4400/6400 PROBLEM SET 9

Recommendation: 4400 students should do at least three problems; 6400 students should do at least four.

Heads up: There is an online applet for finding the fundamental solution to a Pell equation, available at

`http://www.numbertheory.org/php/pell.html`.

When asked for the fundamental solution u to a Pell equation you can use this applet, learn about continued fractions and try to find the solution yourself, or do some combination of the two.

9.1) Find all integral solutions to the Pell equation $x^2 - 53y^2 = 1$.

9.2) a) Show that for any nonsquare positive integer d and any positive integer M there exist infinitely many integral solutions to the Pell equation $x^2 - dy^2 = 1$ with $y \equiv 0 \pmod{M}$. (Suggestion: Write $y = My'$ and "change variables.")
b) Can we always find solutions with $x \equiv 0 \pmod{M}$?

9.3)[1] A *triangular number* is a positive integer of the form $1+2+\ldots+m = \frac{m(m+1)}{2}$. A square number is (of course!) a number of the form n^2. A *square-triangular number* is a number which is simultaneously triangular and square, i.e., a solution of the equation

$$\frac{m(m+1)}{2} = n^2.$$

a) Show that the equation simplifies to

$$8n^2 = (2m+1)^2 - 1.$$

b) Substitute $x = 2m+1$, $y = 2n$ and show that solutions to $x^2 - 2y^2 = 1$ correspond square-triangular numbers, via

$$m = \frac{x-1}{2}, \; n = \frac{y}{2}.$$

c) Use this correspondence to find all square-triangular numbers.

9.4) The **negative Pell equation** is

$$x^2 - dy^2 = -1.$$

Again integral solutions correspond to units in $\mathbb{Z}[\sqrt{d}]$, this time of norm -1. However, the issue of whether solutions exist is more complicated.
a) Find all solutions to the negative Pell equation when d is a square.

[1] Adapted from Chapter 29 of Silverman's text.

b) Suppose that d is not a square and that the negative Pell equation has an integral solution (x, y). Show that d is not divisible by 4, nor by any prime of the form $4k + 3$.
c)** Find a nonsquare value of d satisfying the conditions of part b) for which the negative Pell equation nevertheless has no solutions.

Remark: It can be shown that the negative Pell equation (for nonsquare d) has solutions iff the period length of the continued fraction expansion of $\sqrt{d}$ is even. However this condition is awkward: it does not say much about the set of d's for which there is a solution, and the computation for a given d can be extremely long.

9.5) For d a positive nonsquare and N any nonzero integer, one can consider the **generalized Pell equation**

$$x^2 - dy^2 = N.$$

Suppose that this equation has a positive integer solution (x, y). Show that it has infinitely many positive integral solutions (x_n, y_n), where $x_n + y_n\sqrt{d} = (x + y\sqrt{d})u^n$ and u is the fundamental solution to $x^2 - dy^2 = 1$.

9.6G) Let d be a positive nonsquare integer. Show that the unit group $\mathbb{Z}[\sqrt{d}]^\times$ of the ring $\mathbb{Z}[\sqrt{d}]$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$. (Note that this is true whether or not there is any solution to the negative Pell equation, although what to take as the generator of the $\mathbb{Z}$ factor will depend on this.)