Dedekind Domains with Torsion Class Group

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Let R be a Dedekind domain with fraction field K. An **overring** of R is any ring A such that $R \subseteq A \subseteq K$. I will prove the following result:

The class group of R is a torsion group if and only if every overring of R is a localization of R.

Suppose first that the class group of R is torsion, and let A be any overring of R. We know that

$$A = \bigcap_{\mathfrak{p} \in T} R_{\mathfrak{p}} \qquad (\star)$$

for some set T of prime ideals of R. Explicitly, $T = \{\mathfrak{p} : \mathfrak{p}A \neq A\}$. I will show that $A = S^{-1}R$ where S is the multiplicative set $R \backslash \cup_{\mathfrak{p} \in T} \mathfrak{p}$. It is clear from the definition of localization that $S^{-1}R \subseteq A$. To see the reverse inclusion, fix an element $a \in A$ and consider the ideal $D = \{r \in R : ra \in R\}$ of denominators of a. Since the class group of R is torsion, there is a positive integer n such that D^n is a principal ideal, say $D^n = (s)$. Thus for any prime ideal $\mathfrak{p}$ of R we have that $\mathfrak{p}$ divides D if and only if $s \in \mathfrak{p}$. Since D intersects $R \backslash \mathfrak{p}$ for all $\mathfrak{p} \in T$ (because $a \in R_{\mathfrak{p}}$), then no prime in T divides D, so D is only divisible by primes in T^c. It follows that s only belongs to primes in T^c, so $s \in D \cap S$. Therefore, a can be written with a denominator of $s \in S$, so $a \in S^{-1}R$, as claimed. This proves that the overring A is a localization of R.

Now assume that every overring of R is a localization and let $\mathfrak{p}$ be a maximal ideal of R. Consider the overring $A = \bigcap_{\mathfrak{q} \neq \mathfrak{p}} R_{\mathfrak{q}}$ and let $S \subseteq R$ be a multiplicative set such that $A = S^{-1}R$. I claim that $\mathfrak{p}A = A$. If this were not the case, then we would have $A \subseteq R_{\mathfrak{p}}$ by $(\star)$, and so $\bigcap_{\mathfrak{q} \neq \mathfrak{p}} R_{\mathfrak{q}} \subseteq R_{\mathfrak{p}}$. To reach a contradiction, we use the following result, which follows from the Chinese remainder theorem:

Let R be a Dedekind domain with fraction field K and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct maximal ideals of R. Given any integers $a_1, \ldots, a_n$, there exists an element $x \in K^*$ such that $v_{\mathfrak{p}_i}(x) = a_i$ and $v_{\mathfrak{q}}(x) \geq 0$ for all $\mathfrak{q}$ different from the $\mathfrak{p}_i$'s.

Using this result, it follows that there is an $x \in K^*$ such that $v_{\mathfrak{q}}(x) \geq 0$ for $\mathfrak{q} \neq \mathfrak{p}$ and $v_{\mathfrak{p}}(x) = -1$. In particular, $x \in \bigcap_{\mathfrak{q} \neq \mathfrak{p}} R_{\mathfrak{q}}$ but $x \notin R_{\mathfrak{p}}$. We thus have a contradiction, and hence $\mathfrak{p}A = A$ as claimed.

We therefore have that $S^{-1}\mathfrak{p} = \mathfrak{p}A = A$, so $\mathfrak{p}$ must intersect S. Choose any $s \in S \cap \mathfrak{p}$, so that for any $\mathfrak{q} \neq \mathfrak{p}$ we have $1/s \in R_{\mathfrak{q}}$ and hence $s \notin \mathfrak{q}$. It follows that s belongs to no prime ideal other than $\mathfrak{p}$, so $(s) = \mathfrak{p}^n$ for some positive integer n. This shows that the class of $\mathfrak{p}$ has finite order in the class group, and we conclude that the class group of R is torsion.