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ON HUMBERT-MINKOWSKI'S CONSTANT FOR A NUMBER FIELD

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ABSTRACT. We use Humbert's reduction theory to introduce an obstruction for the unimodularity of minimal vectors of positive definite quadratic forms over totally real number fields. Using this obstruction we obtain an inequality relating the values of a generalized Hermite constant for such fields, which over the field of rational numbers leads to a well-known result of Mordell.

1.

Let $K/\mathbf{Q}$ be a totally real number field of degree m with ring of integers $\mathcal{O}_K$ and real embeddings $\sigma_1, \cdots, \sigma_m : K \to \mathbb{R}$. For $n \geq 1$ let $P_{n,K} \subset \mathbb{R}^{\frac{1}{2}mn(n+1)}$ be the space of all m-tuples $S = (S_1, \cdots, S_m)$, where S_i is an $n \times n$ symmetric positive definite real matrix. We call such tuples Humbert forms. The unimodular group $GL(n, \mathcal{O}_K)$ acts on $P_{n,K}$ as follows. For $S = (S_1, \cdots, S_m) \in P_{n,K}$ and $U \in GL(n, \mathcal{O}_K)$,

$$S[U] := (S_1[U^{(1)}], \cdots, S_m[U^{(m)}]) \tag{1}$$

where $U^{(i)}$ denotes the ith conjugate $\sigma_i(U)$ of U, and $A[B] = B^t AB$, whenever this product of matrices is defined. Two Humbert-forms $S, S' \in P_{n,K}$ are called equivalent $(S \cong S')$ if $S' = S[U]$ for some $U \in GL(n, \mathcal{O}_K)$. In [H] a fundamental domain for this action, say $R_K \subset P_{n,K}$, is constructed. We will call the elements of R_K Humbert reduced forms. For $S \in P_{n,K}$ we define its determinant and minimum as

$$detS = \prod_{i=1}^{m} detS_i \quad \text{and} \quad m(S) = \text{Min}\{\prod_{i=1}^{m} S_i[u^{(i)}] \mid 0 \neq u \in \mathcal{O}_K^n\}.$$

Of course if $S \cong S'$, then $detS = detS'$ and $m(S) = m(S')$. In particular for any $S \in P_{n,K}$ the number

$$\gamma_K(S) = \frac{m(S)}{(detS)^{1/n}}$$

depends only on the class $[S]$ of S.

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Let us associate to $K/\mathbf{Q}$ the following number, which is the analogue of Hermite-Minkowski's constant:

$$\gamma_{K,n} = \sup_{S \in P_{n,K}} \gamma_K(S).$$

If $K = \mathbf{Q}$, we recover indeed Hermite's constant $\gamma_n = \gamma_{Q,n}$ (see [Si], [C-S]). In [I] it is shown that

$$\gamma_{K,n} \leq 4^m w_n^{-\frac{2m}{n}} |d_K|,$$

where d_K is the discriminant of $K/\mathbf{Q}$, and w_n is the volume of the unit sphere in $\mathbb{R}^n$. In particular for $m = 1$ one obtains the classical Minkowski bound. One also has a lower bound for $\gamma_{K,n}$ in terms of some values of the Dedekind zeta function of the field K (see [B]). It is desirable to find better bounds for $\gamma_{K,n}$ and even for low dimensions to get exact values.

The results due to Mordell relating γ_n and γ_{n-1}, namely $\gamma_n^{n-2} \leq \gamma_{n-1}^{n-1}$, allow for $n = 4$ and $n = 8$ to get the exact value of γ_n knowing γ_{n-1} ([Si], [C-S]). For this reason we are interested in extending Mordell's theorem for $\gamma_{K,n}$ and this is the main purpose of this paper. A vector $u \in \mathcal{O}_K^n$ is called a minimal vector of $S \in P_{n,K}$ if $m(S) = S[u] := \prod_{i=1}^m S_i[u^{(i)}]$. For $K = \mathbf{Q}$ it is clear that a minimal vector $u \in \mathbb{Z}^n$ of a quadratic form must be unimodular, i.e. $\sum_{i=1}^n \mathbb{Z} u_i = \mathbb{Z}$. The same argument works also if $\mathcal{O}_K$ is a principal ideal domain, i.e. the class number of K is 1. But in general this is not true, and in fact there is an example over $Q(\sqrt{10})$ found by A. M. Berge of a Humbert form without unimodular minimal vectors (see section 2). Recall that $u = (u_1, \cdots, u_n) \in \mathcal{O}_K^n$ is unimodular if $\sum_{i=1}^n \mathcal{O}_K \cdot u_i = \mathcal{O}_K$. The proof of Mordell's theorem relies heavily on the unimodularity property of minimal vectors over $\mathbb{Z}$. We introduce in section 2 of this paper a constant associated to K which measures the obstruction, for all Humbert forms over K, to having unimodular minimal vectors. Then we prove in section 3 a general version of Mordell's theorem in terms of this constant.

2.

Let $S = (S_1, \cdots, S_m) \subset P_{n,K}$ be a Humbert form. Let $M(S)$ be the set of minimal vectors of S, i.e. $M(S) = \{u \in \mathcal{O}_K^n \mid S[u] = \gamma_K(S)\}$. Since $S[\varepsilon u] = S[u]$ for any unit $\varepsilon \in \mathcal{O}_K^*$, this set is in general infinite, and it is finite only in the case where $\mathcal{O}_K^*$ is finite. Nevertheless the set of classes $[u] = \{\varepsilon u \mid \varepsilon \in \mathcal{O}_K^*\}$, $u \in M(S)$, is a finite set ([I]).

If K has class number one, it is easy to check that any minimal vector $u \in M(S)$ is unimodular. But if $h(K) > 1$, this is no longer true as the following example of A. M. Berge shows.

Example. Let $K = Q(\sqrt{10})$. Then $h(K) = 2$ and the class group of K is $Cl(K) = \{\bar{O}_K, \overline{\langle 2, \sqrt{10} \rangle}\}$. Let $S = (S_1, S_2)$ be the Humbert form defined by $S_1 = \begin{pmatrix} 1 & -\frac{2}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} & c \end{pmatrix}$, $S_2 = \begin{pmatrix} 1 & \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} & c \end{pmatrix}$, where c is a real number with $0.4 = \left(\frac{2}{\sqrt{10}}\right)^2 < c$. Set $d = c - 0.4$. Then for any vector $u = (u_1, u_2) \in \mathcal{O}_K^2$ it holds that

$$S[u] \geq N_{K/Q}(u_1 - \frac{2}{\sqrt{10}} u_2)^2 + d^2 N_{K/Q}(u_2)^2$$

with equality at least when $u_1 = \frac{2}{\sqrt{10}} u_2$.

One can easily check that $N_{K/Q}(u_1 - \frac{2}{\sqrt{10}}u_2)^2 \geq \left(\frac{4}{10}\right)^2$, for $u_1, u_2 \in \mathcal{O}_K$ with $u_1 \neq \frac{2}{\sqrt{10}}u_2$, so in this case it holds that $S[u] \geq \left(\frac{4}{10}\right)^2$. If $u_1 = \frac{2}{\sqrt{10}}u_2$ $(\neq 0)$, then $S[u] \geq d^2 N_{K/Q}(u_2) \geq (10d)^2$, and the equality holds only for the vectors $u = \varepsilon(2, \sqrt{10})$, $\varepsilon \in \mathcal{O}_K^*$. Therefore choosing $0.4 < c < 0.44$ we obtain that

$$m(S) = (10 - 4c)^2$$

is attained only on the class $[(2, \sqrt{10})]$.

Now it is clear that $(2, \sqrt{10})$ is not unimodular and $M(S) = \{\varepsilon \cdot (2, \sqrt{10}) | \varepsilon \in \mathcal{O}_K^*\}$.

Remark. Let S be any Humbert form over the number field K. For any $u = (u_1, \cdots, u_n) \in M(S)$ let $\langle u \rangle = \langle u_1, \cdots, u_n \rangle$ be the ideal in $\mathcal{O}_K$ generated by $u_1, \cdots, u_n$. Then $\langle u \rangle$ is an ideal of minimal norm in its ideal class. In fact, if A is in the class of $\langle u \rangle$, then it holds that $\alpha A = \beta \langle u \rangle$ for some $\alpha, \beta \in \mathcal{O}_K - \{0\}$. Hence $A = \langle \frac{\beta}{\alpha}u_1, \cdots, \frac{\beta}{\alpha}u_n \rangle$ and in particular $(\frac{\beta}{\alpha}u_1, \cdots, \frac{\beta}{\alpha}u_n) \in \mathcal{O}_K^n$. Hence $S\left[\frac{\beta}{\alpha} \cdot u\right] = N_{K/Q}\left(\frac{\beta}{\alpha}\right) S[u] \geq S[u]$, since $u \in M(S)$, and we obtain $N\left(\frac{\beta}{\alpha}\right) \geq 1$. Therefore $N(A) = N\left(\frac{\beta}{\alpha}\right) N(\langle u \rangle) \geq N(\langle u \rangle)$. In particular we conclude that for any S the set of ideals $\{\langle u \rangle \,|\, u \in M(S)\}$ belongs to a fixed finite set of ideals of $\mathcal{O}_K$.

For any $u = (u_1, \cdots, u_n) \in M(S)$ define its norm as

$$N(u) = |N_{K/\mathbf{Q}}(\prod\nolimits_{u_i \neq 0} u_i)|,$$

where $N_{K/\mathbf{Q}} : K \to \mathbf{Q}$ is the usual norm map. Then $N(u)$ is a positive integer. We define

$$N(S) = \mathrm{Inf}_{u \in M(S)} N(u)$$

and for any class $[S]$

$$N[S] = \mathrm{Inf}_{T \in [S]} N(T).$$

Since any unimodular vector in $\mathcal{O}_K^n$ can be transformed through an element of $GL(n, \mathcal{O}_K)$ into the vector $e_1 = (1, 0, \cdots, 0)$, we see that $N[S] = 1$ if and only if S has a unimodular minimal vector. In particular since this is true for any S if the class number of K is 1, we obtain $N[S] = 1$ for all $S \in P_{n,K}$ when $h(K) = 1$.

Proposition. *There exists a constant $M_{K,n}$ depending only on K and n, such that*

$$N[S] \leq M_{K,n}$$

for all $S \in P_{n,K}$.

Proof. Take $S \in P_{n,K}$ and replace S by an equivalent form which is Humbert reduced. We may assume $S \in R_K \subset P_{n,K}$. Then it holds (see [H]) that

$$S_i = (D_i U_i)^t D_i U_i, \qquad 1 \leq i \leq m,$$

where U_i is a unipotent upper triangular matrix with all entries bounded by a constant depending only on K and n, and $D_i = diag(d_{i1}, \cdots, d_{in})$ is a diagonal matrix such that $d_{ij}/d_{i,j+1}$ is bounded for all i,j by a constant depending on K and n. We set $A_i = D_i U_i$ so that $S_i = A_i^t A_i$. Hence for any vector $x \in \mathbb{R}^n$,

$$S_i[x] = \|A_i x\|^2$$

where $\| \; \|$ denotes the euclidean norm in $\mathbb{R}^n$. Let $u = (u_1, \cdots, u_n) \in M(S)$ be a minimal vector of S. Then $S_i[u^{(i)}] = \|A_i u^{(i)}\|^2$ for all $1 \leq i \leq m$. Now for any linear automorphism A of $\mathbb{R}^n$ one has $\|Ax\|^2 \geq \ell(A)^2 \|x\|^2$, where $\ell(A) = \|A^{-1}\|^{-1}$ and $\|B\|$ denotes the usual operator norm of the automorphism B. Therefore

$$\begin{aligned} S_i[u^{(i)}] &\geq \ell(A_i)^2 \|u^{(i)}\|^2, \\ S_i[u^{(i)}] &\geq r\, \ell(A_i)^2 (\textstyle\prod_{u_j \neq 0} u_j^{(i)})^{\frac{2}{r}}, \end{aligned}$$

by the arithmetic-geometric inequality where $r \geq 1$ is the number of $u_i \neq 0$. Taking the product over all $1 \leq i \leq m$ of these inequalities, we obtain with $S[u] = \prod_{i=1}^m S_i[u^{(i)}]$,

$$S[u] \geq r^m \left(\prod_{i=1}^m \ell(A_i)\right)^2 \; N(u)^{\frac{2}{r}}.$$

Since $N(u) \geq 1$ and $n \geq r \geq 1$, we obtain

$$S[u] \geq \left(\prod_{i=1}^m \ell(A_i)\right)^2 \; N(u)^{\frac{2}{n}}$$

and hence

$$(*) \qquad N(u)^2 \leq \frac{(S[u])^n}{(\prod_{i=1}^m \ell(A_i))^{2n}}.$$

Let us estimate $\ell(A_i)$ from below. From the inequality $\|AB\| \leq \|A\| \, \|B\|$ and $\ell(A) = \|A^{-1}\|^{-1}$, we obtain $\ell(AB) \geq \ell(A)\ell(B)$ for any two linear automorphisms. Hence $\ell(A_i) \geq \ell(D_i)\ell(U_i)$ for all $1 \leq i \leq m$. Since D_i is diagonal it follows that $\ell(D_i) = \mathrm{Inf}\{d_{ij}, 1 \leq j \leq n\}$, say $\ell(D_i) = d_{i\ell}$. But $d_{i1}/d_{i\ell} \leq B =$ constant depending on K and n, so that we get $\ell(D_i) \geq B^{-1} d_{i1}$, $1 \leq i \leq m$. Since the matrices U_i are unipotent upper triangular with bounded coefficients for all i, there is a constant C depending only on K and n such that $\ell(U_i) \geq C$ for all i (and all S) (this follows for example from the fact that the function $\ell(U)$ is continuous and hence it attains its minimum on any compact set). Therefore we have

$$\ell(A_i) \geq d_{i1} C B^{-1}, \qquad 1 \leq i \leq m.$$

Inserting this result in inequality $(*)$ we obtain

$$N(u)^2 \leq \frac{m(S)^n B^{2mn}}{C^{2mn}} \cdot \frac{1}{(\prod_{i=1}^m d_{i1})^{2n}}.$$

But $d_{i1}^2 = S_i[e_1]$ where $e_1 = (1, 0, \cdots, 0)$, so that $(\prod_{i=1}^m d_{i1})^2 = S[e_1] \geq m(S)$. From the above inequality, we obtain

$$N(u)^2 \leq \frac{B^{2mn}}{C^{2mn}}.$$

With $M_{K,n} = B^{2mn}/C^{2mn}$ we get $N[S] \leq M_{K,n}$. This proves the proposition. □

According to this result we may now introduce the following number which depends only on K and n, namely

$$M_{n,K} = \sup_{S \in P_{n,K}} N[S].$$

Thus if S is reduced, we have $N(u) \leq M_{n,K}$ for any $u \in M(S)$. As we observed before, if K has class number one, $M_{n,K} = 1$. The constant $M_{n,K}$ measures therefore the obstruction that any form $S \in P_{n,K}$ has a unimodular minimal vector, because $M_{n,K} = 1$ if and only if any Humbert form S has at least one unimodular minimal vector. It would be interesting to find explicit bounds for $M_{n,K}$.

Example. Let $K = Q(\sqrt{10})$ be as in the previous example. Since $\langle 2, \sqrt{10} \rangle$ has norm 2, it is the ideal of smallest norm in its class. Since $Cl(K) = \{\bar{\mathcal{O}}_K, \overline{\langle 2, \sqrt{10} \rangle}\}$, we obtain from the above remark that for any minimal vector u of some Humbert form S it holds that $\langle u \rangle \in \{\mathcal{O}_K, \langle 2, \sqrt{10} \rangle\}$, and the example of A.M. Berge shows that $\langle 2, \sqrt{10} \rangle$ can be realized. In $\langle 2, \sqrt{10} \rangle$ the elements of lowest absolute norm are 2 and $2 + \sqrt{10}$, with norms 4 and 6 respectively. Moreover $\langle 2, \sqrt{10} \rangle = \langle 2, 2+\sqrt{10} \rangle$. Using Lemma 3.5, Chapter I in [Fr] we can find $U \in GL(2, \mathcal{O}_K)$ with $U \begin{pmatrix} 2 \\ \sqrt{10} \end{pmatrix} = \begin{pmatrix} 2 \\ 2+\sqrt{10} \end{pmatrix}$, and we conclude that $(2, 2+\sqrt{10}) \in M(S[U])$, if $(2, \sqrt{10}) \in M(S)$. Since $N((2, 2+\sqrt{10})) = 24$, it follows that $M_{Q(\sqrt{10}),2} = 24$.

3.

In this section we use the previous results to get a general version of Mordell's theorem. Let us first start with some remarks on matrices. If $A = (a_{ij})$ is a symmetric $n \times n$ nonsingular matrix, let $A^* = (A_{ij})$ be its adjoint matrix, i.e. A_{ij} is the $(i,j)-$cofactor of A. Then $AA^* = (detA)I$ and hence $detA^* = (detA)^{n-1}$. If $a = (a_1, \cdots, a_n)$ is a vector with all $a_i \neq 0$, let $D(a)$ be the diagonal matrix with entries $a_1, \cdots, a_n$ on the diagonal. If A is positive definite, then the matrix $\tilde{A} = D(a)AD(a)$ is also positive definite and $det\tilde{A} = N(a)^2 detA$, where $N(a) = a_1 \cdots a_n$. Moreover for $x = (x_1, \cdots, x_n)$ we have $\tilde{A}[x] = A[a \cdot x]$, where $a \cdot x = (a_1x_1, \cdots, a_nx_n)$. Setting $a^* = (a_1^*, \cdots, a_n^*)$ with $a_i^* = N(a) \cdot a_i^{-1}$, one easily checks that

$$(D(a^*)AD(a^*))^* = N(a)^{2(n-2)} D(a)A^*D(a).$$

These notions extend componentwise to Humbert forms in the obvious way. If $S \in P_{n,K}$ and $v = (v_1, \cdots, v_n) \in \mathcal{O}_K^n$, let us set

$$D(v)SD(v) = (D(v^{(1)})S_1D(v^{(1)}), \cdots, D(v^{(n)})S_mD(v^{(m)})).$$

If all $v_i \neq 0$, set $v^* = (N(v)v_1^{-1}, \cdots, N(v)v_n^{-1}) \in \mathcal{O}_K^n$. Then we obtain the relation

$$\begin{aligned}(D(v^*)SD(v^*))^* &= (N(v^{(1)})^{2(n-2)}D(v^{(1)})S_1^*D(v^{(1)}), \cdots, \\ &\qquad N(v^{(m)})^{2(n-2)}D(v^{(m)})S_m^*D(v^{(m)})) \\ &= (N(v^{(1)})^{2(n-2)}, \cdots, N(v^{(m)})^{2(n-2)})D(v)S^*D(v)\end{aligned}$$

(here $(a_1, \cdots, a_m) \cdot (T_1, \cdots, T_n) = (a_1T_1, \cdots, a_mT_m)$).

Let us now consider a Humbert form $S \in P_{n,K}$. Since we want to estimate $\gamma_K(S)$, we can replace S by an equivalent form S^* such that S^* is Humbert-reduced. Let $u = (u_1, \cdots, u_n) \in \mathcal{O}_K^n$ be a minimal vector of S^*, i.e. $m(S^*) = S^*[u]$. By section 2 we have $N(u) \leq M_{n,K}$. Let us write $\mathbf{u} = (\mathbf{u}_1, \cdots, \mathbf{u}_n)$, where $\mathbf{u}_i = u_i$ if $u_i \neq 0$ and $\mathbf{u}_i = 1$ if $u_i = 0$. Then $\mathbf{u} \in \mathcal{O}_K^n$ and $N(\mathbf{u}) = N(u)$. We define $\tilde{S}^* = D(\mathbf{u})S^*D(\mathbf{u})$. Let $e = (\cdots, 1, \cdots)$ be the vector (in $\mathcal{O}_K^n$) with 1 at the places with $u_i \neq 0$ and 0 when $u_i = 0$. Then we have $m(S^*) = S^*[u] = \tilde{S}^*[e]$, since $\mathbf{u} \cdot e = u$. In particular $m(S^*) \geq m(\tilde{S}^*)$. But we have in general, for any $w \in \mathcal{O}_K^n, \tilde{S}^*[w] = S^*[\mathbf{u} \cdot w]$, so that $m(\tilde{S}^*) \leq m(S^*)$. This shows the equalities

$$m(S^*) = m(\tilde{S}^*) = \tilde{S}^*[e].$$

The point is now that $\tilde{S}^*$ has the unimodular vector e as a minimal vector. Then e is also a minimal vector of any multiple $(a_1, \cdots, a_n) \cdot \tilde{S}^*$ of $\tilde{S}^*$, and in particular of the form $(N(\mathbf{u}^{(1)})^{2(n-2)}, \cdots, N(\mathbf{u}^{(m)})^{2(n-2)}) \cdot \tilde{S}^* = (D(\mathbf{u}^*)SD(\mathbf{u}^*))^*$. Let us write $\tilde{S}$ for the form $D(\mathbf{u}^*)SD(\mathbf{u}^*)$, so that $(\tilde{S})^*$ has the minimal vector e. Since $\tilde{S}[x] = S[\mathbf{u}^* \cdot x]$, we have $m(S) \leq m(\tilde{S})$. Now any unimodular vector in $\mathcal{O}_K^n$ can be transformed via $GL(n, \mathcal{O}_K)$ into the vector $e_1 = (1, 0, \cdots, 0)$. Thus we can transform the form $(\tilde{S})^*$ into an equivalent form $(\tilde{S})^{*\prime}$ which has the minimal vector e_1. Then the form $\tilde{S}$ transforms into an equivalent form $\tilde{S}'$ such that $(\tilde{S}')^* = (\tilde{S})^{*\prime}$. Therefore we have

$$m(S) \leq m(\tilde{S}) = m(\tilde{S}').$$

Let $\tilde{S}'_{n-1}$ denote the $(n-1)$-dimensional Humbert form obtained from $\tilde{S}'$ by deleting from each component the first row and column. Then we have

$$m(S) \leq m(\tilde{S}) \leq m(\tilde{S}') \leq m(\tilde{S}'_{n-1})$$

and hence

$$\begin{aligned} m(S) &\leq \gamma_{K,n-1}(det\tilde{S}'_{n-1})^{\frac{1}{n-1}} \\ &\leq \gamma_{K,n-1}((\tilde{S}')^*[e_1])^{\frac{1}{n-1}} \\ &\leq \gamma_{K,n-1}((\tilde{S})^{*\prime}[e_1])^{\frac{1}{n-1}} \\ &\leq \gamma_{K,n-1}(m((\tilde{S})^{*\prime}))^{\frac{1}{n-1}} \\ &\leq \gamma_{K,n-1} m((\tilde{S})^*)^{\frac{1}{n-1}} \\ &\leq \gamma_{K,n-1} \cdot (\gamma_{K,n})^{\frac{1}{n-1}} \cdot [det(\tilde{S})^*]^{\frac{1}{n(n-1)}}. \end{aligned}$$

But $det(\tilde{S})^* = N(u)^{2(n-2)} \cdot det(\tilde{S}^*) = N(u)^{2(n-2)} \cdot N(u)^2 detS^* = N(u)^{2(n-1)} \cdot (detS)^{n-1}$. Hence

$$m(S) \leq \gamma_{K,n-1} \cdot (\gamma_{K,n})^{\frac{1}{n-1}} \cdot N(u)^{\frac{2}{n}} (detS)^{\frac{1}{n}},$$

i.e.

$$\gamma_K(S) \leq \gamma_{K,n-1} \cdot (\gamma_{K,n})^{\frac{1}{n-1}} \cdot M_{n,K}^{\frac{2}{n}},$$

since $N(u) \leq M_{n,K}$. Taking the supremun over $S \in P_{n,K}$ we conclude that

$$\gamma_{K,n} \leq \gamma_{K,n-1} \cdot (\gamma_{K,n})^{\frac{1}{n-1}} \cdot M_{n,K}^{\frac{2}{n}}.$$

We have thus proved the following.

Theorem. *Let* $K/\mathbf{Q}$ *be a totally real number field. Then*

$$\gamma_{K,n}^{n-2} \leq \gamma_{K,n-1}^{n-1} \cdot M_{n,K}^{\frac{2(n-1)}{n}}.$$

Corollary. *If* $h(K) = 1$, *then*

$$\gamma_{K,n}^{n-2} \leq \gamma_{K,n-1}^{n-1}.$$

4.

In this final section we want to mention briefly how one can extend the invariant $\gamma_{K,n}$ to the general case of non-free lattices.

Let us first mention that using the approximation theorem, $\gamma_{K,n}$ can be defined as $\gamma_{K,n} = Sup_f \frac{m(f)}{N_{K/Q}(det f)^{1/n}}$, where f runs over all positive definite quadratic forms $f : \mathcal{O}_K^n \to K$, and $m(f)$ denotes $Min\{N_{K/Q}(f(x)) | 0 \neq x \in \mathcal{O}_K^n\}$.

Let us now consider a general lattice $\Delta = \mathcal{O}_K^{n-1} \oplus A$, where $A \subset \mathcal{O}_K$ is a non-zero ideal, equipped with a positive definite quadratic form $f : \Delta \to K$. We set $m(\Delta) = Min\{N_{K/Q}(f(x)) | 0 \neq x \in \Delta\}$ and $det(\Delta) = N_{K/Q}(vol(\Delta))$, where $vol(\Delta)$ is defined as in [OM], 82E. Then set $\gamma_K(\Delta) = m(\Delta)/det(\Delta)^{1/n}$ and define

$$\gamma'_{K,n} = Sup_\Delta \gamma_K(\Delta)$$

where Δ runs over all lattices as above. It is clear that $\gamma_{K,n} \leq \gamma'_{K,n}$ by the above remark. Now let $\Delta = \mathcal{O}_K^{n-1} \oplus A$ be a given lattice. Then there is $\alpha \in A$ with $\alpha\mathcal{O}_K \subseteq A$ and $|N_{K/Q}(\alpha)| \leq \frac{m!}{m^m}|d_k|^{1/2}N(A)$. Let us consider the free lattice $L = \mathcal{O}_K^{n-1} \oplus \alpha \cdot \mathcal{O}_K \subset \Delta$. Then $[\Delta : L] \leq \frac{m!}{m^m}|d_K|^{1/2}$ and $N_{K/Q}(det L) = [\Delta : L]^2 det(\Delta)$ (see [OM], 81D, 82E). Since $m(L) \geq m(\Delta)$, we conclude that

$$\gamma_K(\Delta) = \frac{m(\Delta)}{det(\Delta)^{1/n}} \leq [\Delta : L]^{\frac{2}{n}} \frac{m(L)}{N_{K/Q}(det L)^{1/n}} = [\Delta : L]^{\frac{2}{n}} \gamma_K(L).$$

Therefore

$$\gamma_K(\Delta) \leq \left(\frac{m!}{m^m}\right)^{\frac{2}{n}} |d_K|^{\frac{1}{n}} \gamma_{K,n}.$$

From this inequality we get finally

$$\gamma_{K,n} \leq \gamma'_{K,n} \leq \left(\frac{m!}{m^m}\right)^{\frac{2}{n}} |d_K|^{\frac{1}{n}} \gamma_{K,n}.$$

It is clear that this relation together with the theorem of the previous section imply a corresponding Mordell-inequality for $\gamma'_{K,n}$.

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