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REPRESENTATIONS OF DEFINITE BINARY QUADRATIC FORMS OVER $\mathbf{F}_q[t]$

JEAN BUREAU AND JORGE MORALES

ABSTRACT. In this paper, we prove that a binary definite quadratic form over $\mathbf{F}_q[t]$, where q is odd, is completely determined up to equivalence by the polynomials it represents up to degree $3m-2$, where m is the degree of its discriminant. We also characterize, when $q > 13$, all the definite binary forms over $\mathbf{F}_q[t]$ that have class number one.

1. Introduction

It is a natural question to ask whether binary definite quadratic forms over the polynomial ring $\mathbf{F}_q[t]$ are determined, up to equivalence, by the set of polynomials they represent. Here $\mathbf{F}_q$ is the finite field of order q and q is odd.

The analogous question over $\mathbf{Z}$ has been answered affirmatively – with the notable exception of the forms X^2+3Y^2 and X^2+XY+Y^2, which have the same representation set but are not equivalent—by Watson [13]. Several related results appear in the literature as far back as the mid-nineteenth century (see [14]).

We begin with the easier question whether the discriminant of a binary definite quadratic form over $\mathbf{F}_q[t]$ is determined by its representation set. In the classical case over $\mathbf{Z}$, Schering [11] showed that this is the case up to powers of 2. The same type of ideas are used here to show in the polynomial context that if Q and Q' represent the same polynomials up to degree $3m-2$, where $m = \max\{\deg \operatorname{disc}(Q), \deg \operatorname{disc}(Q')\}$, then $\operatorname{disc}(Q) = \operatorname{disc}(Q')$ (Proposition 3.5).

The main result of this paper is that if Q and Q' have the same discriminant and represent the same polynomials up to degree equal to their second successive minimum, then they are equivalent (Theorem 4.1). We show that if

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such forms were not equivalent, then there would be an elliptic curve over $\mathbf{F}_q$ that has more rational points than allowed by Hasse's bound. If the condition on the discriminants is omitted, then having the same representation set up to degree $3m-2$ is enough to conclude equivalence (Theorem 4.2).

The same questions can be asked for ternary definite quadratic forms. We show that in this case, the representation *sets* (as opposed to the representation *numbers*), are not enough in general to determine the equivalence class. We do so by constructing a family of counterexamples (Corollary 5.3). It turns out, however, that the representation *numbers*, that is the number of times that each polynomial is represented, are sufficient to determine the equivalence class of a ternary form, as it will be showed in an upcoming paper [2].

Finally, in Section 6, we show, assuming $q > 13$, that if a definite binary quadratic form Q has class number one (i.e., its genus contains only one equivalence class), then $\deg \operatorname{disc}(Q) \leq 2$ (Theorem 6.2).

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2. Notation and terminology

The following notation will be in force throughout the paper:

$\mathbf{F}_q$: The finite field of order q. We always assume q odd.
A: The polynomial ring $\mathbf{F}_q[t]$.
K: The field of rational functions $\mathbf{F}_q(t)$.
δ: A fixed non-square of $\mathbf{F}_q^\times$.

A quadratic form Q over A is a homogeneous polynomial

$$Q = \sum_{1 \leq i,j \leq n} m_{ij} X_i X_j,$$

where $M = (m_{ij})$ is an $n \times n$ symmetric matrix with coefficients in A. The group $\mathbf{GL}_n(A)$ acts by linear change of variables on the set of such forms. Two forms in the same $\mathbf{GL}_n(A)$-orbit are called *equivalent*. Two forms in the same $\mathbf{SL}_n(A)$-orbit are called *properly equivalent*.

The *discriminant* of Q is defined by

$$\operatorname{disc}(Q) = (-1)^{n(n-1)/2} \det(M)$$

as an element of $A/\mathbf{F}_q^{\times 2}$. This is an invariant of the equivalence class of Q.

The *representation set* of Q is the set of polynomials

$$V(Q) = \{Q(\mathbf{x}) : \mathbf{x} \in A^n\},$$

and the *degree k representation set* is

$$V_k(Q) = \{Q(\mathbf{x}) : \mathbf{x} \in A^n, \deg Q(\mathbf{x}) \leq k\}.$$

The form Q is *definite* if it is anisotropic over the field $K_\infty = \mathbf{F}_q((1/t))$. This implies in particular that $n \leq 4$.

A definite quadratic form Q is *reduced* if $\deg m_{ii} \le \deg m_{jj}$ for $i \le j$ and $\deg m_{ij} < \deg m_{ii}$ for $i < j$. Gerstein [5] showed that every definite quadratic form is equivalent to a reduced form and that two reduced forms in the same equivalence class differ at most by a transformation in $\mathbf{GL}_n(\mathbf{F}_q)$. In particular, the increasing sequence of degrees of the diagonal terms of a reduced form

$$(\deg m_{11}, \deg m_{22}, \dots, \deg m_{nn})$$

is an invariant of its equivalence class. This sequence is called the *successive minima* of Q and will be denoted by $(\mu_1(Q), \mu_2(Q), \dots, \mu_n(Q))$.

In the case of binary forms, which are the main topic of this paper, we will often write

$$Q = (a, b, c)$$

for the quadratic form

$$Q = aX^2 + 2bXY + cY^2.$$

For binary forms, it is easy to see that being definite means simply that $\operatorname{disc}(Q) = b^2 - ac$ has either odd degree or has even degree and nonsquare leading coefficient. Also, Q reduced translates into the condition

$$\deg b < \deg a \le \deg c. \tag{2.1}$$

If $Q = (a, b, c)$ is definite and reduced, then

$$\deg Q(x, y) = \max\{2 \deg x + \mu_1, 2 \deg y + \mu_2\} \tag{2.2}$$

for all $x, y \in A$, where μ_1 and μ_2 are the successive minima. When μ_1 and μ_2 have distinct parity, the equality (2.2) follows immediately from (2.1). When μ_1 and μ_2 have the same parity, (2.2) follows from (2.1) together with the fact that the leading coefficient of $-ac$ is a non-square by definiteness.

3. Successive minima and discriminant

LEMMA 3.1. *Let $Q = (a, b, c)$ be a definite reduced form with successive minima $\mu_1 < \mu_2$. If $f \in A$ is represented by Q and $\mu_1 \le \deg f < \mu_2$, then $f = r^2 a$ for some $r \in A$.*

Proof. Write $f = ar^2 + 2brs + cs^2$, with $r, s \in A$. If $\deg f < \mu_2$, then by (2.2) we must have $s = 0$, that is $f = r^2 a$. □

LEMMA 3.2. *Let Q and Q' be definite binary forms over A with discriminants d and d' respectively. Let $m = \max\{\deg d, \deg d'\}$. If $V_m(Q) = V_m(Q')$, then $\mu_i(Q') = \mu_i(Q)$ $(i = 1, 2)$ and $\deg d = \deg d'$. Moreover, there are reduced bases in which the diagonal entries of the matrices of Q and Q' have the same leading coefficients.*

Proof. Let $Q = (a, b, c)$ and $Q' = (a', b', c')$ be in reduced form. Let $\mu_i = \mu_i(Q)$ and $\mu'_i = \mu'_i(Q)$ $(i = 1, 2)$. Since a is represented by Q', we clearly have $\mu'_1 \leq \mu_1$. If $\mu'_2 > \mu_2$, then

$$\mu'_1 \leq \mu_1 \leq \mu_2 < \mu'_2,$$

and applying Lemma 3.1 to Q', we get $a = a'r^2$ and $c = a's^2$ for some $s, r \in A$. In particular, $\mu_1 \equiv \mu_2 \pmod 2$. Let $k = (\mu_2 - \mu_1)/2$ and consider the expression

$$Q(t^k x, y) = t^{2k} ax^2 + 2t^k bxy + cy^2$$

with $x, y \in \mathbf{F}_q$. Using the inequality (2.1), we see that the coefficient of degree μ_2 of $Q(t^k x, y)$ is

$$a_{\mu_1} x^2 + c_{\mu_2} y^2, \tag{3.1}$$

where a_{μ_1} and c_{μ_2} are the leading coefficients of a and c, respectively. Since $a_{\mu_1} c_{\mu_2} \neq 0$, the quadratic form (3.1) is nondegenerate over $\mathbf{F}_q$, and therefore represents all elements of $\mathbf{F}_q^\times$. If we choose in particular x, y so that (3.1) is not in the square class of $a'_{\mu'_1}$, then $Q(t^k x, y)$ cannot be represented by Q', since otherwise it would be of the form $r^2 a'$ by Lemma 3.1. Hence $\mu'_2 \leq \mu_2$, and by symmetry $\mu_1 = \mu'_1$ and $\mu_2 = \mu'_2$. The equality $\deg d = \deg d'$ follows immediately.

We can assume without loss of generality that $a = a'$. It remains to see that the leading coefficients of c and c' are in the same square class. When $\mu_1 \equiv \mu_2 \pmod 2$, the leading coefficients of c and of c' are both in the square class of $-\delta a_{\mu_1}$, where $\delta \in \mathbf{F}_q$ is a nonsquare. When $\mu_1 \not\equiv \mu_2 \pmod 2$, the leading coefficient of any element in $V(Q')$ whose degree has the same parity as μ_2 must be in the same square class as the leading coefficient of c'. This applies in particular to c. $\square$

Lemma 3.3. *Let Q be a primitive definite binary quadratic form over A with discriminant d and let p be an irreducible factor of d. Then Q represents a polynomial not divisible by p of degree $< \deg d$.*

Proof. Write Q in reduced form $Q = (a, b, c)$. Clearly either a or c satisfies the condition. $\square$

Lemma 3.4. *Let Q be a primitive definite binary quadratic form over A with discriminant d. Let $p \in A$. Then each element of $V(Q)$ is congruent modulo p to an element in $V_{2\deg p + \deg d - 2}(Q)$.*

Proof. Let $\{e_1, e_2\}$ be a reduced basis for Q. Each element of $V(Q)$ is congruent modulo p to an element of the form $Q(x_1 e_1 + x_2 e_2)$ with $\deg x_i \leq \deg p - 1$. Clearly $\deg Q(x_1 e_1 + x_2 e_2) \leq 2(\deg p - 1) + \mu_2(Q) \leq 2(\deg p - 1) + \deg d$. $\square$

Corollary 3.5. *Let Q and Q' be definite binary quadratic form over A with discriminants d, d', respectively. Let $m = \max\{\deg d, \deg d'\}$. If $V_{3m-2}(Q) = V_{3m-2}(Q')$, then $d' \in d\,\mathbf{F}_q^{\times 2}$.*

Proof. The statement is trivial if $m = 0$, so we shall assume through the proof that $m \geq 1$.

Notice that the equality of representation sets is preserved by scaling; hence Q and Q' may be assumed primitive.

We shall prove that for each irreducible polynomial $p \in A$:

$$V_{3m-2}(Q) \subset V_{3m-2}(Q') \quad \text{implies } v_p(d') \leq v_p(d),$$

where $v_p(\cdot)$ denotes the p-adic valuation. This will show that $d = ud'$, where $u \in \mathbf{F}_q^\times$, and Lemma 3.2 shows that u must be a square.

Let $n = v_p(d)$ and $n' = v_p(d')$. If $\deg(p) > m$, then trivially $n = n' = 0$, so we may assume $\deg p \leq m$.

Let L be the A-lattice on which Q is defined and let $M = (p^n L^\sharp) \cap L$, where $L^\sharp$ is the dual lattice with respect to Q. Then it is easy to see that the form $Q_0 = p^{-n} Q|_M$ is integral and primitive and has discriminant d. By Lemma 3.3, Q_0 represents a polynomial u relatively prime to p with $\deg u \leq m - 1$. It follows that $p^n u$ is represented by Q and since $\deg p^n u \leq 2m - 1 \leq 3m - 2$ it must also be represented by Q'. In particular, $p^n u$ must be represented p-adically by Q'. Over A_p, the form Q' is equivalent to a diagonal form $(a, 0, p^{n'} b)$ where a, b are p-adic units. Then there exist $x, y \in A_p$ such that

$$p^n u = ax^2 + p^{n'} by^2. \tag{3.2}$$

It follows from (3.2) that if $n' > n$, then $n = v_p(ax^2) \equiv 0 \pmod 2$. Consider now the lattice $N = (p^{n/2} L^\sharp) \cap L$ and let $Q_1 = p^{-n} Q|_N$. One sees immediately that Q_1 is primitive, integral and $\operatorname{disc}(Q_1) = p^{-n} d$, so Q_1 is p-unimodular and thus $V(Q_1)$ contains representatives of all classes modulo p. In particular, Q_1 represents a polynomial w that is relatively prime to p and is in a different square class modulo p as a. Furthermore, by Lemma 3.4, w can be chosen so that $\deg w \leq 2 \deg p + \deg(p^{-n} d) - 2$.

The polynomial $f = p^n w$ is obviously represented by Q and has degree $\leq 2 \deg p + \deg d - 2 \leq 3m - 2$, so it is also represented by Q'. Writing f as in (3.2) and dividing by p^n we see that w is in the same square class as a, which is a contradiction. Hence, $n' \leq n$. □

4. Forms with the same representation sets in small degree

Theorem 4.1. *Assume $q > 3$. Let Q and Q' be two binary definite positive binary quadratic forms over A with the same discriminant and the same successive minima sequence (μ_1, μ_2). Suppose that $V_{\mu_2}(Q) = V_{\mu_2}(Q')$. Then Q and Q' are equivalent.*

Proof. Let $Q=(a,b,c)$ and $Q'=(a',b',c')$ be reduced forms. There is no loss of generality in making the following assumptions: $a=a'$ is monic and c,c' have same leading coefficients. When $\mu_1 \equiv \mu_2 \pmod 2$, the leading coefficients of c and c' can be assumed to be equal to $-\delta$, for the fixed nonsquare $\delta \in \mathbf{F}_q$.

1. Suppose that $\mu_1 \not\equiv \mu_2 \pmod 2$. Since c is also represented by Q, it is represented by Q'; hence, there are $f \in A$ and $\beta\ in \mathbf{F}_q$ such that $c = af^2 + 2b'f\beta + c'\beta^2$. The different parity of the successive minima implies that $\beta = \pm 1$. By changing b' into $-b'$ if necessary, we can assume that $\beta = 1$. Let $\varphi = \left(\begin{smallmatrix}1 & f\\ 0 & 1\end{smallmatrix}\right) \in \mathbf{GL}_2(\mathbf{F}_q)$. Then $Q'' := Q \circ \varphi = (a, b'', c')$, for some $b'' \in A$. Since $\det(\varphi)=1$, it follows that $\operatorname{disc}(Q'') = \operatorname{disc}(Q) = \operatorname{disc}(Q')$; hence, $ac' - b''^2 = ac' - b'^2$. This leads to $b'' = \pm b'$.

2. Suppose that $\mu_1 \equiv \mu_2 \pmod 2$ and that $\mu_1 < \mu_2$. It follows from the equality of the discriminants that $\deg(c'-c) < \max\{\deg b, \deg b'\} < \deg a$.

If $b = b' = 0$, we conclude immediately that $c = c'$ by the equality of the discriminants. So we may assume $b \neq 0$.

Consider all the elements $au^2+2bu+c \in V(Q)$ with $u \in \mathbf{F}_q$. By assumption, the equation

$$au^2 + 2bu + c = ax^2 + 2b'xy + c'y^2 \tag{4.1}$$

is always solvable for some $x = x_k t^k + x_{k-1}t^{k-1} + \cdots + x_0 \in A$, where $k = (\mu_2 - \mu_1)/2$, and $y \in \mathbf{F}_q$.

Notice that for degree reasons, the polynomials a, b and c are linearly independent over $\mathbf{F}_q$ (recall that we are assuming $b \neq 0$), hence the left hand side of (4.1) takes exactly q values as u runs over $\mathbf{F}_q$. The equality of the leading coefficients in (4.1) gives

$$-\delta = x_k^2 - \delta y^2. \tag{4.2}$$

It is a standard fact that the number of pairs (x_k, y) satisfying (4.2) is $q+1$ (see e.g., [6, Theorem 2.59]). Notice that if (x_k, y) is a solution of (4.2), then so is $(-x_k, y)$, thus the number of possible y's appearing in a solution of (4.2) is $(q-1)/2 + 2 = (q+3)/2$.

Since $q > (q+3)/2$ by hypothesis, there must be two different values of u on the left-hand side of (4.1) with the same y on the right-hand side. In other words, there exist $u, v \in \mathbf{F}_q$, $u \neq v$, such that the system

$$\begin{cases} au^2 + 2bu + c = ax^2 + 2b'xy + c'y^2, \\ av^2 + 2bv + c = az^2 + 2b'zy + c'y^2 \end{cases} \tag{4.3}$$

has a solution (x, y, z), with $x, z \in A$ and $y \in \mathbf{F}_q$. By subtracting the two lines of (4.3), we get

$$a(u^2 - v^2) + 2b(u - v) = a(x^2 - z^2) + 2b'(x - z)y.$$

By degree considerations $x^2 - z^2 = u^2 - v^2$ and hence x and z are constant. In particular $x_k = 0$ (since $k = (\mu_2 - \mu_1)/2 > 0$) and hence, by (4.2), we have $y^2 = 1$.

Going back to (4.1), we get

$$a(u^2 - x^2) + 2(bu - b'xy) = c' - c.$$

As observed earlier, $\deg(c' - c) < \max\{\deg b, \deg b'\} < \deg a$. Thus, the above equality implies $u^2 = x^2$. Thus, $2(bu - b'xy) = 2u(b \pm b') = c' - c$. Replacing b' by $-b'$ if necessary, we can assume $2u(b + b') = c' - c$. Multiplying by $b - b'$ gives $2ua(c - c') = 2u(b^2 - b'^2) = (c' - c)(b - b')$ by the equality of the discriminants. Degree considerations again imply $c = c'$ and $b = \pm b'$.

3. Suppose that $\mu_1 = \mu_2 = n$. Write

$$\begin{aligned} a &= t^n + a_{n-1}t^{n-1} + \cdots + a_0, \\ c &= -\delta t^n + c_{n-1}t^{n-1} + \cdots + c_0, \\ c' &= -\delta t^n + c'_{n-1}t^{n-1} + \cdots + c'_0, \\ b &= b_k t^k + \cdots + b_0, \\ b' &= b'_k t^k + \cdots + b'_0, \end{aligned}$$

where $k = \max\{\deg b, \deg b'\}$. If $b = b' = 0$, we are done, so we may assume $k \geq 0$ and $b'_k \neq 0$. Note that since $\mathrm{disc}(Q) = \mathrm{disc}(Q')$, we have $\deg(c - c') < k$ as in the previous case.

Since $V_n(Q) = V_n(Q')$, for any pair $(u, v) \in \mathbf{F}_q^2$, there exists a pair $(x, y) \in \mathbf{F}_q^2$ such that

$$Q(u, v) = Q'(x, y). \tag{4.4}$$

Taking the coefficients of t^n and t^k in the above polynomials, we get the system of quadrics:

$$\begin{cases} u^2 - \delta v^2 = x^2 - \delta y^2, \\ a_k u^2 + 2b_k uv + c_k v^2 = a_k x^2 + 2b'_k xy + c_k y^2, \end{cases} \tag{4.5}$$

which defines an algebraic curve E in $\mathbf{P}^3$. For every $(u, v) \in \mathbf{F}_q^2 \setminus \{0\}$, there is $(x, y) \in \mathbf{F}_q^2 \setminus \{0\}$ satisfying (4.5). Notice also that if a quadruplet (u, v, x, y) satisfies (4.5), so does $(u, v, -x, -y)$ and that the two sides of the first equation are forms anisotropic over $\mathbf{F}_q$, so $|E(\mathbf{F}_q)| \geq 2(q + 1)$.

If the curve E given by (4.5) were smooth, then it would be an elliptic curve and by the Hasse estimate [12, Chapter V] we would have $|E(\mathbf{F}_q)| \leq 2\sqrt{q} + q + 1$, which would contradict the above count. Thus, E cannot be a smooth curve.

It is also known that the intersection of two quadric hypersurfaces, say $Q_1 = 0, Q_2 = 0$, in $\mathbf{P}^m$ is a smooth variety of codimension 2 if and only if the binary form $\det(XQ_1 + YQ_2)$ of degree $m + 1$ has no multiple factor (see e.g., [4, Remark 1.13.1] or [7, Chapter XIII, Section 11]). In the case of our system

(4.5), by computing explicitly the discriminant of $\det(XQ_1 + YQ_2)$, where Q_1, Q_2 are the two quaternary quadratic forms of (4.5), we get the condition

$$\delta^4(b_k - b'_k)^4(b_k + b'_k)^4\big((a_k\delta + c_k)^2 - 4\delta b_k'^2\big)\big((a_k\delta + c_k)^2 - 4\delta b_k^2\big) = 0. \tag{4.6}$$

Since δ is not a square in $\mathbf{F}_q$ and $b'_k \neq 0$ by assumption, we must have either $b_k = \pm b'_k$ or $b_k = 0$ and $a_k\delta + c_k = 0$. We shall rule out the second possibility.

Since $V_n(Q) = V_n(Q')$, these sets span the same $\mathbf{F}_q$ -subspace of A; in particular b' must be an $\mathbf{F}_q$ -linear combination of a, b and c. Write

$$b' = \alpha a + \beta b + \gamma c,$$

with $\alpha, \beta, \gamma \in \mathbf{F}_q$. Taking terms of degree n gives

$$0 = \alpha - \delta\gamma,$$

which implies

$$b' = \gamma(\delta a + c) + \beta b.$$

Taking now terms of degree k we get

$$b'_k = \gamma(\delta a_k + c_k) + \beta b_k.$$

If $b_k = 0$ and $a_k\delta + c_k = 0$, then $b'_k = 0$, which is a contradiction with our assumption.

Thus, $b_k = \pm b'_k$ is the only possibility. Replacing b by $-b$ if needed, we shall assume $b_k = b'_k$.

We shall now show that $b = b'$. Suppose by contradiction that $b \neq b'$ and let $m = \deg(b - b') < k$. Then, by the equality of the discriminants, $\deg(b^2 - b'^2) = m + k = n + \deg(c - c')$, which implies $\deg(c - c') < m$ and in particular $c_m = c'_m$.

Exactly the same argument that showed $b_k^2 = b'^2_k$ (just replace k by m in (4.5)) shows that $b_m^2 = b_m'^2$. Now consider the system

$$\begin{cases} a_m u^2 + 2b_m uv + c_m v^2 = a_m x^2 + 2b'_m xy + c_m y^2, \\ a_k u^2 + 2b_k uv + c_k v^2 = a_k x^2 + 2b'_k xy + c_k y^2. \end{cases} \tag{4.7}$$

Adding the two equations and combining the result with the first equation in (4.5) we get the system

$$\begin{cases} u^2 - \delta v^2 = x^2 - \delta y^2, \\ (a_k + a_m)u^2 + 2(b_k + b_m)uv + (c_k + c_m)v^2 \\ \qquad = (a_k + a_m)x^2 + 2(b'_k + b'_m)xy + (c_k + c_m)y^2, \end{cases} \tag{4.8}$$

Applying one more time the rational-point counting argument, this time to the above system, we conclude that $(b_m - b_k)^2 = (b'_m - b'_k)^2$, which yields $b_m b_k = b'_m b'_k$. Since $b_k = b'_k \neq 0$, we conclude $b_m = b'_m$, which contradicts the hypothesis that $m = \deg(b - b')$. Hence, $b = b'$ as claimed. □

Finally, putting together Proposition 3.5, Lemma 3.2, and Theorem 4.1, we get our main result.

THEOREM 4.2. *Assume $q > 3$. Let Q and Q' be definite binary quadratic forms over A with discriminants d and d' respectively. Let $m = \max\{\deg d, \deg d'\}$. If $V_{3m-2}(Q) = V_{3m-2}(Q')$, then Q and Q' are equivalent.*

5. The Ternary case

In this section, we give an example showing that in the case of ternary definite forms over A, the representation *sets* in general do not determine the discriminant, much less the equivalence class of the form.[1]

LEMMA 5.1. *Let $Q_a = X^2 + tY^2 - \delta(t + a^2)Z^2$, where $a \in \mathbf{F}_q^\times$. Then a polynomial $f \in A$ is represented by Q_a over A if and only if it is represented by Q_a over $A_{(t)} = \mathbf{F}_q[[t]]$.*

Proof. By [3, Theorem 3.5], the form Q_a has class number one, so a polynomial $f \in A$ is represented by Q_a over A if and only if it is represented locally everywhere. At primes $\mathfrak{p}$ not dividing $\mathrm{disc}(Q_a) = \delta t(t + a^2)$, Q_a is unimodular and isotropic, hence represents everything. At $\mathfrak{p} = (t + a^2)$, since $t \equiv -a^2 \pmod{\mathfrak{p}}$, Q_a is equivalent to $X^2 - Y^2 - \delta(t + a^2)Z^2$ which also represents everything since $X^2 - Y^2$ already does so. Thus, the only condition is at the prime $\mathfrak{p} = (t)$ (the condition at ∞ is automatic by reciprocity). □

COROLLARY 5.2. *For each $a \in \mathbf{F}_q^\times$, let Q_a be as in Lemma 5.1. The representation set $V(Q_a)$ does not depend upon the choice of a.*

Proof. By virtue of Lemma 5.1, it is enough to notice that Q_a is equivalent to $X^2 + tY^2 - \delta Z^2$ over $\mathbf{F}_q[[t]]$, which is independent of a. □

COROLLARY 5.3. *Assume $q \geq 5$ and choose $a, b \in \mathbf{F}_q^\times$ such that $a^2 \neq b^2$. Then $V(Q_a) = V(Q_b)$ but $\mathrm{disc}(Q_a) \neq \mathrm{disc}(Q_b)$.*

Proof. Clear by Corollary 5.2. □

6. Primitive binary forms of class number one

In this section, we characterize primitive binary quadratic forms over $A = \mathbf{F}_q[t]$ of class number one. Although it should be possible, in principle, to deduce the results below from general formulas such as the ones in [9], we prefer to give here a direct argument.

We begin by a statement on orders in quadratic extensions of $K = \mathbf{F}_q(t)$.

COROLLARY 6.1. *Let $D = f^2 D_0 \in A$, where D_0 is a square-free polynomial of either odd degree or of even degree and nonsquare leading coefficient, and $f \in A$ is a monic polynomial. Let $B = A[\sqrt{D}]$. Assume that $\mathrm{Pic}(B)$ is an Abelian 2-group and has at most one cyclic component of order 4 and all other components of order 2. Then*

[1] However, the representation *numbers* do determine the equivalence class of such forms as showed in [1], [2].

(1) *If* $\deg D_0 > 0$ *and* $q > 13$, *then* D *is square-free* (*i.e.*, $f = 1$) *and* $\deg D \leq 2$.

(2) *If* $\deg D_0 = 0$ *and* $q > 5$, *then* $\deg D \leq 2$.

Proof. Let $\mathfrak{O} = A[\sqrt{D_0}]$. Notice that $\mathfrak{O}$ is the maximal A-order in the field $E = K(\sqrt{D_0})$ and that f is the conductor of B in $\mathfrak{O}$.

There is an exact sequence

$$1 \longrightarrow \frac{\mathfrak{O}^\times}{B^\times} \longrightarrow \frac{(\mathfrak{O}/f\mathfrak{O})^\times}{(A/fA)^\times} \longrightarrow \mathrm{Pic}(B) \longrightarrow \mathrm{Pic}(\mathfrak{O}) \longrightarrow 1. \tag{6.1}$$

1. Assume $\deg D_0 > 0$. Then $\mathfrak{O}^\times = B^\times = \mathbf{F}_q^\times$ and we get a shorter exact sequence

$$1 \longrightarrow \frac{(\mathfrak{O}/f\mathfrak{O})^\times}{(A/fA)^\times} \longrightarrow \mathrm{Pic}(B) \longrightarrow \mathrm{Pic}(\mathfrak{O}) \longrightarrow 1. \tag{6.2}$$

Let h be the radical of f (i.e., the product of all irreducible monic divisors of f). The subgroup $(1 + h\mathfrak{O}/f\mathfrak{O})/(1 + hA/fA)$ of $(\mathfrak{O}/f\mathfrak{O})^\times/(A/fA)^\times$ has order $q^{\deg f - \deg h}$ and is a 2-group by the exact sequence (6.2), so we must have $f = h$, i.e., f is square-free.

Let π be an irreducible factor of f of degree d. Then $(\mathfrak{O}/\pi\mathfrak{O})^\times/(A/\pi A)^\times$ is a direct factor of $(\mathfrak{O}/f\mathfrak{O})^\times/(A/fA)^\times$ and is cyclic of order $q^d - 1$ or $q^d + 1$ (according to whether π is split or inert in E) or is isomorphic to the additive group $\mathbf{F}_{q^d}$ when π is ramified. Clearly, the latter case is impossible since q is odd and in the first two cases we must have $q^d \pm 1 = 2$ or 4, which is also impossible when $q > 5$. Hence, $f = 1$, D is square-free and $B = \mathfrak{O}$.

Let r be the number of irreducible factors of D. It is well known that the 2-rank of $\mathrm{Pic}(\mathfrak{O})$ is $r - 1$. Hence, under our present hypotheses, $|\mathrm{Pic}(\mathfrak{O})| \leq 2^r$. The order of $\mathrm{Pic}(\mathfrak{O})$ is essentially the class number h_E of E; more precisely $|\mathrm{Pic}(\mathfrak{O})| = h_E$ if $\deg D$ is odd and $|\mathrm{Pic}(\mathfrak{O})| = 2h_E$ if $\deg D$ is even [10, Proposition 14.7].

Using the lower bound for h_E given by the Riemann Hypothesis [10, Proposition 5.11], we get

$$\left(\sqrt{q} - 1\right)^{\deg D - 1} \leq 2^r \quad \text{if deg is odd;}$$
$$\left(\sqrt{q} - 1\right)^{\deg D - 2} \leq 2^{r-1} \quad \text{if } \deg D \text{ is even.}$$

When $\deg D \geq 3$, using the above inequalities and the obvious fact that $r \leq \deg D$, we get easily the inequality $\log_2(\sqrt{q} - 1) \leq 3/2$, which is impossible if $q > 13$.

2. Assume $\deg D_0 = 0$ and $\deg f > 0$. Then $\mathfrak{O} = \mathbf{F}_{q^2}[t]$, so $\mathrm{Pic}(\mathfrak{O}) = \{1\}$, $\mathfrak{O}^\times = \mathbf{F}_{q^2}^\times$ and $B^\times = \mathbf{F}_q^\times$. The exact sequence (6.1) becomes

$$1 \longrightarrow \frac{\mathbf{F}_{q^2}^\times}{\mathbf{F}_q^\times} \longrightarrow \frac{(\mathfrak{O}/f\mathfrak{O})^\times}{(A/fA)^\times} \longrightarrow \mathrm{Pic}(B) \longrightarrow 1. \tag{6.3}$$

Let p be the characteristic of $\mathbf{F}_q$. Taking p-parts in the sequence above (i.e., tensoring by $\mathbf{Z}_p$), we get $[(\mathfrak{O}/f\mathfrak{O})^\times/((A/fA)^\times)]_p = 0$. Exactly the same argument as in Case 1 shows that f must be square-free. Hence,

$$\frac{(\mathfrak{O}/f\mathfrak{O})^\times}{(A/fA)^\times} = \prod_{\pi|f} \frac{(\mathfrak{O}/\pi\mathfrak{O})^\times}{(A/\pi A)^\times}, \tag{6.4}$$

where π runs over all irreducible monic divisors of f.

Notice that the factors on the right-hand side of (6.4) are cyclic of order $q^{\deg \pi}+1$ if $\deg \pi$ is odd, and $q^{\deg\pi} - 1$ if $\deg \pi$ is even.

Let π be an irreducible factor of f of even degree, say $\deg \pi = 2m$, then by the exact sequence (6.3), $(q^{2m}-1)/(q+1)$ must be a 2-power ≤ 4. This is possible only when $m = 1$ and $q = 3$ or $q = 5$. Similarly, if $\deg\pi$ is odd, say $\deg \pi = 2m+1$, then $(q^{2m+1}+1)/(q+1)$ must be a 2-power, but it is always an odd number, so the only possibility is $m = 0$, i.e. $\deg \pi = 1$. Thus, when $q > 5$, f is a product of linear factors.

If $q+1$ is divisible by an odd prime ℓ, then, since $\mathrm{Pic}(B)$ is a 2-group, taking ℓ-parts in (6.3) shows that there must be only one factor in the decomposition (6.4), i.e., f is irreducible (necessarily linear as shown above).

The only case left is when $q+1$ is a 2-power. Notice that the factors on the right-hand side of (6.4) are all cyclic of order $q+1$, since all the π's are linear. By the hypothesis on $\mathrm{Pic}(B)$, if there is more than one factor in (6.4), then $q+1$ is a 2-power ≤ 4. This is impossible if $q > 3$. Thus, also in this case, f has only one irreducible, necessarily linear, factor. ☐

Let (V, Q) be a quadratic space over the field $K = \mathbf{F}_q(t)$. Let $L \subset V$ be an A-lattice and let $\mathrm{Gen}(L)$ be the set of lattices of V in the genus of L. The orthogonal group $\mathbf{O}(V,Q)$ acts on $\mathrm{Gen}(L,Q)$ and the number of orbits (which is well known to be finite) is called the *class number* of L and will be denoted by $h(L,Q)$, or simply $h(Q)$ when the underlying lattice is obvious. The number of orbits of the action of the subgroup $\mathbf{SO}(V,Q)$ on $\mathrm{Gen}(L,Q)$ will be denoted by $h^+(L,Q)$. Since $\mathbf{SO}(V,Q)$ has index 2 in $\mathbf{O}(V,Q)$, we have $h^+(L,Q) \leq 2h(L,Q)$.

If (L,Q) is primitive of rank 2, then $h^+(L,Q)$ depends only on $D = \mathrm{disc}(L, Q)$. Indeed, let G_D be the set of classes of primitive binary quadratic forms of discriminant D up to orientation-preserving (i.e., of determinant 1) linear transformation. This set is a group for Gaussian composition [8] and there is a natural exact sequence relating G_D and $\mathrm{Pic}(B)$, where $B = A[\sqrt{D}]$, (see [8, Section 6]), which in our situation is

$$1 \longrightarrow \mathbf{F}_q^\times/\mathbf{F}_q^{\times 2} \longrightarrow G_D \longrightarrow \mathrm{Pic}(B) \to 1. \tag{6.5}$$

The *principal genus* consists of forms in the genus of the norm form $X^2 - DY^2$ of B, and their classes in G_D form a subgroup G_D^0. The different genera are cosets for this subgroup and hence they have all the same number of classes, i.e. $h^+(L,Q) = |G_D^0|$ for all primitive quadratic lattices (L,Q) of discriminant D. It is also well known (and easy to see) that G_D/G_D^0 is 2-elementary.

THEOREM 6.2. *Let Q be a definite primitive binary quadratic form over A, where $q > 13$. If $h(Q) = 1$, then* $\deg \operatorname{disc}(Q) \leq 2$.

Proof. If $h(Q) = 1$, then $h^+(Q) = |G_D^0| \leq 2$ and by the remarks above G_D is an Abelian 2-group with at most one cyclic component of order 4 and all others of order 2. So is $\operatorname{Pic}(B)$ by the exact sequence (6.5), and we conclude by Proposition 6.1. ☐

REMARK. Theorem 6.2 is incorrect without the assumption $q > 13$. Here is a counterexample for $q = 13$.

Let $D = t(t^2 - 1)$ and let E be the elliptic curve over $\mathbf{F}_{13}$ given by the equation $s^2 = D$. Let $B = A[\sqrt{D}]$. Then $\operatorname{Pic}(B) = E(\mathbf{F}_{13}) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}$. It is easy to see that the exact sequence (6.5) is split in this case, so $G_D^0 = 2G_D = 2E(\mathbf{F}_{13}) \cong \mathbf{Z}/2\mathbf{Z}$. Let Q_0 be a form whose class $[Q_0]$ generates G_D^0. Then the genus of any form Q of discriminant D consists of the classes $[Q]$ and $[Q'] = [Q] + [Q_0]$ in G_D. If $[Q]$ has order 4, then $[Q'] = -[Q]$ i.e., Q and Q' are (improperly) equivalent, and thus $h(Q) = 1$. An explicit example is $Q = (t - 5, 4, -(t^2 + 5t + 11))$, which corresponds to the point $P = (5, 4)$ of order 4 in $E(\mathbf{F}_{13})$.

Theorem 6.2 gives a converse of a result of Chan–Daniels [3]. We summarize this in the following statement.

COROLLARY 6.3. *Assume $q > 13$. A binary definite quadratic form Q over A of discriminant D has class number one if and only if it satisfies one of the following conditions*:

(1) $\deg D \leq 1$.
(2) $\deg D = 2$ *and* $\mu_1(Q) = 1$.
(3) $\deg D = 2$, $\mu_1(Q) = 0$ *and D is reducible.*

Proof. The "if" part follows from [3, Lemma 3.7] and the ensuing remark. The "only if" part is a consequence of Theorem 6.2. ☐

REFERENCES

[1] J. Bureau, *Representation properties of definite lattices in function fields*, Ph.D. thesis, Louisiana State University, 2006.

[2] J. Bureau and J. Morales, *Isospectral definite ternary* $\mathbf{F}[t]$*-lattices*, to be published in J. Number Theory, 2009.

[3] W. K. Chan and J. Daniels, *Definite regular quadratic forms over* $\mathbb{F}_q[T]$, Proc. Amer. Math. Soc. **133** (2005), 3121–3131 (electronic). MR 2160173

[4] J.-L. Colliot-Thélène, J.-J. Sansuc and P. Swinnerton-Dyer, *Intersections of two quadrics and Châtelet surfaces. I*, J. Reine Angew. Math. **373** (1987), 37–107. MR 0870307

[5] L. J. Gerstein, *Definite quadratic forms over* $\mathbb{F}_q[x]$, J. Algebra **268** (2003), 252–263. MR 2005286

[6] ———, *Basic quadratic forms*, Graduate Studies in Mathematics, vol. 90, Amer. Math. Soc., Providence, RI, 2008. MR 2396246

[7] W. V. D. Hodge and D. Pedoe, *Methods of algebraic geometry. Vol. II. Book III: General theory of algebraic varieties in projective space. Book IV: Quadrics and Grassmann varieties*, Cambridge, Univ. Press, 1952. MR 0048065

[8] M. Kneser, *Composition of binary quadratic forms*, J. Number Theory **15** (1982), 406–413. MR 0680541

[9] U. Korte, *Class numbers of definite binary quadratic lattices over algebraic function fields*, J. Number Theory **19** (1984), 33–39. MR 0751162

[10] M. Rosen, *Number theory in function fields*, Graduate Texts in Mathematics, vol. 210, Springer, New York, 2002. MR 1876657

[11] M. Schering, *Théorèmes relatifs aux formes quadratiques qui représentent les mêmes nombres*, J. Math. Pures Appl. **2** (1859), 253–269.

[12] J. H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 106, Springer, New York, 1986. MR 0817210

[13] G. L. Watson, *Determination of a binary quadratic form by its values at integer points*, Mathematika **26** (1979), 72–75. MR 0557128

[14] ———, *Acknowledgement: "Determination of a binary quadratic form by its values at integer points" [Mathematika* **26** *(1979), 72–75; MR 81e:10019]*, Mathematika **27** (1980), 188 (1981). MR 0557128

Jean Bureau, Mathematics Department, Louisiana State University, Baton Rouge, LA 70803-4918, USA

E-mail address: jbureau@math.lsu.edu

Jorge Morales, Mathematics Department, Louisiana State University, Baton Rouge, LA 70803-4918, USA

E-mail address: morales@math.lsu.edu