4400/6400 PROBLEM SET 4

Recommendations: Both 4400 and 6400 students should, ideally, solve all of the first eight problems. 4400 students should make reasonable guesses at answers to at least two of the four questions (I mean, questions 4Q.1-4.Q4, not "Why is this night unlike all other nights?" and so forth). 6400 students should make reasonable guesses for at least three of these questions. If anyone has extra time, try solving $aX^2 + bY^2 = cZ^2$ for various values of a, b and c: it's good fun!

4.1: State Lagrange's Four Squares Theorem and the Legendre-Gauss Three Squares Theorem. (These were stated in class, and of course not yet proved. But I want you to be able to quote these theorems from memory by the end of the course!)

4.2: Deduce "half" of the Four Squares Theorem directly from the Three Squares Theorem: show that every *odd* positive integer n is of the form $0^2 + x^2 + y^2 + z^2$ or of the form $1^2 + x^2 + y^2 + z^2$.

4.3: Prove Euler's Identity: for integers $a_1, \ldots, a_4, b_1, \ldots, b_4$ – and in fact, for any eight elements in a commutative ring R – we have

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) = (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4)^2 +$$

$$(a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3)^2 + (a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2)^2 + (a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1)^2.$$

Conclude that – as *is* the case for sums of two squares and **is not** the case for sums of three squares – the product of two sums of four squares is again a sum of four squares.

4.4: Combine the previous two problems to deduce the Four Squares Theorem. (Of course this leaves the Three Squares Theorem unproved!)

4.5: In any primitive Pythaogorean triple (x, y, z) show that exactly one of x, y and z is divisible by 5.

The next few questions pertain to finding all the integer solutions (x, y, z) to on a Diophantine equation $aX^2 + bY^2 = cZ^2$ for $a, b, c \in \mathbb{Z}$. Note that as with the $a = b = c = 1$ case, if (X, Y, Z) is a solution, so is (CX, CY, CZ) for any solution. In the following problems we will regard two solutions as *equivalent* if one can be obtained from the other as scaling: this simplifies matters considerably.

4.6: Show that if $a = c$, there are always nontrivial integer solutions. Find the general integer solution (up to scaling) of $3X^2 + 7Y^2 = 3Z^2$. (Suggestion: Reduce to finding all rational points on the ellipse $x^2 + 7/3y^2 = 1$; find one rather easy solution, and then draw all lines with rational slope through it.)

4.7: Show that there are no nontrivial solutions if $a, b > 0$, $c < 0$.

4.8: Show that although $3X^2 + 5Y^2 = Z^2$ has real solutions, there is no nontrivial integer solution. (Suggestion: if there is any nontrivial integer solution, then after rescaling there is an integer solution (x, y, z) with x, y and z not all divisible by 5. Now work modulo 5 and obtain a contradiction.)

Remark: If you have the time, I highly encourage you to experiment with other values of a, b and c. In each case, try to find either a nontrivial integer solution or an "obstruction" either in the form of an incompatibility of signs (as in 4.7) or a contradiction using congruences (as in 4.8), or both! A fascinating question: will you always succeed?

The following problems are all closely related to each other and to the last two topics we have discussed in class. To prove the answers will take more time than you probably have to devote to this one class in the next six days. So here's a fun alternative: I will write down the problems and you will try to figure out what the answers are, either computationally or by any means necessary. Then you will turn in *conjectures*, and later on we will try to prove your conjectures.

4.Q1: Which integers n are sums of two *nonzero* squares?

4.Q2: What are the possible integer lengths z of the hypotenuse of a right triangle with integers legs x and y?

(Note that these two questions are really the same!)

4.Q3: Call a representation of N as $a^2 + b^2$ **primitive** if a and b are relatively prime. Which integers n admit a primitive representation as a sum of two squares?

4.Q4: In how many ways can we write an integer n as a sum of two squares? I will leave the exact statement to you but you should explain how many times you are counting minor variations in the representation, e.g. does

$$a^2+b^2, (-a)^2+b^2, a^2+(-b)^2, (-a)^2+(-b)^2, b^2+a^2, (-b)^2+a^2, b^2+(-a)^2, (-b)^2+(-a)^2$$

count as one representation or 8 representations?