4400/6400 PROBLEM SET 8

Recommendations: All students should do 8.1, 8.2, 8.5, and 8.7. 6400 students should do at least one additional problem. Students should not hold their breath while working on 8.8.

8.1: Define a sequence $q_1, q_2, \ldots$ as follows: put $q_1 = 5$; having defined $q_1, \ldots, q_n$, let q_{n+1} be the smallest prime factor of

$$N_n = (2 \cdot q_1 \cdots q_n)^2 + 1.$$

Prove that this sequence consists of distinct primes all of which are congruent to 1 modulo 4. (Hint: When is -1 a square modulo p?)

8.2: a) Let $F_n = 2^{2^n} + 1$ (the nth Fermat number). Prove that for all $n \in \mathbb{N}$,

$$\prod_{i=0}^{n} F_n = F_{n+1} - 2.$$

b) Recall, briefly, why this shows there are infinitely many primes.

8.3: a) Let S be the set of positive integers whose first decimal digit ends in 1. In class we showed that S does not have a density; our argument showed that $\overline{d}(S) \geq \frac{1}{2}$ and $\underline{d}(S) \leq \frac{1}{5}$. However, it was noted in class that the lower density is smaller than $\frac{1}{5}$. What are the upper and lower densities of S?
b)(O) State and prove a more general result about densities of subsets of integers defined by "digital" conditions (not necessarily to base 10).
c)(O) The nonexistence of the density of S makes one especially curious as to how to interpret "the probability" that a "randomly chosen" positive integer has first digit 1. The question is a little too concrete for the "no solution" answer to be satisfactory. A better answer is given by **Benford's Law**: what is it?

8.4**: Let S be a subset of the integers with positive upper density. Show that S is substantial: $\sum_{s \in S} \frac{1}{s} = \infty$.

8.5: Recall

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

a) Show that $\mathrm{Li}(x) \sim \frac{x}{\log x}$.
b) Show that for any $a > 0$,

$$\lim_{x \to \infty} \frac{x e^{-a\sqrt{\log x}}}{\mathrm{Li}(x)} = 0.$$

8.6: Let

$$M(N) = \sum_{n=1}^{N} \mu(n).$$

Compute $M(N)$ for $1 \le N \le 100$. (Yes, just write down the sequence of values, or get a computer to do it for you.) What are the largest and smallest values obtained?

8.7: (Computer problem) Find the least value of N such that $M(N) = 10^k$ for $k = 1, 2, 3, 4, \ldots$ (as far out as you can get your computer to go). What can you guess about the upper order of $M(N)$? Can you find a value of N for which $M(N) > \sqrt{N}$?[1]

8.8***: Prove or disprove: For every $\epsilon > 0$, there exists a constant C_ϵ (i.e., possibly depending on ϵ) such that for all N,

$$M(N) \le C_\epsilon N^{\frac{1}{2}+\epsilon}.$$

Remark: An affirmative solution of this problem will win you US $ 1 million. Really.

[1] An automatic A^+ if you can.