ADELES AND IDELES

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Contents

7. Adeles

7.1. **What should the adeles do?**

We have now come to what is probably the most important topic in our course: the systematic study of global fields using their locally compact completions.

Let K be a global field – we call that this means that K is either a finite extension of $\mathbb{Q}$ (a number field) or a finite separable extension of $\mathbb{F}_q(t)$ (a function field).

In the case of a locally compact field K, the additive group $(K, +)$ and the multiplicative group $(K^\times, \cdot)$ play key roles in the theory. Each is a locally compact abelian group, hence amenable to the methods of Fourier analysis. Moreover, the additive group is self-Pontrjagin dual, and the multiplicative group $K^\times$ is a target group for class field theory on K: that is, there is a bijective correspondence between the finite abelian extensions L/K and the finite index open subgroups H_L of $K^\times$ such that $K^\times/H_L \cong \mathrm{Gal}(L/K)$.

We seek global analogues of all of these facts. That is, for K a global field, we will define a commutative topological ring $\mathbb{A}_K$, the **adele ring**, which is locally compact and self-Pontrjagin dual. (This allows us to do harmonic analysis on global fields, as was first done in John Tate's 1950 Princeton thesis. We will not actually do this in our course, but it is an all-important topic in modern number theory, and I wish to be aware of it and be prepared to learn it!) Moreover, the group of units, suitably topologized, is called the **idele group** $\mathbb{I}_K$. It is again a locally compact abelian group.

There are further important topological properties that we do not see in the local case: namely, we will have canonical embeddings $K \hookrightarrow \mathbb{A}_K$ and $K^\times \hookrightarrow \mathbb{I}_K$. In the additive case, K is discrete (hence closed) as a subgroup of the adele ring, and the quotient $\mathbb{A}_K$ is compact. In the multiplicative case, $K^\times$ is again a discrete subgroup of $\mathbb{I}_K$; the quotient group is denoted $C(K)$. $C(K)$ need not be compact, but it is again a target group for class field theory on K.

7.2. **The adele ring.**

For each place v of K, the completion K_v is a locally compact field, so it seems natural not to proceed merely by analogy but actually to use the fields K_v in the construction of our putative $\mathbb{A}_K$. The first idea is simply to take the product of all the completions: $\prod_v K_v$. However, this will not work:

Exercise 6.1: Let $\{X_i\}_{i\in I}$ be an indexed family of nonempty topological spaces. Show that TFAE:
(i) $X = \prod_{i\in I} X_i$ is locally compact.
(ii) Each X_i is locally compact, and $\{i \in I \mid X_i \text{ is not compact}\}$ is finite.

So the next try is cut down by taking only compact groups except in finitely many places. In our case this makes sense: all but finitely many places v (all of them in the function field case!) are non-Archimedean, so that we have the valuation ring R_v of K_v, a compact subgroup (subring, even) of K_v. Thus the product $\prod_{v Arch} K_v \times \prod_{v NA} R_v$ is a locally compact abelian group having something to do with the global field K. But not enough: we would like to have an embedding $K \hookrightarrow \mathbb{A}_K$ and this is clearly not the case with the above product. For instance, taking $K = \mathbb{Q}$ we see that an element $x \in K$ lies in the direct product iff $\operatorname{ord}_p(x) \geq 0$ for all primes p, i.e., iff $x \in \mathbb{Z}$. So this is a reasonable "global completion" of $\mathbb{Z}$: indeed it is precisely $\hat{\mathbb{Z}} \times \mathbb{R}$, i.e., the direct product of the usual profinite completion of $\mathbb{Z}$ with $\mathbb{R}$.

Suppose we wanted to make an analogous construction that included the nonzero rational number $\frac{x}{y}$. Then y is divisible only by a finite set of primes, say S. Thus $\frac{x}{y}$ naturally lives in $\mathbb{R} \times \prod_{\ell\in S} \mathbb{Q}_\ell \times \prod_{\ell\notin S} \mathbb{Z}_\ell$, which is still a locally compact group since it has only finitely many noncompact factors. Of course the finite set S that we need to take depends on $\frac{x}{y}$: indeed, to get all possible denominators we will need to use all the groups

$$\mathbb{A}_{\mathbb{Q}}(S) = \mathbb{R} \times \prod_{\ell\in S} \mathbb{Q}_\ell \times \prod_{\ell\notin S} \mathbb{Z}_\ell.$$

But can we get one locally compact abelian group out of this family of locally compact groups indexed by the finite subsets S of non-Archimedean places of $\mathbb{Q}$? Indeed yes! Observe that these locally compact groups naturally form a directed system: if $S \subset S'$, then $\mathbb{A}_{\mathbb{Q}}(S) \hookrightarrow \mathbb{A}_{\mathbb{Q}}(S')$ embeds as an open subgroup. Therefore we may define

$$\mathbb{A}_{\mathbb{Q}} = \lim_{S} \mathbb{A}_{\mathbb{Q}}(S).$$

Thus, as a set, $\mathbb{A}_{\mathbb{Q}}$ is the subset of $\mathbb{R} \times \prod_\ell \mathbb{Q}_\ell$ consisting of sequences (r, x_ℓ) such that $x_\ell \in \mathbb{Z}_\ell$ for all but finitely many primes ℓ. (Note that this is strongly reminiscent of a direct sum, except that instead of requiring all but finitely many components to be zero, we require that they lie in a fixed compact subgroup.)

How do we topologize $\mathbb{A}_{\mathbb{Q}}$? Well, how do we topologize a direct limit of topological spaces? There is a standard recipe for this. More generally, suppose we have a directed system of topological spaces $\{X_i\}_{i\in I}$ – i.e., I is a directed set and for each pair of indices with $i \leq j$ we have continuous maps $\varphi(i,j) : X_i \to X_j$ such that $\varphi(i,k) = \varphi(j,k) \circ \varphi(i,j)$ when $i \leq j \leq k$ – a set X, and maps $f_i : X_i \to X$ which are compatible in the sense that $i \leq j$ implies $f_i = f_j \circ \varphi(i,j)$. Then the canonical topology to put on X is the **final topology**, i.e., the finest topology which makes all the maps f_i continuous. (To see that such a topology exists, note that the trivial topology on X makes all the maps continuous and given any family of topologies which makes all the maps continuous, their union is such a topology.) This topology can also be characterized by a universal mapping property. To make use of this, the following exercise is critical.

Exercise 6.2: a) Show that the final topology can be characterized as follows: a subset $U \subset X$ is open in the final topology iff for all $i \in I$, $f_i^{-1}(U)$ is open in U.
b) Deduce that for each finite subset S of prime numbers, $\mathbb{A}_{\mathbb{Q}}(S)$ is an open, locally compact subring of $\mathbb{A}_{\mathbb{Q}}$, and therefore that $\mathbb{A}_{\mathbb{Q}}$ is a compact topological ring.

Of course we want the same sort of construction with $\mathbb{Q}$ replaced by an arbitrary global field K. In fact, we may as well develop things in "proper generality" (it costs nothing extra): the key concept is that of the **restricted direct product**.

Suppose we are given the following data: (i) a nonempty index set I, (ii) for all $i \in I$, a topological group G_i, (iii) for all $i \in I$ an subgroup H_i of G_i. Then we define the **restricted direct product** $G = \prod' G_i$ to be the subset of $\prod G_i$ consisting of tuples (x_i) such that $x_i \in H_i$ for all but finitely many i's. For each finite subset $J \subset I$, let $G^J = \prod_{i\in J} G_i \times \prod_{i\notin J} H_i$. Then $G = \lim_J G^J$, and we give it the direct limit topology.

Exercise 6.3: Is it true that the direct limit topology on G is the same topology as it inherits as a subset of the direct product $\prod_{i\in I} G_i$?

Exercise 6.4: Show that G is compact iff each G_i is compact.

Exercise 6.5: Show that G is locally compact iff: each G_i is locally compact and all but finitely many H_i's are compact.

We now give a more technical discussion of the relation of Haar measure on each of a family $\{G_i\}$ of locally compact abelian groups and the Haar measure on the restricted direct product. This will be used (only) in the proof of the Adelic Blichfeldt-Minkowski Lemma.

We place ourselves in the following situation: we are given a family $\{G_i\}_{i\in I}$ of commutative topological groups together with, on the complement of some finite subset I_∞ of I, a compact open subgroup H_i of G_i. Let $G = \prod' G_i$ be the restricted direct product of the G_i's with respect to the H_i's. Then, for each $i \in I$, there is a Haar measure mu_i on G_i. Moreover, for each $i \in I \setminus I_\infty$, we can normalize μ_i by decreeing $\mu_i(H_i) = 1$. We define a **product measure** on G to be a measure whose σ-algebra is generated by cylindrical sets $\prod_i M_i$ such that each $M_i \subset G_i$ is μ_i measurable and has $\mu_i(M_i) < \infty$ and such that $M_i = G_i$ for all but finitely many i. There is a unique measure μ on G such that $\mu(\prod_i M_i) = \prod_i \mu_i(M_i)$. Note that the restriction of μ to any subgroup $G_S = \prod_{i \in S} G_i \times \prod_{i \in I \setminus S} H_i$ is the usual product measure, a Haar measure.

Definition: Let K be any global field. We define the **adele ring** $\mathbb{A}_K$ as the restricted direct product of the topological fields $G_v := K_v$ as v ranges over all places of K and with the following chosen subgroups: if v is Archimedean, we put $H_v = K_v$ (no restriction), and if v is non-Archimedean, we put $H_v = R_v$, the valuation ring, a compact subring. Thus $\mathbb{A}_K$ is a locally compact ring.

Notation: Despite some misgivings, we introduce the following notation. For a global field K, we let Σ_K denote the set of all places of K, i.e., equivalence classes of nontrivial norms on K. (Recall that at this point we have associated a canonical normalized Artin absolute value to each place in K.) Further we write Σ_K^{NA} for the subset of non-Archimeden places of K, and let Σ_K^{Arch} denote the subset of Archimedean places of K, which is nonempty iff K is a number field.

For each finite S with $\Sigma_K^{\mathrm{Arch}} \subset S \subset \Sigma_K$, let $\mathbb{A}_K(S) = \prod_{v \in S} K_v \times \prod_{v \notin S} R_v$ be the ring of **S-adeles**.

Exercise 6.6: a) Show that $\mathbb{A}_K(S)$ is both open and closed in $\mathbb{A}_K$.
b) Let $\iota : K \to \prod_v K_v$ be the natural embedding. Show that $\iota(K) \subset \mathbb{A}_K$.

Exercise 6.7: a) Show that the adele ring $\mathbb{A}_K$ is *not* an integral domain.
b)* Compute $\operatorname{Spec} \mathbb{A}_K$.[1]

7.3. Basic results on the topology of the adeles.

Lemma 1. *Let L/K be a finite separable extension of global fields. Then we have a canonical isomorphism of topological rings $\mathbb{A}_L = \mathbb{A}_K \otimes_K L$.*

Proof. The main idea of the proof is the following familiar fact: for every place v of K we have an isomorphism of topological K_v-algebras $L \otimes_K K_v \cong \prod_{w \mid v} L_w$. Compiling these local isomorphisms gives the global isomorphism. For the topological isomorphism, we check that both sides are topologized by the same restricted direct product topology. Details are left to the reader as a good exercise. □

Corollary 2. *Maintain the notation of the previous lemma. Then, as additive groups, $(\mathbb{A}_L, +) \cong (\mathbb{A}_K, +)^{[L:K]}$.*

Proof. This follows from the isomorphism of topological groups $\mathbb{A}_K \otimes_K L \cong \mathbb{A}_K^{[L:K]}$. □

Theorem 3. *Let K be a global field. As a subspace of $\mathbb{A}_K$, K is discrete. Moreover, the quotient $\mathbb{A}_K/K$ is compact.*

Proof. By Corollary 2, we may assume that $K = \mathbb{Q}$ or $K = \mathbb{F}_q(t)$. In the number field case, we let ∞ denote the Archimedean place, whereas in the function field

[1] Hint/Warning: This involves ultrafilters and such.

case, we let (as usual) ∞ denote the place at infinity, for which $\frac{1}{t}$ is a uniformizer. Let $R = \mathbb{Z}$ or $\mathbb{F}_q[t]$, accordingly.
Let us show the discreteness of K in $\mathbb{A}_K$. Because we are in a topological group, it is enough to find a neighborhood of zero $U \subset \mathbb{A}_K$ such that $U \cap K = \{0\}$. Let U be the set of adeles x with $|x_\infty|_\infty < 1$ and $|x_p|_p \leq 1$ for all finite places p. Certainly U is an open set. When $K = \mathbb{Q}$ the intersection $U \cap K$ consists of rational numbers which are integral at all places – i.e., integers – and have standard absolute value strictly less than 1. Clearly this intersection is 0. Similarly, in the function field case, the integrality conditions force $x \in K \cap U \implies x \in R$, i.e., x is a polynomial, whereas $|x|_\infty < 1$ means that x is, if not zero, a rational function of negative degree. So $x = 0$. This proves the discreteness in both cases.
Now let $W \subset \mathbb{A}_K$ be the compact subset defined by $|x_\infty|_\infty \leq \frac{1}{2}$, $|x_v|_v \leq 1$ for all finite places v. We claim that every adele y can be written in the form $b + x$, with $b \in K$, $x \in W$. This follows easily by weak approximation. Therefore the quotient map $\mathbb{A} \to \mathbb{A}/K$ restricted to W is surjective, done. □

Lemma 4. *(Adelic Blichfeldt-Minkowski Lemma) Let K be a global field. There is a constant $C = C(K)$ such that: if $x = \{x_v\} \in \mathbb{A}_K$ is such that $|x_v|_v = 1$ for almost every v and $\prod_v |x_v|_v > C$, then there is a nonzero $y \in K$ such that for all $v \in \Sigma_K$, $|y|_v \leq |x_v|_v$.*

Proof. This proof uses the product measure μ on $\mathbb{A}_K$ defined above. Moreover, since K is countable and$\mathbb{A}_K/K$ is compact, it has a finite, positive, total measure c_0 with respect to the Haar measure – in other words, this is the measure of a fundamental region for the coset space $\mathbb{A}_K/K$ in K. Let c_1 be the measure of the subset of $\mathbb{A}_K$ defined by the inequalities $|x_v|_v \leq \frac{1}{10}$ at the Archimedean places and $|x_v|_v \leq 1$ at the non-Archimedean places. It is easy to see that $0 < c_1 < \infty$. We show that we may take $C = \frac{c_0}{c_1}$.

Now fix an adele $\alpha = (\alpha_v)$ such that $|\alpha_v|_v = 1$ for almost every v and $\prod_v |\alpha_v|_v > C$. Let T be the set of adeles (x_v) with $|x_v|_v \leq \frac{1}{10}|\alpha_v|_v$ at Archimedean places and $|x_v|_v \leq |\alpha_v|_v$ at non-Archimedean places. The set T has measure $c_1 \prod_v |\alpha_v| > c_1 C = c_0$, hence there must exist distinct elements of T with the same image in the quotient $\mathbb{A}_K/K$, say τ' and τ'' so that $\beta := \tau' - \tau''$ is a nonzero element of K such that $|\beta|_v = |\tau'_v - \tau''_v| \leq |\alpha_v|_v$, qed. □

Exercise: What we the point of inserting the factor $\frac{1}{10}$ at the Archimedean places?

Remark: For a statement of the classical Blichfeldt Lemma, see e.g. Theorem 5 of `http://math.uga.edu/~pete/4400Minkowski.pdf`.

For future reference, a constant C as in the statement of Lemma 4 will be called an **adelic Blichfeldt constant**.

Corollary 5. *Let v_0 be a normalized valuation on K. Choose a sequence $\{\delta_v\}_{v \neq v_0}$ such that $\delta_v > 0$ for all v and $\delta_v = 1$ for all but finitely many v. Then there exists $x \in K^\times$ such that $|x|_v \leq \delta_v$ for all $v \neq v_0$.*

Proof. Choose $\alpha_v \in K_v$ with $0 < |\alpha_v| \leq \delta_v$ and $|\alpha_v|_v = 1$ if $\delta_v = 1$. Choose $\alpha_{v_0} \in K_{v_0}$ such that $\prod_v |\alpha_v|_v > C$. Apply the Lemma. □

Theorem 6. *(Strong Approximation) Fix any valuation v_0 of the global field K. Define $\mathbb{A}_K^{v_0}$ to be the restricted direct product of the K_v's (with $v \neq v_0$) with respect to the subrings R_v for the non-Archimedean places. Then the natural embedding $K \hookrightarrow \mathbb{A}_K^{v_0}$ has dense image.*

Proof. The theorem is equivalent to the following statement: suppose we are given (i) a finite set S of valuations $v \neq v_0$, (ii) elements $\alpha_v \in K_v$ for all $v \in S$ and (iii) $\epsilon > 0$. Then there exists $\beta \in K$ such that $|\beta - \alpha_v|_v \leq \epsilon$ for all $v \in S$ and $|\beta|_v \leq 1$ for all $v \neq v_0$. By the proof of Theorem 3, there exists $W \subset \mathbb{A}_K$ defined by inequalities $|\alpha_v|_v \leq \delta_v$ for all v, $\delta_v = 1$ for all but finitely many v, such that every adele $\alpha \in \mathbb{A}_K$ is of the form $\alpha = y + w$, $y \in K$, $w \in W$. By Corollary 5, there exists $x \in K^\times$ such that $|x|_v < \delta_v^{-1}\epsilon$ for all $v \in S$ and $|x|_v \leq \delta_v^{-1}$ for all $v \neq v_0$. Let α be any adele. Write $x^{-1}\alpha = y + w$ with $y \in K$, $w \in W$, and multiply by x to get $\alpha = xy + xw$. Finally, choose α to be the adele with v component the given α_v for all $v \in S$ and 0 elsewhere. Then we may take $\beta = xy$. $\square$

Exercise 6.8: Give a hands-on proof of Theorem 6 when $K = \mathbb{Q}$, $v_0 = \infty$.

7.4. **Ideles.**

Let R be a topological ring. Then the group of units $R^\times$ need not be a topological group under the induced topology: the problem is that inversion need not be continuous.

It will be helpful to use the following snippet from the theory of topological groups: a **paratopological group** is a group G endowed with a topology with respect to which the group law is "jointly continuous", i.e., $\cdot : G \times G \to G$ is continuous. A **semitopological group** is a group endowed with a topology with respect to which the group law is "separately continuous": for all $y \in G$, the maps $y\bullet : G \to G$, $x \mapsto yx$ and $\bullet y : G \to G$, $x \mapsto xy$ are continuous. Thus a topological group is precisely paratopological group in which the inversion map $x \mapsto x^{-1}$ is continuous, and a paratopological group is a semitopological group.

Theorem 7. *(Ellis)*
a) Every locally compact paratopological group is a topological group [**?**].
b) Every locally compact semitopological group is a topological group [**?**].

Exercise 6.9:
a) Suppose R is a locally compact topological ring in which $R^\times$ is an open subgroup. Show that $R^\times$ is a topological group: i.e., $x \mapsto x^{-1}$ is a homeomorphism of $R^\times$ under the subspace topology.
b) (T. Trimble) Take on $\mathbb{Q}$ the topology τ_T for which a neighborhood base at $x \in \mathbb{Q}$ is given by $\{x + n\mathbb{Z}\}_{n=1}^\infty$. Show that τ_T is Hausdorff, so in particular $\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$ is open. Show that inversion on $\mathbb{Q}^\times$ is not continuous.

There is a general method for endowing $R^\times$ with the structure of a topological group. Namely, we think of $R^\times$ as a subset of $R \times R$ via the injection $x \mapsto (x, x^{-1})$. Let us call this the **multiplicative topology**.

Proposition 8. *Let R be a Hausdorff topological ring.*
a) Show that the multiplicative topology on $R^\times$ is at least as fine as the subspace topology.

b) Show that if we endow $R^\times$ with the multiplicative topology, it is a Hausdorff topological group.

Exercise 6.10: Prove Proposition 8.

Of course the case we have in mind is $R = \mathbb{A}_K$, the adele ring of a global field K. We define the **idele group** $\mathbb{I}_K$ to be the unit group of the adele ring.

Proposition 9. *Let $\mathbb{I}_K = \mathbb{A}_K^\times$ be the idele group.*
a) $\mathbb{I}_K$ is the set of all adeles $(x_v)_v$ with $x_v \neq 0$ for all v and $x_v \in R_v^\times$ for almost all v.
b) The idele group $\mathbb{I}_K$ is not *an open subgroup of $\mathbb{A}_K$.*
c) The multiplicative topology on $\mathbb{I}_K$ is strictly finer than the subspace topology.
d) The multiplicative topology on $\mathbb{I}_K$ makes it into a locally compact (Hausdorff) topological group. In fact, it is nothing else than the restricted direct product of the spaces $K_v^\times$ with respect to the compact subgroups $R_v^\times$ (and the locally compact subgroups $K_v^\times$ at the Archimedean places, if any).

Exercise 6.11: Prove Proposition 9.

Exercise 6.12: For finite S, $\Sigma_K^{\text{Arch}} \subset S \subset \Sigma_K$, define the group of S-ideles

$$\mathbb{I}_K(S) = \prod_{v \in S} K_v^\times \times \prod_{v \notin S} R_v^\times.$$

a) Show that each $\mathbb{I}_K(S)$ is open and closed as a subgroup of $\mathbb{I}_K$.
b) Show that the natural (diagonal) map from $K^\times$ to $\mathbb{I}_K(S)$ is an injection. Henceforth we identify $K^\times$ with its image in $\mathbb{I}_K$ or $\mathbb{I}_K(S)$ for some S.

Lemma 10. *$K^\times$ is discrete in $\mathbb{I}_K$.*

Proof. By the definition of the topology on the adeles, it is enough to show that discreteness of $K^\times$ as embedded in $\mathbb{A}_K \times \mathbb{A}_K$ via $x \mapsto (x, x^{-1})$. But this follows immediately from the discreteness of K in $\mathbb{A}_K$ and the easy fact that a product of two discrete spaces is discrete. □

Exercise 6.13: a) Give a direct proof of Lemma 10 when $K = \mathbb{Q}$ or $\mathbb{F}_p(t)$: i.e., exhibit a compact neighborhood U of 1 in $\mathbb{I}_K$ such that $U \cap K^\times = \{1\}$.
b) Let K/K_0 be a separable extension of global fields. Can you deduce the discreteness of $K^\times$ in $\mathbb{I}_K$ from the discreteness of $K_0^\times$ in $\mathbb{A}_K$ as we did in the additive case?
Stop for a moment and think what should be coming next. The natural question is – isn't it? – whether $K^\times$ is cocompact in $\mathbb{I}_K$ as was the case for K and $\mathbb{A}_K$. The answer is no, and this is a fundamental difference between the ideles and the adeles. To see this, we will construct a continuous map from the **idele class group** $C(K) = \mathbb{I}_K/K^\times$ to a noncompact subgroup of $R^\times$, a kind of "norm map".

Normalized valuations: let K_0 be $\mathbb{Q}$ or $\mathbb{F}_p(t)$, the prime global field. For v a place of K, we choose a particular norm as follows: say $v \mid v_0$ a place of K_0, let $| \, |_{v_0}$ be the standard norm on the $(K_0)_{v_0}$– i.e. which gives a uniformizer norm $\frac{1}{p}$ – and define $|| \, ||_v$ by $||x||_v = |N_{K/K_0}(x)|$. (This coincides with the canonical norm given by the Haar measure.)

Theorem 11. *(Product formula) Let $x \in K^\times$. Then*

$$\prod_v ||x||_v = 1.$$

Proof. Step 1: Suppose $K = \mathbb{Q}$ or $\mathbb{F}_p(t)$. In this case it is straightforward to verify the product formula directly. Indeed, in both cases it takes the special form that the norm at the infinite place is exactly the reciprocal of the product of the norms at all the finite places. This verification was done in class with the help of the students, and we leave it to you, the reader, now.
Step 2: Let L/K be a finite degree separable field extension. We wish to reduce the product formula for L to the product formula for K, a task which involves little more than careful bookkeeping. First we recall the following appealing formula for the normalized Artin absolute values. Let L_w/K_v be a finite exension of locally compact fields. Then for all $x \in L_w$, we have

$$||x||_{L_w} = |N_{L/K}(x)|_{K_v}.$$

Thus, for $x \in K$, we have

$$||x|| = \prod_{v \in \Sigma_K} \prod_{w \mid v} ||x||_w = \prod_{v \in \Sigma_K} \prod_{w \mid v} |N_{L_w/K_v} x|.$$

On the other hand, as we well know,

$$L \otimes_K K_v \cong \prod_{w \mid v} L_w,$$

so that

$$\prod_{w \mid v} N_{L_w/K_v}(x) = N_{L/K}(x).$$

Using this identity, we get

$$||x|| = \prod_{v \in \Sigma_K} |N_{L/K}(x)| = 1$$

by Step 1. □

We define a norm map $|\ | : \mathbb{I}_K \to \mathbb{R}^{>0}$ by $x \in \mathbb{I}_K \mapsto |x| = \prod_v |x_v|_v$. It is immediate that this is a group homomorphism.

Exercise 6.14: Show that the norm map $||\ || : \mathbb{I}_K \to \mathbb{R}^{>0}$ is continuous.

Since our normalized Artin absolute values on $\mathbb{R}$ and $\mathbb{C}$ are both surjective onto $\mathbb{R}^{>0}$, if K is a number field, looking only at ideles which have component 1 except at one fixed Archimedean place shows that the norm map is surjective. If K has characteristic $p > 0$, then the image of the norm map lies somewhere in between $p^{\mathbb{Z}}$ and $p^{\mathbb{Q}}$: i.e., it is not surjective, but its image is an unbounded – hence noncompact – subset of $\mathbb{R}^\times$. Thus we have shown:

Lemma 12. *The idele class group $C(K) = \mathbb{I}_K/K^\times$ is* not *compact.*

We also include, for future use, the following result to the effect that the disparity between the subspace and multiplicative topologies is eliminated by passage to norm one ideles:

Lemma 13. *The norm one ideles $\mathbb{I}_K^1$ are closed as a subset of the adele ring $\mathbb{A}_K$. Moreover, the topology that $\mathbb{I}_K^1$ inherits from $\mathbb{A}_K$ is the same as the topology it inherits from $\mathbb{I}_K$.*

Proof. First we show that $\mathbb{I}_K^1$ is closed in $\mathbb{A}_K$. So, let $alpha \in \mathbb{A}_K \setminus \mathbb{I}_K^1$. We must, of course, find a neighborhood W of α which does not meet $\mathbb{I}_K^1$.
Case 1: $\prod_v |\alpha_v|_v < 1$. Let $S \subset \sigma_K$ be a finite set containing all places v with $|\alpha_v|_v > 1$ and such that $\prod_{v\in S} |\alpha_v|_v < 1$. Let W_ϵ be the set of all $\beta \in \mathbb{A}_K$ such that $|\beta_v - \alpha_v|_v < \epsilon$ for all $v \in S$ and $|\beta_v|_v \leq 1$ for all other v. Then W_ϵ does the job for all sufficiently small ϵ.
Case 2: $\prod_v |\alpha_v|_v = C > 1$. Then there is a finite set S of places v containing each place v with $|\alpha_v|_v > 1$ and if $v \notin S$ then $|\beta_v|_v < 1 \implies \frac{C}{2}$. (For instance, in the number field case we may take S to contain all Archimedean places and all finite places of residue characteristic $p \leq 2C$. Similarly, in the function field case we may take S to contain all places extending places of $K_0 = \mathbb{F}_p(t)$ with residue cardinality at most $2C$.) Then defining W_ϵ as above does the job for all sufficiently small ϵ.

Now we show that the adelic and idelic topologies on $\mathbb{I}_K^1$ coincide. Fix $\alpha \in \mathbb{I}_K^1$. Let $W \subset \mathbb{I}_K^1$ be an adelic neighborhood of α. Then it contains an adelic neighborhood of type $W_\epsilon(S)$ above, for some finite subset S. The corresponding set $W'_\epsilon(S)$ defined by β such that $|\beta_v - \alpha_v|_v < \epsilon$ for all $v \in S$ and $|\beta_v|_v = 1$ for all other v, is an idelic neighborhood of α and $W'_\epsilon(S) \subset W_\epsilon(S)$. Conversely, let $H \subset \mathbb{I}_K^1$ be an idelic neighborhood. Then it contains a neighborhood of type $W_\epsilon(S)$ where S contains all Archimedean places and all v such that $|\alpha_v|_v \neq 1$. Since by assumption $\prod_v |\alpha_v|_v = 1$, by taking ϵ sufficiently small we get $\prod_v |\beta_v|_v < 2$. Then $W_\epsilon(S) \cap \mathbb{I}_K^1 = W'_\epsilon(S) \cap \mathbb{I}_K^1$, qed. ☐

It is natural to try to "fix" the noncompactness of $C(K)$ by passing to the kernel of the norm map. So we make another key definition: put $C^1(K) = \ker(|\ | : \mathbb{I}_K/K^\times \to \mathbb{R}^{>0})$, the **norm one idele class group**.

Theorem 14. *The norm one idele class group $C^1(K)$ is compact.*

Proof. Using Lemma 13, it suffices to find a compact subset $W \subset \mathbb{A}_K$ such that the map $W \cap \mathbb{I}_K^1 \to \mathbb{I}_K^1/K^\times$ is surjective. Let C be an adelic Blichfeldt constant for K; let $\alpha = (\alpha_v)$ be an idele with $||\alpha|| > C$, and let W be the set of adeles β such that for all places v of K, $|\beta_v|_v \leq |\alpha_v|_v$. W is easily seen to be compact. Now let $\beta \in \mathbb{I}_K^1$. Then, by the adelic Blichfeldt-Minkowski Lemma, there exists $x \in K^\times$ such that for all places v, $|x|_v \leq |\beta_v^{-1}\alpha_v|$. Then $x\beta \in W$, qed. ☐