Mh. Math. 123, 299–308 (1997)

Monatshefte für
Mathematik

Approximation of Lebesgue Integrals by Riemann Sums and Lattice Points in Domains with Fractal Boundary

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(Received 30 November 1995; in revised form 12 June 1996)

Abstract. Sets thrown at random in space contain, on average, a number of integer points equal to the measure of these sets. We determine the mean square error in the estimate of this number when the sets are homothetic to a domain with fractal boundary. This is related to the problem of approximating Lebesgue integrals by random Riemann sums.

1. Introduction

It is known that one can evaluate a Lebesgue integral using generalized Riemann sums. See, for example, the monograph of R. M. McLeod [6]. Here we want to approximate a Lebesgue integral $\int_{\mathbb{R}^n} \phi(x)dx$ by random Riemann sums $s^{-n}\sum_{n\in\mathbb{Z}^n} \phi(s^{-1}\sigma(t+n))$, where $s>0$ is a dilation, $\sigma \in \mathbb{SO}(N)$ a rotation, $t\in\mathbb{R}^n$ a translation. Since these Riemann sums are periodic with respect to translations, one may restrict the variable t to the torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$.

Our motivation is the following lattice point problem.

Let $\mathscr{D}$ be a bounded and measurable set in $\mathbb{R}^N$, and let $\chi_{s\sigma^{-1}\mathscr{D}-t}$ be the characteristic function of $s\sigma^{-1}\mathscr{D}-t$, rotated, dilated, and translated of $\mathscr{D}$. This set contains $\sum_{n\in\mathbb{Z}^N}\chi_{s\sigma^{-1}\mathscr{D}-t}(n)$ points with integer coordinates, and when $s\to+\infty$ it is natural to compare the sum with the measure of $s\sigma^{-1}\mathscr{D}-t$, that is $s^N|\mathscr{D}|$. Let us define

$$\mathscr{E}(s,\sigma,t) = \sum_{n\in\mathbb{Z}^N}\chi_{s\sigma^{-1}\mathscr{D}-t}(n) - s^N|\mathscr{D}|.$$

The problem of estimating the number of integer points in large domains is a generalization of the circle problem of C. F. Gauss and of the divisor problem of P. G. L. Dirichlet. In particular, a theorem of V. Jarník and H. Steinhaus [10] states that for planar domains bounded by rectifiable Jordan curves of length greater than one the error function is strictly smaller than the length of the curves:

$$|\mathscr{E}(s,\sigma,t)| < s|\partial\mathscr{D}|.$$

When $\mathscr{D}$ is the disk of center 0 and radius 1 in the plane and when $s\to+\infty$, it is known that $|\mathscr{E}(s,\sigma,0)|\leqslant cs^{2/3-\varepsilon}$. It is also known that $|\mathscr{E}(s,\sigma,0)|$ for some s can

1991 Mathematics Subject Classification: 11P21, 26A42.
Key words: Fractals, Integer Points, Riemann Sums.

be much larger than $s^{1/2}$, but the conjecture is that $|\mathscr{E}(s,\sigma,0)|\leqslant cs^{1/2+\varepsilon}$. In this direction G. H. HARDY [2] proved that

$$\left\{\frac{1}{s}\int_0^s |\mathscr{E}(z,\sigma,0)|^2 dz\right\}^{1/2} \leqslant cs^{1/2},$$

while D. G. KENDALL [41] proved that

$$\left\{\int_{\mathbb{T}^2} |\mathscr{E}(s,\sigma,t)|^2 dt\right\} \leqslant cs^{1/2}.$$

Therefore the estimate $|\mathscr{E}(s,\sigma,t)|\leqslant cs^{1/2}$ is true on the average.

These results, valid for a disk, have an analog for convex domains bounded by surfaces with strictly positive curvature, but do not extend to domains with flat boundaries. In particular, a point of the boundary where the curvature vanishes and the normal has rational direction may give a substantial contribution to the error. The idea that allows to overcome this problem consists in rotating the domain. Indeed, B. RANDOL [8], A. N. VARCHENKO [12], and others, considered the mean square error with respect to rotations and translations, and in the case of a domain in $\mathbb{R}^N$ with smooth boundary they proved the estimate

$$\left\{\int_{\mathbb{SO}(N)}\int_{\mathbb{T}^N} |\mathscr{E}(s,\sigma,t)|^2 dt\, d\sigma\right\}^{1/2} \leqslant cs^{(N-1)/2}.$$

One of our purposes is to give simple but sharp extensions of the above quoted results to domains with fractal boundaries. In particular, we show that it is possible to estimate the growth of the error function in terms of the Minkowski dimension of the boundary.

Assume that for some $c > 0, 0\leqslant\alpha\leqslant 1$, and for all $0 < \varepsilon < 1$,

$$\left|\left\{x \in \mathbb{R}^N : d(x,\partial\mathscr{D})\leqslant\varepsilon\right\}\right| \leqslant c\varepsilon^{\alpha}.$$

The parameter α measures the regularity of $\partial\mathscr{D}$ and $N-\alpha$ is the dimension of $\partial\mathscr{D}$: if $\alpha = 1$ the boundary is "regular", while if $\alpha < 1$ the boundary is "fractal". Under the above condition it is possible to prove an analog for domains with fractal boundary of the results of V. Jarník, G. H. Hardy, D. G. Kendall, B. Randol, A. N. Varchenko.

Theorem 1.

1) $|\mathscr{E}(s,\sigma,t)|\leqslant c(1+s)^{N-\alpha}$.

2) $\lim_{s\to+\infty} s^{-N/2}\{\frac{1}{s}\int_0^s\int_{\mathbb{SO}(N)}\int_{\mathbb{T}^N}|\mathscr{E}(z,\sigma,t)|^2 dt\, d\sigma\, dz\}^{1/2} = 0$.

3) $\{\frac{1}{s}\int_0^s\int_{\mathbb{SO}(N)}\int_{\mathbb{T}^N}|\mathscr{E}(z,\sigma,t)|^2 dt\, d\sigma\, dz\}^{1/2}\leqslant cs^{(N-\alpha)/2}$.

Observe that the estimate of the mean square error is much smaller than the one of the maximum error. Also observe that when $\alpha > 0$, then 3) implies 2), but in the case $\alpha = 0$, that is a completely arbitrary domain with no conditions on the boundary, 2) is a bit better than 3). The estimates 2) and 3) are particular cases of a more general result on the approximation of Lebesgue integrals by Riemann sums, which may have some independent interest.

Given an integrable function ϕ on $\mathbb{R}^N$, let us define

$$\mathbb{E}\phi(s,\sigma,t) = s^{-N}\sum_{n\in\mathbb{Z}^N}\phi(s^{-1}\sigma(t+n)) - \int_{\mathbb{R}^N}\phi(x)dx.$$

Observe that when $\phi = \chi_{\mathscr{D}}$ is a characteristic function, then $\mathscr{E}(s,\sigma,t) = s^N\mathbb{E}\chi_{\mathscr{D}}(s,\sigma,t)$, and the lattice points problem we have considered before reduces to an estimate of the difference between Riemann sums and integral.

It is not difficult to see that if ϕ is integrable on $\mathbb{R}^N$, then $\mathbb{E}\phi(s,\sigma,t)$ is defined almost everywhere and it is integrable on $(s,2s)\times\mathbb{SO}(N)\times\mathbb{T}^N$ (Lemma 3). If $s\longrightarrow+\infty$, we also have

$$\frac{1}{s}\int_s^{2s}\int_{\mathbb{SO}(N)}\int_{\mathbb{T}^N}|\mathbb{E}\phi(z,\sigma,t)|dt\,d\sigma\,dz\longrightarrow 0.$$

The natural conjecture is that the rate of convergence to zero of $\mathbb{E}\phi(s,\sigma,t)$ may be related to the smoothness of ϕ. Indeed this is the case, but more is true. The rate of convergence depends not only on the smoothness, but also on the L^p class to which ϕ belongs.

Assume that $\left\{\int_{\mathbb{R}^N}|\phi(x)|^p dx\right\}^{1/p}<+\infty$ and denote by $\omega(r)$ the modulus of continuity of order m in the norm of $L^p(\mathbb{R}^N)$ of ϕ,

$$\omega(r) = \sup_{|y|\leqslant r}\left\{\int_{\mathbb{R}^N}\left|\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\phi(x+jy)\right|^p dx\right\}^{1/p}.$$

The following result shows that when $1\leqslant p\leqslant 2$ it is possible to control the size of $\mathbb{E}\phi(s,\sigma,t)$ with $s^{N(1/p-1)}\omega(s^{-1})$.

Theorem 2. *Let ϕ be a function in $L^1(\mathbb{R}^N)\cap L^p(\mathbb{R}^N), 1\leqslant p\leqslant 2$. Then*

$$\left\{\frac{1}{s}\int_s^{2s}\int_{\mathbb{SO}(N)}\int_{\mathbb{T}^N}|\mathbb{E}\phi(z,\sigma,t)|^p dt\,d\sigma\,dz\right\}^{1/p}\leqslant cs^{N(1/p-1)}\omega(s^{-1}).$$

This theorem not only suggests that it is possible to evaluate a Lebesgue integral using Riemann sums, but also gives the rate of approximation. Observe that when $p>1$, even in the case of almost zero smoothness, the approximation is non trivial. The estimate of this approximation is the product of the two factors $s^{N(1/p-1)}$ and $\omega(s^{-1})$. We shall see that the second factor is related to the possibility of approximating the function ϕ by test functions ψ satisfying $\mathbb{E}\psi(s,\sigma,t)=0$. When $p=2$ the first factor becomes $s^{-N/2}$. This is the approximation that one has to expect by a Monte Carlo method with s^N sampling points in a unit cube.

We conclude with a remark on the almost everywhere convergence of Riemann sums of Lebesgue integrable functions. For every Riemann integrable function ϕ and every (σ,t) in $\mathbb{SO}(N)\times\mathbb{T}^N$, we have that

$$\lim_{s\longrightarrow+\infty}\left\{s^{-N}\sum_{n\in\mathbb{Z}^N}\phi(s^{-1}\sigma(t+n))\right\} = \int_{\mathbb{R}^N}\phi(x)dx.$$

Of course for Lebesgue integrable functions this may be no longer true, and the natural question is about convergence for almost every (σ, t). On this problem see, for example, the papers of B. JESSEN [3], J. MARCINKIEWICZ and R. SALEM [5], and W. RUDIN [9].

Our small contribution to this subject is the following.

Corollary 1. *Let $1 < p \leqslant 2$ and let $\{s(k)\}$ be a positive increasing sequence with $\sum_k s(k)^{N(1-p)} < +\infty$. Then for every function ϕ in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ and for almost every (σ, t) in $\mathbb{SO}(N) \times \mathbb{T}^N$ we have*

$$\lim_{k \to +\infty} \left\{ \frac{1}{s(k)} \int_{s(k)}^{2s(k)} \left[z^{-N} \sum_{n \in \mathbb{Z}^N} \phi(z^{-1}\sigma(t+n)) \right] dz \right\} = \int_{\mathbb{R}^N} \phi(x) dx.$$

2. Proof of the Theorems

We now present the proof of theorems and corollary.

Proof of Theorem 1. Let us prove 1). Here and in the sequel we may assume s to be large. Let us divide $\mathbb{R}^N$ into cubes centered at points of $\mathbb{Z}^N$ and with side one. $\mathscr{E}(s, \sigma, t)$ is bounded by the number of cubes that intersect the boundary of $s\sigma^{-1}\mathscr{D} - t$, and since a cube has diameter $\sqrt{N}$, by the assumption on $\partial\mathscr{D}$ we have

$$|\mathscr{E}(s, \sigma, t)| \leqslant \left| \left\{ x \in \mathbb{R}^N : d(x, \partial(s\sigma^{-1}\mathscr{D} - t)) \leqslant \sqrt{N} \right\} \right|$$
$$= s^N \left| \left\{ y \in \mathbb{R}^N : d(y, \partial\mathscr{D}) \leqslant \frac{\sqrt{N}}{s} \right\} \right| \leqslant c s^{N-\alpha}.$$

1) is thus proved. Items 2) and 3) are direct consequences of Theorem 2 and of the equality $\mathscr{E}(s, \sigma, t) = s^N \mathbb{E}\chi_{\mathscr{D}}(s, \sigma, t)$. We shall give the idea of the proof, but we shall skip some details.

To prove 2), divide the interval $(0, s)$ in dyadic pieces, $[s/2, s), [s/4, s/2), \ldots, [2^{-j}s, 2^{1-j}s)$, and a last piece $(0, 2^{-j}s)$ with $2^{-j}s \simeq 1$. One can estimate the contribution of the last piece directly, and then estimate the contribution of the dyadic pieces using Theorem 2. Finally, a sum gives the desired result.

Similarly, to prove 3) one only needs to verify that if $\partial\mathscr{D}$ has dimension $N - \alpha$, then the first modulus of continuity of $\chi_{\mathscr{D}}$ in the norm of $L^2(\mathbb{R}^N)$ satisfies the estimate $\omega(r) \leqslant c r^{\alpha/2}$. Indeed, if $|\chi_{\mathscr{D}}(x+y) - \chi_{\mathscr{D}}(x)|$ is different from zero, then the point x is in $\{x \in \mathbb{R}^N : d(x, \partial\mathscr{D}) \leqslant |y|\}$, and

$$\left\{ \int_{\mathbb{R}^N} |\chi_{\mathscr{D}}(x+y) - \chi_{\mathscr{D}}(x)|^2 dx \right\}^{1/2} \leqslant |\{x \in \mathbb{R}^N : d(x, \partial\mathscr{D}) \leqslant |y|\}|^{1/2}$$
$$\leqslant c|y|^{\alpha/2}.$$

□

We split the proof of Theorem 2 in some easy lemmas. First we compute the Fourier expansion of the error function $\mathbb{E}\phi(s, \sigma, t)$ using Poisson's summation formula, and as an immediate consequence we obtain an exact quadrature formula for entire functions of exponential type s. Then we estimate the size of $\mathbb{E}\phi(s, \sigma, t)$ in terms of the best approximation of the function ϕ by entire functions of exponential type s. Finally we show that, by the D. Jackson's approximation theorem, it is possible to control this best approximation by the modulus of continuity of ϕ. For some background on entire functions and approximation see, for example, the monograph of S. M. Nikol'skii [7]. In the sequel we denote the Fourier transform of a function ϕ by $\hat{\phi}(\xi) = \int_{\mathbb{R}^N} \phi(x) \exp(-2\pi i\xi \cdot x)dx$.

Lemma 1.

$$s^{-N} \sum_{n\in\mathbb{Z}^N} \phi(s^{-1}\sigma(t+n)) = \sum_{n\in\mathbb{Z}^N} \hat{\phi}(s\sigma n) \exp(2\pi i n\cdot t).$$

Proof.

$$\int_{\mathbb{T}^N} \left[s^{-N} \sum_{n\in\mathbb{Z}^N} \phi(s^{-1}\sigma(t+n)) \right] \exp(-2\pi i k\cdot t)dt$$
$$= s^{-N} \int_{\mathbb{R}^N} \phi(s^{-1}\sigma x)\exp(-2\pi i k\cdot x)dx = \hat{\phi}(s\sigma k). \qquad \square$$

Lemma 2. *If the function ψ is integrable and its Fourier transform vanishes outside $\{|\xi| \leqslant s\}$, then*

$$s^{-N} \sum_{n\in\mathbb{Z}^N} \psi(s^{-1}\sigma(t+n)) = \int_{\mathbb{R}^N} \psi(x)dx.$$

In particular, $\mathbb{E}\psi(s,\sigma,t) = 0$.

Proof. Since $\hat{\psi}(0) = \int_{\mathbb{R}^N} \psi(x)dx$, and $\hat{\psi}(s\sigma n) = 0$ if $n \neq 0$, this lemma follows from the previous one. $\square$

Lemma 3.

$$\frac{1}{s}\int_s^{2s} \int_{\mathbb{SO}(N)} \int_{\mathbb{T}^N} |\mathbb{E}\phi(z,\sigma,t)|dt\, d\sigma\, dz \leqslant 2\int_{\mathbb{R}^N} |\phi(x)|dx.$$

Proof.

$$\frac{1}{s}\int_s^{2s} \int_{\mathbb{SO}(N)} \int_{\mathbb{T}^N} |\mathbb{E}\phi(z,\sigma,t)|dt\, d\sigma\, dz$$
$$\leqslant \frac{1}{s}\int_s^{2s} \int_{\mathbb{SO}(N)} \left[z^{-N} \sum_{n\in\mathbb{Z}^N} \int_{\mathbb{T}^N} |\phi(z^{-1}\sigma(t+n))|dt \right] d\sigma\, dz +$$
$$+ \frac{1}{s}\int_s^{2s} \int_{\mathbb{SO}(N)} \int_{\mathbb{T}^N} \left| \int_{\mathbb{R}^N} \phi(x)dx \right| dt\, d\sigma\, dz$$
$$\leqslant \frac{1}{s}\int_s^{2s} \int_{\mathbb{SO}(N)} \left[z^{-N} \int_{\mathbb{R}^N} |\phi(z^{-1}\sigma x)|dx \right] d\sigma\, dz +$$
$$+ \frac{1}{s}\int_s^{2s} \int_{\mathbb{SO}(N)} \int_{\mathbb{T}^N} \int_{\mathbb{R}^N} |\phi(x)|dx\, dt\, d\sigma\, dz$$
$$= 2\int_{\mathbb{R}^N} |\phi(x)|dx. \qquad \square$$

Lemma 4.

$$\left\{\frac{1}{s}\int_s^{2s}\int_{\mathbb{SO}(N)}\int_{\mathbb{T}^N}|\mathbb{E}\phi(z,\sigma,t)|^2 dt\,d\sigma\,dz\right\}^{1/2}$$
$$\leqslant cs^{-N/2}\left\{\int_{\{|\xi|\geqslant s\}}|\hat{\phi}(\xi)|^2 d\xi\right\}^{1/2}\leqslant cs^{-N/2}\left\{\int_{\mathbb{R}^N}|\phi(x)|^2 dx\right\}^{1/2}.$$

Proof.

$$\left\{\frac{1}{s}\int_s^{2s}\int_{\mathbb{SO}(N)}\int_{\mathbb{T}^N}|\mathbb{E}\phi(z,\sigma,t)|^2 dt\,d\sigma\,dz\right\}^{1/2}$$
$$=\left\{\frac{1}{s}\int_s^{2s}\int_{\mathbb{SO}(N)}\int_{\mathbb{T}^N}\left|\sum_{n\in\mathbb{Z}^N-\{0\}}\hat{\phi}(z\sigma n)\exp(2\pi i n\cdot t)\right|^2 dt\,d\sigma\,dz\right\}^{1/2}$$
$$=\left\{\sum_{n\in\mathbb{Z}^N-\{0\}}\left[\frac{1}{s}\int_s^{2s}\int_{\mathbb{SO}(N)}|\hat{\phi}(z\sigma n)|^2 d\sigma\,dz\right]\right\}^{1/2}$$
$$=\left\{\sum_{n\in\mathbb{Z}^N-\{0\}}\left[\frac{c}{s}\int_{\{s<|\xi|<2s\}}|\xi|^{1-N}|\hat{\phi}(|n|\xi)|^2 d\xi\right]\right\}^{1/2}$$
$$=\left\{\sum_{n\in\mathbb{Z}^N-\{0\}}\left[cs^{-1}|n|^{-1}\int_{\{s|n|<|\xi|<2s|n|\}}|\xi|^{1-N}|\hat{\phi}(\xi)|^2 d\xi\right]\right\}^{1/2}$$
$$=\left\{\int_{\mathbb{R}^N}|\hat{\phi}(\xi)|^2\left[cs^{-1}|\xi|^{1-N}\sum_{(2s)^{-1}|\xi|<|n|<s^{-1}|\xi|}|n|^{-1}\right]d\xi\right\}^{1/2}$$
$$\leqslant c\left\{s^{-N}\int_{\{|\xi|\geqslant s\}}|\hat{\phi}(\xi)|^2 d\xi\right\}^{1/2}.$$

□

Lemma 5. *If* $1\leqslant p\leqslant 2$, *then*

$$\left\{\frac{1}{s}\int_s^{2s}\int_{\mathbb{SO}(N)}\int_{\mathbb{T}^N}|\mathbb{E}\phi(z,\sigma,t)|^p dt\,d\sigma\,dz\right\}^{1/p}$$
$$\leqslant cs^{N(1/p-1)}\inf\left\{\left\{\int_{\mathbb{R}^N}|\phi(x)-\psi(x)|^p dx\right\}^{1/p}\right\},$$

where the infimum is taken with respect to all functions ψ *in* $L^1(\mathbb{R}^N)\cap L^p(\mathbb{R}^N)$ *with Fourier transform vanishing outside* $\{|\xi|\leqslant s\}$.

Proof. By Lemma 3, Lemma 4, and the Riesz–Thorin interpolation theorem ([1], 1.1), we have for $1 \leqslant p \leqslant 2$

$$\left\{ \frac{1}{s} \int_s^{2s} \int_{\mathbb{SO}(N)} \int_{\mathbb{T}^N} |\mathbb{E}\phi(z,\sigma,t)|^p dt\, d\sigma\, dz \right\}^{1/p} \leqslant c s^{N(1/p-1)} \left\{ \int_{\mathbb{R}^N} |\phi(x)|^p dx \right\}^{1/p}.$$

Substituting ϕ with $\phi - \psi$ and recalling that, by Lemma 2, $E\psi(s,\sigma,t) = 0$, we obtain the desired result. □

Proof of Theorem 2. The proof is an immediate consequence of the previous lemma and of the Jackson's approximation theorem ([7], 5.2). The approximation in the norm of $L^p(\mathbb{R}^N)$ of the function ϕ by entire functions of exponential type s is controlled by the modulus of continuity $\omega(s^{-1})$ of ϕ. More precisely, given ϕ in $L^p(\mathbb{R}^N)$, there exists ψ in $L^p(\mathbb{R}^N)$ and entire of exponential type at most s, that is with Fourier transform vanishing outside $\{|\xi| \leqslant s\}$, such that

$$\left\{ \int_{\mathbb{R}^N} |\phi(x) - \psi(x)|^p dx \right\}^{1/p} \leqslant c\omega(s^{-1}).$$

The proof of the approximation theorem shows that when ϕ is in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, then one can choose ψ in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. In fact one may obtain ψ by convolving ϕ with an integrable kernel K_s. □

Proof of Corollary 1. It is enough to show that if $1 < p \leqslant 2$ and $\sum_k s(k)^{N(1-p)} < +\infty$, then for every function ϕ in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ the series

$$\sum_k \left[\frac{1}{s(k)} \int_{s(k)}^{2s(k)} |\mathbb{E}\phi(z,\sigma,t)| dz \right]^p$$

converges for almost every (σ, t) in $\mathbb{SO}(N) \times \mathbb{T}^N$. Indeed by Theorem 2,

$$\begin{aligned} &\int_{\mathbb{SO}(N)} \int_{\mathbb{T}^N} \left(\sum_k \left[\frac{1}{s(k)} \int_{s(k)}^{2s(k)} |\mathbb{E}\phi(z,\sigma,t)| dz \right]^p \right) dt\, d\sigma \\ &\leqslant \sum_k \left[\frac{1}{s(k)} \int_{s(k)}^{2s(k)} \int_{\mathbb{SO}(N)} \int_{\mathbb{T}^N} |\mathbb{E}\phi(z,\sigma,t)|^p dt\, d\sigma\, dz \right] \\ &\leqslant c \sum_k [s(k)^{N(1/p-1)} \omega(s(k)^{-1})]^p < +\infty. \end{aligned}$$

□

We conclude with some remarks.

Remark 1. The estimates of the maximum and of the mean square error in Theorem 1 are essentially sharp, at least for regular domains with $\alpha = 1$. As an example it is enough to consider a polyhedron. Actually we expect that this theorem is sharp also in the case $\alpha < 1$, this because of the relation between modulus of continuity of $\chi_{\mathscr{D}}$ and regularity of $\partial\mathscr{D}$. See also the Remarks 2 and 3.

Remark 2. Interpolating between the estimates 1) and 3) in Theorem 1, we obtain for $2 \leqslant p \leqslant +\infty$

$$\left\{ \frac{1}{s} \int_0^s \int_{\mathbb{SO}(N)} \int_{\mathbb{T}^N} |\mathscr{E}(z,\sigma,t)|^p dt\, d\sigma\, dz \right\}^{1/p} \leqslant c(1+s)^{(N-\alpha)(1-1/p)}.$$

Since these L^p means decrease with p, it is natural to enquire if one can extrapolate something interesting even when $p < 2$. The answer to this question must depend on the domain, and for example it is different for a polyhedron or for a sphere.

When the domain $\mathscr{D}$ is a polyhedron in $\mathbb{R}^N$, the L^1 norm of the error is logarithmically small. Indeed it is possible to prove that

$$\left\{ \frac{1}{s} \int_0^s \int_{\mathbb{SO}(N)} \int_{\mathbb{T}^N} |\mathscr{E}(z,\sigma,t)|^p dt\, d\sigma\, dz \right\}^{1/p} \leqslant \begin{cases} c\, Lg^N(2+s) & \text{if } p = 1, \\ c(1+s)^{(N-1)(1-1/p)} & \text{if } 1 < p \leqslant +\infty. \end{cases}$$

The L^1 estimate has been essentially proved by M. TARNOPOLSKA-WEISS [11]. In her paper the exponent of the logarithm is $2+\varepsilon$, but in dimension N the proof gives the exponent $N+\varepsilon$ with $\varepsilon > 0$. Actually G. Travaglini communicated to me that one can get $\varepsilon = 0$.

When $\mathscr{D}$ is the unit ball in $\mathbb{R}^N$, the Fourier transform involves a Bessel function, $\hat{\chi}_{\mathscr{D}}(\xi) = |\xi|^{-N/2} J_{N/2}(2\pi|\xi|)$, and by Lemma 1 we have

$$\mathscr{E}(s,\sigma,t) = s^N \sum_{n \in \mathbb{Z}^N - \{0\}} |s\sigma n|^{-N/2} J_{N/2}(2\pi|s\sigma n|) \exp(2\pi i n \cdot t).$$

Since for every $1 \leqslant p \leqslant +\infty$ and $n \neq 0$ we have

$$\left\{ \int_{\mathbb{T}^N} |\mathscr{E}(s,\sigma,t)^p| dt \right\}^{1/p} \geqslant s^N |s\sigma n|^{-N/2} |J_{N/2}(2\pi|s\sigma n|)|$$
$$\approx \pi^{-1} |n|^{-(N+1)/2} s^{(N-1)/2} |\cos(2\pi|n|s - \pi(N+1)/4)|,$$

we also have

$$\left\{ \frac{1}{s} \int_0^s \int_{\mathbb{SO}(N)} \int_{\mathbb{T}^N} |\mathscr{E}(z,\sigma,t)^p| dt\, d\sigma\, dz \right\}^{1/p} \geqslant c s^{(N-1)/2}.$$

Therefore, if the domain is a ball and if $p < 2$, the estimate of the L^p mean of the error is not essentially better than the estimate of the mean square error. By the way, this example again shows that the estimate of the mean square error in Theorem 1 is essentially sharp.

Remark 3. Theorem 2 is sharp, at least when $p = 2$. In fact the chain of equalities in the proof of Lemma 4 shows that the case $p = 2$ in the theorem is more or less equivalent to an estimate of the average decay of the Fourier transform $\hat{\phi}$. This decay is related to the degree of approximation of ϕ by functions of exponential type and, by Bernstein's theorem ([7], 5.4), this gives a bound on the modulus of continuity of ϕ.

We also observe that in order to obtain the desired result in Theorem 2 it is necessary to average with respect to all $(s, 2s) \times \mathbb{SO}(N) \times \mathbb{T}^N$, that is one cannot avoid a triple integral. Indeed if, for example, ϕ is concentrated around the origin, then $s^{-N} \sum_{n\in\mathbb{Z}^N} \phi(s^{-1}\sigma n) \approx s^{-N}\phi(0)$ can be much larger than $\int_{\mathbb{R}^N} \phi(x)dx$, so that $\int_s^{2s} \int_{\mathbb{SO}(N)} |\mathbb{E}\phi(z,\sigma,0)|^2 d\sigma\, dz$ can be very large. Similarly, since $\int_{\mathbb{T}^N} |\mathbb{E}\phi(s,\sigma,t)|^2 dt = \sum_{n\in\mathbb{Z}^N-\{0\}} |\hat{\phi}(s\sigma n)|^2$, if $\hat{\phi}$ is concentrated along some particular directions, then for suitable rotations $\int_s^{2s} \int_{\mathbb{T}^N} |\mathbb{E}\phi(z,\sigma,t)|^2 dt\, dz$ can be very large. If $\hat{\phi}$ is concentrated on some spheres, then for suitable dilations $\int_{\mathbb{SO}(N)} \int_{\mathbb{T}^N} |\mathbb{E}\phi(s,\sigma,t)|^2 d\sigma\, dt$ can be very large.

Remark 4. A function in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N), 2 \leqslant p \leqslant +\infty$, also belongs to $L^2(\mathbb{R}^N)$. Therefore we may apply Theorem 2 to this function. However, for a generic ϕ in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, with $p > 2$, $\mathbb{E}\phi(s,\sigma,t)$ cannot decrease much faster than $s^{-N/2}$. Indeed

$$\left\{\frac{1}{s}\int_s^{2s}\int_{\mathbb{SO}(N)}\int_{\mathbb{T}^N} |\mathbb{E}\phi(z,\sigma,t)|^p dt\, d\sigma\, dz\right\}^{1/p}$$

$$\geqslant \left\{\frac{1}{s}\int_s^{2s}\int_{\mathbb{SO}(N)}\int_{\mathbb{T}^N} |\mathbb{E}\phi(z,\sigma,t)|^2 dt\, d\sigma\, dz\right\}^{1/2}$$

$$\approx c\left\{s^{-N}\int_{\{|\xi|\geqslant s\}} |\hat{\phi}(\xi)|^2 d\xi\right\}^{1/2},$$

and it turns out that the average decay of the Fourier transform of a generic function in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N), p > 2$, is not better than the decay in $L^2(\mathbb{R}^N)$.

Remark 5. Lemma 2 has a converse. If ψ is in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, and if for every $t \in \mathbb{T}^N, \sigma \in \mathbb{SO}(N)$, and every $s > S$, one has the exact quadrature formula $s^{-N} \sum_{n\in\mathbb{Z}^N} \psi(s^{-1}\sigma(t+n)) = \int_{\mathbb{R}^N} \psi(x)dx$, then ψ is an entire function of exponential type at most S. This follows from lemma 4.

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