4400/6400 PROBLEM SET 7

Recommendations: All students should do 7.1a), 7.2, 7.3, 7.7 and 7.8. 6400 students should do 7.1b). Math education students should do 7.9. Problems 7.4-7.6 are optional for all students.

7.1: a) Exactly how many many lattice points in the square $[-10, 10]^2$ are visible from the origin? Divide by 400 and compare to $\frac{6}{\pi^2}$: what is the error?
b)* Use a computer to extend the above calculation to $N = 100$ and $N = 1000$; in each case, compute the error in the corresponding approximation to $\frac{6}{\pi^2}$ and, the corresponding approximation to π (obtained by dividing by 6, taking the reciprocal and then the square root).

7.2: Suppose $f : [0, \infty) \rightarrow (0, \infty)$ is a continuous function. We define its average value function to be

$$f_{\text{ave}}(n) = \frac{\int_0^n f dx}{n}.$$

Note that this is, by definition, the average value of f on the interval $[0, n]$ that one meets in calculus. Suppose f is monotone (either increasing or decreasing). Show that as $n \rightarrow \infty$,

$$f_{\text{ave}}(n) \sim \frac{f(1) + \ldots + f(n)}{n}.$$

In fact, obtain explicit upper and lower bounds on $f_{\text{ave}}(n)$ in terms of averages over integer values of f.[1]

7.3: Let α be a positive real number. Find a simple asymptotic expression for the average value of $f(n) = n^\alpha$.[2]

7.4(O): Find a formula for the (k-dimensional) volume of the unit ball in $\mathbb{R}^k$. (Interpret "find" either in the weaker sense of looking it up, or in the stronger sense of deriving it.) Use this formula to give an asymptotic expression for the average value of $r_k(n)$, the number of ways to represent n as a sum of k squares.

7.5(O): A positive integer n is said to be **kth power free** if for every prime p, $\text{ord}_p(n) < k$. What is the probability that a randomly chosen positive integer is kth power free? (It is enough to make a probabilistic argument.)

7.6(O): What is the probability that a lattice point in $\mathbb{R}^3$ is visible from the origin? A lattice point in $\mathbb{R}^k$? (Same parenthetical remark as in the previous problem.)

7.7: A beautiful theorem of **Pick** gives a formula for the area of a **lattice polygon** – i.e., a polygon all of whose vertices (x, y) are lattice points in the plane – in terms

[1] Hint: look up the proof of the "integral test" in any calculus book.
[2] Hint: $f(x) = x^\alpha$ is the one function whose antiderivative every calculus student knows.

of the number of interior lattice points and lattice points on the boundary. What is the precise statement?

7.8: Show that there is no equilateral triangle all of whose vertices are lattice points. (Apply Pick's theorem.)

7.9(O): The previous exercise would make a nice project for a high school geometry class: i.e., first asking the students to find a triangle, letting them come around to the belief that it is impossible, and then telling them about Pick's theorem (even just for triangles). Can you think of other interesting classroom exercises involving Pick's theorem?