4400/6400 PROBLEM SET 5

Recommendations: 4400 students should do at least 6 problems. 6400 students should do all but, perhaps, 5.8b).

5.1: Compute $\omega(2007)$, $\mu(2007)$, $\varphi(2007)$, $d(2007)$ and, for each $k \geq 1$, $\sigma_k(2007)$. (Of course the answer to the last will be in terms of k.)

5.2: Find all n for which $\varphi(n)$ is odd.

5.3: For each i, $1 \leq i \leq 10$, find all n for which $\varphi(n) = i$.

Comment: Part of the point of this problem is that it swiftly becomes a real pain to prove that your lists are complete. We will soon learn a better method...

5.4: Define the **Liouville λ function** for $n = p_1^{a_1} \cdots p_r^{a_r}$ as

$$\lambda(p_1^{a_1} \cdots p_r^{a_r}) = (-1)^{a_1 + \ldots + a_r}.$$

a)(E) Show that λ is completely multiplicative.
b) Give, with proof, a nice description of the divisor sum function $L(n) = \sum_{d|n} \lambda(d)$.[1]

5.5: Prove the Generalized Chinese Remainder Theorem, from §2.4 of the handout on multiplicative functions.

5.6: What is the largest currently known perfect number?

5.7: Show that $\sigma = \varphi * d$. (Hint: use $d = \mathbf{1} * \mathbf{1}$ and Möbius inversion.)

5.8G): We say that an arithmetical function g is the **Dirichlet inverse** of an arithmetical function f if $f * g = \delta$, i.e., if $g = f^{-1}$ with respect to the convolution multiplicative structure of the ring, let's say $\mathcal{D}$, of arithmetical functions. Thus, the functions which have Dirichlet inverses are precisely the units $\mathcal{D}^{\times}$ of $\mathcal{D}$.
a) Show that an arithmetical function f has a Dirichlet inverse f^{-1} iff $f(1) \neq 0$. (In particular, multiplicative functions are invertible.) In fact, show that the inverse is given recursively by $f^{-1}(1) = \frac{1}{f(1)}$ and

$$f^{-1}(n) = \frac{-1}{f(1)} \sum_{d|n,\ d<n} f\left(\frac{n}{d}\right) f^{-1}(d),\ n > 1.$$

b*) Suppose f and $f * g$ are multiplicative. Show that g is multiplicative.
c) Deduce that if f is a multiplicative function, then f^{-1} is also multiplicative, and in particular that the multiplicative functions form a subgroup of $\mathcal{D}^{\times}$.

[1] The letter Λ is reserved for a different arithmetical function, named after von Mangoldt, most pithily described as $\log * \mu$. It is an important auxiliary function in the study of $\pi(n)$, but we will probably not encounter it in our course.