

Now suppose that $\alpha \in \text{R.H.S.}$ and that

$$\beta = \sum_1^N b_n \omega_n \in \text{R.H.S.} \quad (b_n \in k_v). \quad (12.2)$$

Then for any m , $1 \leq m \leq N$ we have

$$D(\omega_1, \dots, \omega_{m-1}, \beta, \omega_{m+1}, \dots, \omega_N) = b_m^2 D(\omega_1, \dots, \omega_N),$$

and so

$$db_m^2 \in \mathfrak{o}_v \quad (1 \leq m \leq N)$$

where

$$d = D(\omega_1, \dots, \omega_N) \in k.$$

But (Appendix B) we have $d \neq 0$, and so $|d|_v = 1$ for almost all v . For almost all v the condition (12.2) thus implies

$$b_m \in \mathfrak{o}_v \quad (1 \leq m \leq N),$$

i.e.

$$\text{R.H.S.} \subset \text{L.H.S.}$$

This proves the lemma.

[COROLLARY. *Almost all v are unramified in the extension K/k .*

For by the results of Chapter I a necessary and sufficient condition for v to be unramified is that there are $\gamma_1, \dots, \gamma_N \in \text{R.H.S.}$ with $|D(\gamma_1, \dots, \gamma_N)|_v = 1$. And for almost all v we can put $\gamma_n = \alpha^{n-1}$.]

13. Restricted Topological Product

We describe here a topological tool which will be needed later:

DEFINITION. *Let Ω_λ ($\lambda \in \Lambda$) be a family of topological spaces and for almost all λ let $\Theta_\lambda \subset \Omega_\lambda$ be an open subset of Ω_λ . Consider the space Ω whose points are sets $\alpha = \{\alpha_\lambda\}_{\lambda \in \Lambda}$, where $\alpha_\lambda \in \Omega_\lambda$ for every λ and $\alpha_\lambda \in \Theta_\lambda$ for almost all λ . We give Ω a topology by taking as a basis of open sets the sets*

$$\prod \Gamma_\lambda$$

where $\Gamma_\lambda \subset \Omega_\lambda$ is open for all λ and $\Gamma_\lambda = \Theta_\lambda$ for almost all λ . With this topology Ω is the restricted topological product of the Ω_λ with respect to the Θ_λ .

COROLLARY. *Let S be a finite subset of Λ and let Ω_S be the set of $\alpha \in \Omega$ with $\alpha_\lambda \in \Theta_\lambda$ ($\lambda \notin S$), i.e.*

$$\Omega_S = \prod_{\lambda \in S} \Omega_\lambda \prod_{\lambda \notin S} \Theta_\lambda. \quad (13.1)$$

Then Ω_S is open in Ω and the topology induced in Ω_S as a subset of Ω is the same as the product topology.

Beweis. Klar.

The restricted topological product depends on the totality of the Θ_λ but not on the individual Θ_λ :

† i.e. all except possibly finitely many.

LEMMA. Let $\Theta'_\lambda \subset \Omega_\lambda$ be open sets defined for almost all λ and suppose that $\Theta_\lambda = \Theta'_\lambda$ for almost all λ . Then the restricted product of the Ω_λ with respect to the Θ'_λ is the same as† the restricted product with respect to the Θ_λ .

Beweis. Klar.

LEMMA. Suppose that the Ω_λ are locally compact and that the Θ_λ are compact. Then Ω is locally compact.

Proof. The Ω_S are locally compact by (13.1) since S is finite. Since $\Omega = \cup \Omega_S$ and the Ω_S are open in Ω , the result follows.

DEFINITION. Suppose that measures μ_λ are defined on the Ω_λ with $\mu_\lambda(\Theta_\lambda) = 1$ when Θ_λ is defined. We define the product measure μ on Ω to be that for which a basis of measurable sets is the

$$\prod_\lambda M_\lambda$$

where $M_\lambda \subset \Omega_\lambda$ has finite μ_λ -measure and $M_\lambda = \Theta_\lambda$ for almost all λ and where

$$\mu\left(\prod_\lambda M_\lambda\right) = \prod_\lambda \mu_\lambda(M_\lambda).$$

COROLLARY. The restriction of μ to Ω_S is just the ordinary product measure.

14. Adele Ring (or Ring of Valuation Vectors)

Let k be a global field. For each normalized valuation $|\cdot|_v$ of k denote by k_v the completion of k . If $|\cdot|_v$ is non-archimedean denote by \mathfrak{o}_v the ring of integers of k_v . The adèle ring V_k of k is the topological ring whose underlying topological space is the restricted product of the k_v with respect to the \mathfrak{o}_v and where addition and multiplication are defined componentwise:

$$(\alpha\beta)_v = \alpha_v\beta_v \quad (\alpha + \beta)_v = \alpha_v + \beta_v \quad \alpha, \beta \in V_k. \quad (14.1)$$

It is readily verified (i) that this definition makes sense, i.e. if $\alpha, \beta \in V_k$ then $\alpha\beta, \alpha + \beta$ whose components are given by (14.1) are also in V_k and (ii) that addition and multiplication are continuous in the V_k -topology, so V_k is a topological ring, as asserted.

V_k is locally compact because the k_v are locally compact and the \mathfrak{o}_v are compact (§ 7).

There is a natural mapping of k into V_k which maps $\alpha \in k$ into the adèle every one of whose components is α : this is an adèle because $\alpha \in \mathfrak{o}_v$ for almost all v . The map is an injection, because the map of k into any k_v is an injection. The image of k under this injection is the ring of *principal adèles*. It will cause no trouble to identify k with the principal adèles, so we shall speak of k as a subring of V_k .

LEMMA. Let K be a finite (separable) extension of the global field k . Then

$$V_k \otimes_k K = V_K \quad (14.2)$$

† A purist would say "canonically isomorphic to".

algebraically and topologically. In this correspondence $k \otimes_k K = K \subset V_k \otimes_k K$, where $k \subset V_k$, is mapped identically on to $K \subset V_K$.

Proof. We first established an isomorphism of the two sides of (14.2) as topological spaces. Let $\omega_1, \dots, \omega_N$ be a basis for K/k and let v run through the normalized valuations of k . It is easy to see that the L.H.S. of (14.2), with the tensor product topology, is just the restricted product of the

$$k_v \otimes_k K = k_v \omega_1 \oplus \dots \oplus k_v \omega_N \tag{14.3}$$

with respect to the

$$\mathfrak{o}_v \omega_1 \oplus \dots \oplus \mathfrak{o}_v \omega_N. \tag{14.4}$$

But now (cf. §10), (14.3) is just

$$K_{V_1} \oplus \dots \oplus K_{V_J}, \quad (V_1|v, \dots, V_J|v) \tag{14.5}$$

where $V_1, \dots, V_J, J = J(v)$ are the normalized extensions of v to K . Further (§ 12) the identification of (14.3) with (14.5) identifies (14.4) with

$$\mathfrak{D}_{V_1} \oplus \dots \oplus \mathfrak{D}_{V_J} \tag{14.6}$$

for almost all† v . Hence the L.H.S. of (14.2) is the restricted product of (14.3) with respect to (14.4), which is clearly the same thing as the restricted product of the K_v with respect to the \mathfrak{D}_v , where V runs through all the normalized valuations of K . This is just the R.H.S. of (14.2). This establishes an isomorphism between the two sides of (14.2) as topological spaces. A moment's consideration shows that it is also an algebraic isomorphism.

Q.E.D.

COROLLARY. *Let V_k^+ denote the topological group obtained from V_k by forgetting the multiplicative structure. Then*

$$V_K^+ = \underbrace{V_k^+ \oplus \dots \oplus V_k^+}_{N \text{ summands}} \quad (N = [K : k]).$$

In this isomorphism the additive group $K^+ \subset V_K^+$ of the principal adeles is mapped into $k^+ \oplus \dots \oplus k^+$, in an obvious notation.

Proof. $\omega V_k^+ \subset V_K^+$, for any non-zero $\omega \in K$, is clearly isomorphic to V_k^+ as a topological group. Hence we have the isomorphisms

$$V_K^+ = V_k^+ \otimes_k K = \omega_1 V_k^+ \oplus \dots \oplus \omega_N V_k^+ = V_k^+ \oplus \dots \oplus V_k^+.$$

THEOREM. *k is discrete‡ in V_k and $V_k^+|k^+$ is compact in the quotient topology.*

Proof. The preceding corollary (with k for K and \mathbb{Q} or $\mathbb{F}(t)$ for k) shows that it is enough to verify the theorem for \mathbb{Q} or $\mathbb{F}(t)$ and we shall do it for \mathbb{Q} .

To show that \mathbb{Q}^+ is discrete in $V_{\mathbb{Q}}^+$ it is enough because of the group

† This was proved there only when $\omega_n = \alpha^{n-1}$, where $K = k(\alpha)$. We should therefore take this choice of ω_n .

‡ It is impossible to conceive of any other uniquely defined topology in k . This meta-mathematical reason is more persuasive than the argument that follows!

structure to find a neighbourhood U of 0 which contains no other elements of k^+ . We take for U the set of $\alpha = \{\alpha_v\} \in V_{\mathbf{Q}}^+$ with

$$\begin{aligned} |\alpha_\infty|_\infty &< 1 \\ |\alpha_p|_p &\leq 1 \quad (\text{all } p), \end{aligned}$$

where $|\cdot|_p, |\cdot|_\infty$ are respectively the p -adic and the absolute values on \mathbf{Q} .

If $b \in \mathbf{Q} \cap U$ then in the first place $b \in \mathbf{Z}$ (because $|b|_p \leq 1$ for all p) and then $b = 0$ because $|b|_\infty < 1$.

Now let $W \subset V_{\mathbf{Q}}^+$ consists of the $\alpha = \{\alpha_v\}$ with

$$|\alpha_\infty|_\infty \leq \frac{1}{2}, \quad |\alpha_p|_p \leq 1 \quad (\text{all } p).$$

We show that every adele β is of the shape

$$\beta = b + \alpha, \quad b \in \mathbf{Q}, \quad \alpha \in W. \tag{14.7}$$

For each p we can find an

$$r_p = z_p/p^{x_p} \quad (z_p \in \mathbf{Z}, \quad x_p \in \mathbf{Z}, \quad x_p \geq 0)$$

such that

$$|\beta_p - r_p|_p \leq 1$$

and since α is an adele we may take

$$r_p = 0 \quad (\text{almost all } p).$$

Hence $r = \sum_p r_p$ is well defined and

$$|\beta_p - r| \leq 1 \quad (\text{all } p).$$

Now choose $s \in \mathbf{Z}$ such that

$$|\beta_\infty - r - s| \leq \frac{1}{2}.$$

Then $b = r + s, \beta = \alpha - b$ do what is required.

Hence the continuous map $W \rightarrow V_{\mathbf{Q}}^+/\mathbf{Q}^+$ induced by the quotient map $V_{\mathbf{Q}}^+ \rightarrow V_{\mathbf{Q}}^+/\mathbf{Q}^+$ is surjective. But W is compact (topological product of $|\alpha_\infty|_\infty \leq \frac{1}{2}$ and the \mathfrak{o}_p) and hence so is $V_{\mathbf{Q}}^+/\mathbf{Q}^+$.

As already remarked, V_k^+ is a locally compact group and so it has an invariant (Haar) measure. It is easy to see that in fact this Haar measure is the product of the Haar measures on the k_v in the sense described in the previous section.

COROLLARY 1. *There is a subset W of V_k defined by inequalities of the type $|\xi_v|_v \leq \delta_v$, where $\delta_v = 1$ for almost all v , such that every $\varphi \in V_k$ can be put in the form*

$$\varphi = \theta + \gamma, \quad \theta \in W, \quad \gamma \in k$$

Proof. For the W constructed in the proof is clearly contained in some W of the type described above.

COROLLARY 2. *V_k^+/k^+ has finite measure in the quotient measure induced by the Haar measure on V_k^+ .*

Note. This statement is, of course, independent of the particular choice of the multiplicative constant in the Haar measure on V_k^+ . We do not here go into the question of finding the measure of V_k^+/k^+ in terms of our explicitly given Haar measure. (See Tate's thesis, Chapter XV of this book.)

Proof. This can be reduced similarly to the case of \mathbf{Q} or $\mathbf{F}(t)$, which is almost immediate: thus W defined above has measure 1 for our Haar measure.

Alternatively finite measure follows from compactness. For cover V_k^+/k^+ with the translates of F , where F is an open set of finite measure. The existence of a finite subcover implies finite measure.

[We give an alternative proof of the product formula $\prod |\xi|_v = 1$ for $\xi \in k$, $\xi \neq 0$. We have seen that if $\beta_v \in k_v$ then multiplication by β_v magnifies the Haar measure in k_v^+ by the factor $|\beta_v|_v$. Hence if $\beta = \{\beta_v\} \in V_k$, multiplication by β magnifies Haar measure in V_k^+ by $\prod |\beta_v|_v$. In particular multiplication by the principal adèle ξ magnifies Haar measure by $\prod |\xi|_v$. But now multiplication by ξ takes $k^+ \subset V_k^+$ into k^+ and so gives a well-defined $1 - 1$ map of V_k^+/k^+ onto V_k^+/k^+ which magnifies the measure by the factor $\prod |\xi|_v$. Hence $\prod |\xi|_v = 1$ by the Corollary.]

In the next section we shall need the

LEMMA. *There is a constant $C > 0$ depending only on the global field k with the following property:*

Let $\alpha = \{\alpha_v\} \subset V_k$ be such that

$$\prod_v |\alpha_v|_v > C. \tag{14.8}$$

Then there is a principal adèle $\beta \in k \subset V_k$, $\beta \neq 0$ such that

$$|\beta|_v \leq |\alpha_v|_v \quad (\text{all } v).$$

Proof. This is modelled on Blichfeldt's proof of Minkowski's Theorem in the Geometry of Numbers and works in quite general circumstances.

Note that (14.8) implies $|\alpha_v|_v = 1$ for almost all v because $|\alpha_v|_v \leq 1$ for almost all v .

Let c_0 be the Haar measure of V_k^+/k^+ and let c_1 be that of the set of $\gamma = \{\gamma_v\} \subset V_k^+$ with

$$\begin{aligned} |\gamma_v|_v &\leq \frac{1}{10} && \text{if } v \text{ is arch.} \\ |\gamma_v|_v &\leq 1 && \text{if } v \text{ is non-arch.} \end{aligned}$$

Then $0 < c_0 < \infty$ and $0 < c_1 < \infty$ because the number of arch. v 's is finite. We show that

$$C = c_0/c_1$$

will do.

The set T of $\tau = \{\tau_v\} \subset V_k^+$ with

$$\begin{aligned} |\tau_v|_v &\leq \frac{1}{10} |\alpha_v|_v && \text{if } v \text{ is arch.} \\ |\tau_v|_v &\leq |\alpha_v|_v && \text{if } v \text{ is non-arch.} \end{aligned}$$

has measure

$$c_1 \prod_v |\alpha_v|_v > c_1 C = c_0.$$

Hence in the quotient map $V_k^+ \rightarrow V_k^+/k^+$ there must be a pair of distinct points of T which have the same image in V_k^+/k^+ , say

$$\tau' = \{\tau'_v\} \in T, \quad \tau'' = \{\tau''_v\} \in T$$

and

$$\tau' - \tau'' = \beta \text{ (say)} \in k^+.$$

Then

$$|\beta|_v = |\tau'_v - \tau''_v|_v \leq |\alpha_v|_v$$

for all v , as required.

COROLLARY. *Let v_0 be a normalized valuation and let $\delta_v > 0$ be given for all $v \neq v_0$ with $\delta_v = 1$ for almost all v . Then there is a $\beta \in k$, $\beta \neq 0$ with*

$$|\beta|_v \leq \delta_v \quad (\text{all } v \neq v_0).$$

Proof. This is just a degenerate case. Choose $\alpha_v \in k_v$ with $0 < |\alpha_v|_v \leq \delta_v$ and $|\alpha_v|_v = 1$ if $\delta_v = 1$. We can then choose $\alpha_{v_0} \in k_{v_0}$ so that $\prod_{\text{all } v \text{ inc. } v_0} |\alpha_v|_v > C$.

Then the lemma does what is required.

[The character group of the locally compact group V_k^+ is isomorphic to V_k^+ and k^+ plays a special role. See Chapter XV (Tate's thesis), Lang: "Algebraic Numbers" (Addison-Wesley), Weil: "Adeles and Algebraic Groups" (Princeton lecture notes) and Godement: Bourbaki seminars 171 and 176. This duality lies behind the functional equation of ζ and L -functions. Iwasawa has shown (*Annals of Math.*, 57 (1953), 331-356) that the rings of adèles are characterized by certain general topologico-algebraic properties.]

15. Strong Approximation Theorem

The results of the previous section, in particular the discreteness of k in V_k depend critically on the fact that *all* normalized valuations are used in the definition of V_k :

THEOREM. (Strong approximation theorem.) *Let v_0 be any valuation of the global field k . Define \mathcal{V} to be the restricted topological product of the k_v with respect to the v_0 , where v runs through all normalized $v \neq v_0$. Then k is everywhere dense in \mathcal{V} .*

Proof.† It is easy to see that the theorem is equivalent to the following statement. Suppose we are given (i) a finite set S of valuations $v \neq v_0$, (ii) elements $\alpha_v \in k_v$ for all $v \in S$ and (iii) $\varepsilon > 0$. Then there is a $\beta \in k$ such that $|\beta - \alpha_v|_v < \varepsilon$ for all $v \in S$ and $|\beta|_v \leq 1$ for all $v \notin S$, $v \neq v_0$.

By Corollary 1 to the Theorem of § 14 there is a $W \subset V_k$ defined by inequalities of the type $|\xi_v|_v \leq \delta_v$ ($\delta_v = 1$ for almost all v) such that every

† Suggested by Prof. Kneser at the Conference.

$\varphi \in V_k$ is of the form

$$\varphi = \theta + \gamma, \quad \theta \in W, \quad \gamma \in k. \quad (15.1)$$

By the corollary to the last lemma of §14, there is a $\lambda \in k$, $\lambda \neq 0$ such that

$$\begin{aligned} |\lambda|_v &< \delta_v^{-1}\varepsilon \quad (v \in S), \\ |\lambda|_v &\leq \delta_v^{-1} \quad (v \notin S, v \neq v_0). \end{aligned} \quad (15.2)$$

Hence, on putting $\varphi = \lambda^{-1}\alpha$ in (15.1) and multiplying by λ we see that every $\alpha \in V_k$ is of the shape

$$\alpha = \psi + \beta, \quad \psi \in \lambda W, \quad \beta \in k, \quad (15.3)$$

where λW is the set of $\lambda\xi$, $\xi \in W$. If now we let α have components the given α_v at $v \in S$ and (say) 0 elsewhere, it is easy to see that β has the properties required.

[The proof clearly gives a quantitative form of the theorem (i.e. with a bound for $|\beta|_{v_0}$). For an alternative approach, see K. Mahler: Inequalities for ideal bases, *J. Australian Math. Soc.* 4 (1964), 425-448.]

16. Idele Group

The set of invertible elements of any commutative topological ring R form a group R^\times under multiplication. In general, R^\times is not a topological group if it is endowed with the subset topology because inversion need not be continuous. It is usual therefore to give R^\times the following topology. There is an injection

$$x \rightarrow (x, x^{-1}) \quad (16.0)$$

of R^\times into the topological product $R \times R$. We give to R^\times the corresponding subset topology. Clearly R^\times with this topology is a topological group and the inclusion map $R^\times \rightarrow R$ is continuous.

DEFINITION. *The idele group J_k of k is the group V_k^\times of invertible elements of the adèle ring V_k with the topology just defined.*

We shall usually speak of J_k as a subset of V_k and will have to distinguish between the J_k - and V_k -topologies.†

We have seen that k is naturally embedded in V_k and so k^\times is naturally embedded in J_k . We shall call k^\times considered as a subgroup of J_k the *principal* ideles.

LEMMA. *k^\times is a discrete subgroup of J_k .*

Proof. For k is discrete in V_k and so k^\times is injected into $V_k \times V_k$ by (16.0) as a discrete subset.

LEMMA. *J_k is just the restricted topological product of the k_v^\times with respect to the units $U_v \subset k_v$ (with the restricted product topology).*

Beweis. Klar.

† Let $\alpha^{(q)}$ for a rational prime q be the element of $J_{\mathbb{Q}}$ with components $\alpha_v^{(q)} = q$, $\alpha_v^{(q)} = 1$ ($v \neq q$). Then $\alpha^{(q)} \rightarrow 1$ ($q \rightarrow \infty$) in the $V_{\mathbb{Q}}$ -topology, but not in the $J_{\mathbb{Q}}$ -topology.

DEFINITION. For $\alpha = \{\alpha_v\} \in J_k$ we define $c(\alpha) = \prod_{\text{all } v} |\alpha_v|_v$ to be the content of α .

LEMMA. The map $\alpha \rightarrow c(\alpha)$ is a continuous homomorphism of the topological group J_k into the multiplicative group of the (strictly) positive real numbers.

Beweis. Klar.

[Lemma. Let $\alpha \in J_k$. Then the map $\xi \rightarrow \alpha\xi$ of V_k^+ onto itself multiplies Haar measure on V_k^+ by a factor $c(\alpha)$.

Beweis. Klar.

Note also that the J_k -topology is that appropriate to a group of operators on V_k^+ : a basis of open sets is the $S(C, O)$ where $C, O \subset V_k^+$ are respectively V_k -compact and V_k -open and S consists of the $\alpha \in J_k$ such that $(1 - \alpha)C \subset O, (1 - \alpha^{-1})C \subset O$.]

Let J_k^1 be the kernel of the map $\alpha \rightarrow c(\alpha)$ with the topology as a subset of J_k . We shall need the

LEMMA. J_k^1 considered as a subset of V_k is closed and the V_k -subset topology on J_k^1 coincides with the J_k -topology.

Proof. Let $\alpha \in V_k, \alpha \notin J_k^1$. We must find a V_k -neighbourhood W of α which does not meet J_k^1 .

1st Case. $\prod |\alpha_v|_v < 1$ (possibly = 0). Then there is a finite set S of v such that

- (i) S contains all the v with $|\alpha_v|_v > 1$ and
- (ii) $\prod_{v \in S} |\alpha_v|_v < 1$. Then the set W can be defined by

$$\begin{aligned} |\xi_v - \alpha_v|_v &< \varepsilon & v \in S \\ |\xi_v|_v &\leq 1 & v \notin S \end{aligned}$$

for sufficiently small ε .

2nd Case. $\prod_v |\alpha_v|_v = C$ (say) > 1 . Then there is a finite set S of v such that (i) S contains all the v with $|\alpha_v|_v > 1$ and (ii) if $v \notin S$ an inequality $|\xi_v|_v < 1$ implies $\dagger |\xi_v|_v < \frac{1}{2}C$. We can choose ε so small that $|\xi_v - \alpha_v|_v < \varepsilon$ ($v \in S$) implies $1 < \prod_{v \in S} |\xi_v|_v < 2C$. Then W may be defined by

$$\begin{aligned} |\xi_v - \alpha_v|_v &< \varepsilon & (v \in S) \\ |\xi_v|_v &\leq 1 & (v \notin S). \end{aligned}$$

We must now show that the J_k - and V_k -topologies on J_k^1 are the same. If $\alpha \in J_k^1$ we must show that every J_k -neighbourhood of α contains a V_k -neighbourhood and vice-versa.

Let $\dagger W \subset J_k^1$ be a V_k -neighbourhood of α . Then it contains a V_k -neigh-

† If $k \supset \mathbb{Q}$ and v is a normalized extension of the p -adic valuation then the value group of v consists of (some of) the powers of p . Hence it is enough for (ii) to include in S all the arch. v and all the extensions of p -adic valuations with $p \leq 2C$. Similarly if $k \supset \mathbb{F}(t)$.

‡ This half of the proof of the equality of the topologies makes no use of the special properties of ideles. It is only an expression of the fact noted above that the inclusion $R^* \rightarrow R$ is continuous for any topological ring R .

bourhood of the type

$$\left. \begin{array}{l} |\xi_v - \alpha_v|_v < \varepsilon \quad (v \in S) \\ |\xi_v|_v \leq 1 \quad (v \notin S) \end{array} \right\} \quad (16.1)$$

where S is a finite set of v . This contains the J_k -neighbourhood in which \leq in (16.1) is replaced by $=$.

Now let $H \subset J_k^1$ be a J_k -neighbourhood. Then it contains a J_k -neighbourhood of the type

$$\left. \begin{array}{l} |\xi_v - \alpha_v|_v < \varepsilon \quad (v \in S) \\ |\xi_v|_v = 1 \quad (v \notin S) \end{array} \right\} \quad (16.2)$$

where the finite set S contains at least all arch. v and all v with $|\alpha_v|_v \neq 1$. Since $\prod |\alpha_v|_v = 1$ we may also suppose that ε is so small that (16.2) implies

$$\prod_v |\xi_v|_v < 2.$$

Then the intersection of (16.2) with J_k^1 is the same† as that of (16.1) with J_k^1 , i.e. (16.2) defines a V_k -neighbourhood.

By the product formula we have $k^\times \subset J_k^1$. The following result is of vital importance in class-field theory.

THEOREM. J_k^1/k^\times with the quotient topology is compact.

Proof. After the preceding lemma it is enough to find a V_k -compact set $W \subset V_k$ such that the map

$$W \cap J_k^1 \rightarrow J_k^1/k^\times$$

is surjective.

We take for W the set of $\xi = \{\xi_v\}$ with

$$|\xi_v|_v \leq |\alpha_v|_v$$

where $\alpha = \{\alpha_v\}$ is any idele of content greater than the C of the last lemma of § 14.

Let $\beta = \{\beta_v\} \in J_k^1$. Then by the lemma just quoted there is a $\eta \in k^\times$ such that

$$|\eta|_v \leq |\beta_v^{-1} \alpha_v|_v \quad (\text{all } v).$$

Then $\eta\beta \in W$, as required.

$[J_k/k^\times$ is totally disconnected in the function field case. For the structure of its connected component in the number theory case see papers of Artin and Weil in the "Proceedings of the Tokyo Symposium on Algebraic Number Theory, 1955" (Science Council of Japan) or Artin-Tate: "Class Field Theory", 1951/2 (Harvard, 1960(?)). The determination of the character group of J_k/k^\times is global class field theory.]

17. Ideals and Divisors

Suppose that k is a finite extension of \mathbf{Q} . We define the ideal group I_k of k to be the free abelian group on a set of symbols in 1-1 correspondence

† See previous footnote.

with the *non-arch.* valuations v of k , i.e. formal sums

$$\sum_{v \text{ non-arch.}} n_v \cdot v \tag{17.1}$$

where $n_v \in \mathbb{Z}$ and $n_v = 0$ for almost all v , addition being defined component-wise. We call (17.1) an ideal and call it integral if $n_v \geq 0$ for all v . This language is justified by the existence of a 1-1 correspondence between integral ideals and the ideals (in the ordinary sense) in the Dedekind ring

$$\mathfrak{o} = \bigcap_{\text{non-arch.}} \mathfrak{o}_v:$$

cf. Chapter I, §2, Prop. 2.

There is a natural continuous map

$$J_k \rightarrow I_k$$

of the idele group on to the ideal group† given by

$$\alpha = \{\alpha_v\} \rightarrow \sum (\text{ord}_v \alpha) \cdot v.$$

The image of $k^\times \subset J_k$ is the group of principal ideals.

THEOREM. *The group of ideal classes, i.e. I_k modulo principal ideals, is finite.*

Proof. For the map $J_k^1 \rightarrow I_k$ is surjective and so the group of ideal classes is the continuous image of the compact group J_k^1/k^\times and hence compact. But a compact discrete group is finite.

When k is a finite separable extension of $\mathbb{F}(t)$ we define the divisor group D_k of k to be the free group on all the v . For each v the number of elements in the residue class field of v is a power, say q^{d_v} of the number q of elements in \mathbb{F} . We call d_v the degree of v and similarly define $\sum n_v d_v$ to be the degree of $\sum n_v \cdot v$. The divisors of degree 0 form a group D_k^0 . One defines the principal divisors similarly to principal ideals and then one has the

THEOREM. *D_k^0 modulo principal divisors is a finite group.*

For the quotient group is the continuous image of the compact group J_k^1/k^\times .

18. Units

In this section we deduce the structure theorem for units from our results about idele classes.

Let S be any finite non-empty set of normalized valuations and suppose that S contains all the archimedean valuations. The set of $\eta \in k$ with

$$|\eta|_v = 1 \quad (v \notin S) \tag{18.1}$$

are a group under multiplication, the group H_S of S -units. When $k \supset \mathbb{Q}$ and S is just the archimedean valuations, then H_S is the group of units *tout court*.

† I_k being given the discrete topology.

LEMMA 1. Let $0 < c < C < \infty$. Then the set of S -units η with

$$c \leq |\eta|_v \leq C \quad (v \in S) \tag{18.2}$$

is finite.

Proof. The set W of ideles $\alpha = \{\alpha_v\}$ with

$$|\alpha_v|_v = 1 \quad (v \notin S), \quad c \leq |\alpha_v|_v \leq C \quad (v \in S) \tag{18.3}$$

is compact (product of compact sets with the product topology). The required set of units is just the intersection of W with the discrete subset k of J_k and so is both discrete and compact, hence finite.

LEMMA 2. There are only finitely many $\varepsilon \in k$ such that $|\varepsilon|_v = 1$ for every v . They are precisely the roots of unity in k .

Proof. If ε is a root of unity it is clear that $|\varepsilon|_v = 1$ for every v . Conversely, by the previous lemma (with any S and $c = C = 1$) there are only finitely many $\varepsilon \in k$ with $|\varepsilon|_v = 1$ for all v . They form a group under multiplication and so are all roots of 1.

THEOREM. (Unit theorem.) H_S is the direct sum of a finite cyclic group and a free abelian group of rank $s-1$.

Proof. To avoid petty notational troubles we treat only the case when $Q \subset k$ and S is the set of arch. valuations.

Let J_S consist of the ideals $\alpha = \{\alpha_v\}$ with $|\alpha_v|_v = 1$ ($v \notin S$) and put

$$J_S^1 = J_S \cap J_k^1.$$

Clearly J_S^1 is open in J_k^1 and so

$$J_S^1/H_S = J_S^1/(J_S^1 \cap k^\times) \tag{18.4}$$

is open in J_k^1/k^\times . Since it is a subgroup, it is also closed, and so compact (§ 16).

Consider the map

$$\lambda : J_S \rightarrow \underbrace{\mathbf{R}^+ \oplus \mathbf{R}^+ \oplus \dots \oplus \mathbf{R}^+}_{s \text{ times}}$$

where \mathbf{R}^+ is the additive group of reals, given by

$$\alpha \rightarrow (\log |\alpha_1|_1, \log |\alpha_2|_2, \dots, \log |\alpha_s|_s),$$

where $1, 2, \dots, s$ are the valuations in S . Clearly λ is both continuous and surjective.

The kernel of λ restricted to H_S consists just of the ε with $|\varepsilon|_v = 1$ for every v , so is a finite cyclic group by Lemma 2. By Lemma 1 there are only finitely many $\eta \in H_S$ with

$$\frac{1}{2} \leq |\eta|_v \leq 2 \quad v \in S. \tag{18.5}$$

Hence the group Λ (say) $= \lambda(H_S)$ is discrete.

Further, $T = \lambda(J_S^1)$ is just the set of (x_1, \dots, x_s) with

$$x_1 + x_2 + \dots + x_s = 0,$$

i.e. an $s-1$ dimensional real vector space. Finally, T/Λ is compact, being the continuous image of the compact set (18.4). Hence Λ is free on $s-1$ generators, as asserted.

Of course this structure-theorem (Dirichlet) and the finiteness of the class-number (Minkowski) are older than ideles. It is more usual to deduce the compactness of J_k^1/k^\times from these theorems instead of vica versa.

19. Inclusion and Norm Maps for Adeles, Ideles and Ideals

Let K be a finite extension of the global field k . We have already seen (§ 14, Lemma) that there is a natural isomorphism

$$V_k \otimes_k K = V_K \tag{19.1}$$

algebraically and topologically. Hence $V_k = V_k \otimes_k k$ can naturally be regarded as a subring of V_K which is closed in the topology of V_K . This injection of V_k into V_K is called the injection map or the conorm map and is written

$$\text{con: } \alpha \rightarrow \text{con } \alpha = \text{con}_{K/k} \alpha \in V_K \quad (\alpha \in V_k).$$

Explicitly if $\mathbf{A} = \text{con } \alpha$, then the components satisfy

$$A_V = \alpha_v \in k_v \subset K_V \tag{19.2}$$

where V runs through the normalized valuations of K and v is the normalized valuation of k which extends to V . If $k \subset L \subset K$ it follows that

$$\text{con}_{K/k} \alpha = \text{con}_{L/k} (\text{con}_{K/L} \alpha). \tag{19.3}$$

Finally, for principal adeles the conorm map is just the usual injection of k into K .

It is customary, and usually leads to no confusion, to identify $\text{con}_{K/k} \alpha$ with α .

One can also define norm and trace maps from V_K to V_k by imitating the usual procedure (cf. Appendix A). Let $\omega_1, \dots, \omega_n$ be a basis for K/k . Then by (19.1) every $\mathbf{A} \in V_K$ is uniquely of the shape

$$\mathbf{A} = \sum \alpha_j \omega_j \quad \alpha_j \in V_k \tag{19.4}$$

and the map $\mathbf{A} \rightarrow \alpha_j$ of V_K into V_k is continuous by the very definition of the tensor product topology (§ 9). Hence if we define

$$\alpha_{ij} = \alpha_{ij}(\mathbf{A}) \in V_k$$

by

$$\mathbf{A} \omega_i = \sum_j \alpha_{ij} \omega_j \tag{19.5}$$

the $n \times n$ matrices (α_{ij}) give a continuous representation of the ring V_K over V_k . In particular, the

$$S_{K/k} \mathbf{A} = \sum \alpha_{ii} \tag{19.6}$$

$$N_{K/k} \mathbf{A} = \det (\alpha_{ij}) \tag{19.7}$$

are *continuous* functions of A and have the usual formal properties

$$S_{K/k}(A_1 + A_2) = S_{K/k}A_1 + S_{K/k}A_2 \tag{19.8}$$

$$S_{K/k} \text{ con}_{K/k} \alpha = n\alpha \tag{19.9}$$

$$N_{K/k}(A_1 A_2) = N_{K/k}A_1 N_{K/k}A_2 \tag{19.10}$$

$$N_{K/k} \text{ con}_{K/k} \alpha = \alpha^n. \tag{19.11}$$

Further, the norm and trace operations are compatible with the embedding of k, V in V_k, V_K respectively, i.e. if $A \in K \subset V_K$ we get the same answer whether we compute $N_{K/k}A, S_{K/k}A$ in K or in V_K , so there is no ambiguity in the notation.

Finally if $K \supset L \supset k$ we have $V_k \subset V_L \subset V_K$ (on regarding conorm as an identification), and so the usual relations (cf. Appendix A)

$$S_{L/k}(S_{K/L}A) = S_{K/k}A \tag{19.12}$$

and

$$N_{L/k}N_{K/L}A = N_{K/k}A. \tag{19.13}$$

We can express the maps (19.6), (19.7) componentwise if we like. Let V_1, \dots, V_J be the extensions of any given valuation v of k to K . Then (§ 9)

$$K_v \text{ (say)} = \bigoplus_{1 \leq j \leq J} K_j = k_v \otimes_k K = \bigoplus_{1 \leq i \leq n} k_v \omega_i \tag{19.14}$$

where k_v, K_j are the completions of k, K with respect to v, V_j respectively. Any $A \in V_K$ can be regarded as having components

$$A_{V_1} \oplus \dots \oplus A_{V_J} = A_v \tag{19.15}$$

in the K_v and then the components in the matrix representation (19.5) of A are just the representations of the A_v . In particular

$$S_{K/k}(A) = \{S_{K_v/k_v}A_v\} \tag{19.16}$$

and

$$N_{K/k}A = \{N_{K_v/k_v}A_v\}. \tag{19.17}$$

Finally, making use of the final remarks of § 9, we deduce that

$$S_{K/k}A = \left\{ \sum_{V|v} S_{K_V/k_v}(A_V) \right\}_v \tag{19.18}$$

and

$$N_{K/k}A = \left\{ \prod_{V|v} N_{K_V/k_v}A_V \right\}_v \tag{19.19}$$

where $V|v$ means “ V is a continuation of v ”.

We now consider the consequences for ideles. If α is an idele, it is clear from the definition (19.2) that $\text{con}_{K/k} \alpha$ is an idele, so we have an injection

$$\text{con}_{K/k} : J_k \rightarrow J_K$$

which is clearly a homomorphism of J_k with a closed subset of V_K . Further,

if $\mathbf{A} \in J_K \subset V_K$, so \mathbf{A} is invertible, it follows from (19.9) that $N_{K/k} \mathbf{A}$ is invertible, i.e. is an element of J_k . Hence we have a map

$$N_{K/k} : J_K \rightarrow J_k$$

which is continuous by the definition of the idele topology (§16) and which clearly satisfies (19.10), (19.11), (19.13) and (19.19). On the other hand, the definition of trace does not go over to ideles.

Finally, we consider the conorm and norm maps for ideals, where k is a finite extension of \mathbf{Q} . The kernel of the map (§17)

$$J_k \rightarrow I_k$$

of the idele group into the ideal group is just the group U_k (say) of ideles $\alpha = \alpha_v$ which have $|\alpha_v|_v = 1$ for every non-archimedean v . If K is a finite extension of k , it is clear that

$$\text{con}_{K/k} U_k \subset U_K$$

and from the Lemma of §11 and (19.17) we have

$$N_{K/k} U_K \subset U_k.$$

Hence on passing to the quotient from J_k we have the induced maps

$$\text{con}_{K/k} : I_k \rightarrow I_K$$

$$N_{K/k} : I_K \rightarrow I_k$$

with the usual properties (19.10), (19.11) and (19.13); and these maps are compatible with the norm and conorm maps for elements of K and k on taking principal ideals. By definition (19.2) we have

$$\text{con}_{K/k} v = \sum_{V|v} e_V V \tag{19.20}$$

where the positive integers e_V are defined by

$$|\pi_v|_V = |\Pi_V|_V^{e_V}, \tag{19.21}$$

π_v and Π_V being prime elements of k_v , K_V respectively. Similarly, it follows from (19.19) that

$$N_{K/k} V = f_V v, \tag{19.22}$$

where f_V is the degree of the residue class field of V over that of v . We note in passing that (19.11), (19.20) and (19.22) imply that

$$\sum_{V|v} e_V f_V = n,$$

as it should since

$$e_V f_V = [K_V : k_V].$$

Similarly, when k is a finite extension of $\mathbf{F}(t)$ one defines conorm and norm of divisors, with the appropriate properties.