i \leq n \), define \( h_i : X \to \mathbb{R}^m \) by
\[
h_i(x) = f_i(x) \cdot \iota_i(x), \quad x \in U_i
\]
\[
= 0, \quad x \in X \setminus A_i.
\]
This function is well-defined because the two prescriptions agree on the intersection and is continuous by the Pasting Lemma. Now consider the function
\[
F : X \to \mathbb{R}^{n+mn}
given by
\[
F(x) = (f_1(x), \ldots, f_n(x), h_1(x), \ldots, h_n(x)).
\]
The characteristic property of the product topology shows that \( F \) is continuous. Suppose \( F(x) = F(y) \). Since \( \sum_{i=1}^n f_i(x) = 1 \) we have \( f_i(x) > 0 \) for some \( i \); thus \( f_i(y) = f_i(x) > 0 \), so \( x, y \in U_i \). We have
\[
f_i(x) \iota_i(x) = h_i(x) = h_i(y) = f_i(y) \iota_i(y),
\]
so \( \iota_i(x) = \iota_i(y) \). But \( \iota_i : U_i \to \mathbb{R}^m \) is an embedding, so \( x = y \). Thus \( F \) is injective. Being an injective continuous map from a compact space to a Hausdorff space, \( F \) is an embedding.

**Remark.** Theorem 3 can be improved in several ways (which are unfortunately beyond the scope of our ambitions).

a) The word “compact” can be removed entirely [Mu, p. 315]. The proof given there uses topological dimension theory.

b) Every smooth \( n \)-manifold can be smoothly embedded in \( \mathbb{R}^{2n} \). This gives sharper results in small dimensions, since (as it happens: this is certainly not an easy result!) every manifold of dimension 3 admits a smooth structure. In particular we deduce that all surfaces can be embedded in \( \mathbb{R}^4 \), a fact which follows more directly by classifying all surfaces and finding explicit embeddings.

**References**