1. Metric Geometry

A metric on a set $X$ is a function $d : X \times X \to [0, \infty)$ satisfying:

(M1) $d(x, y) = 0 \iff x = y$.
(M2) For all $x, y \in X$, $d(x, y) = d(y, x)$.
(M3) (Triangle Inequality) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

A metric space is a pair $(X, d)$ consisting of a set $X$ and a metric $d$ on $X$. By the usual abuse of notation, when only one metric on $X$ is under discussion we will typically refer to “the metric space $X$.”

Example 1.1. (Discrete Metric) Let $X$ be a set, and for any $x, y \in X$, put

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}.$$  

This is a metric on $X$ which we call the discrete metric. We warn the reader that we will later study a property of metric spaces called discreteness. A set endowed with the discrete metric is a discrete space, but there are discrete metric spaces which are not endowed with the discrete metric.

In general showing that a given function $d : X \times X \to \mathbb{R}$ is a metric is nontrivial. More precisely verifying the Triangle Inequality is often nontrivial; (M1) and (M2) are usually very easy to check.

Example 1.2. a) Let $X = \mathbb{R}$ and take $d(x, y) = |x - y|$. This is the most basic and important example.

b) More generally, let $N \geq 1$, let $X = \mathbb{R}^N$, and take $d(x, y) = ||x - y|| = \sqrt{\sum_{i=1}^{N} (x_i - y_i)^2}$. It is very well known but not very obvious that $d$ satisfies the triangle inequality. This is a special case of Minkowski’s Inequality, which will be studied later. c) More generally let $p \in [1, \infty)$, let $N \geq 1$, let $X = \mathbb{R}^N$ and take

$$d_p(x, y) = ||x - y||_p = \left( \sum_{i=1}^{N} (x_i - y_i)^p \right)^{\frac{1}{p}}.$$  

Thanks to Kaj Hansen for pointing out typos in these notes.
Now the assertion that $d_p$ satisfies the triangle inequality is precisely Minkowski’s Inequality.

**Exercise 1.1.** a) Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Show that the function $d_{X \times Y} : (X \times Y) \times (X \times Y) \to \mathbb{R}$, $((x_1, y_1), (x_2, y_2)) \mapsto \max(d_X(x_1, x_2), d_Y(y_1, y_2))$ is a metric on $X \times Y$.

b) Extend the result of part a) to finitely many metric spaces $(X_1, d_{X_1}), \ldots, (X_n, d_{X_n})$.

c) Let $N \geq 1$, let $X = \mathbb{R}^N$ and take $d_\infty(x, y) = \max_{1 \leq i \leq N} |x_i - y_i|$. Show that $d_\infty$ is a metric.

d) For each fixed $x, y \in \mathbb{R}^N$, show $d_\infty(x, y) = \lim_{p \to \infty} d_p(x, y)$.

Use this to give a second (more complicated) proof of part c).

**Example 1.3.** Let $(X, d)$ be a metric space, and let $Y \subset X$ be any subset. Show that the restricted function $d : Y \times Y \to \mathbb{R}$ is a metric function on $Y$.

**Example 1.4.** Let $a \leq b \in \mathbb{R}$. Let $C[a, b]$ be the set of all continuous functions $f : [a, b] \to \mathbb{R}$. For $f \in C[a, b]$, let $||f|| = \sup_{x \in [a, b]} |f(x)|$.

Then $d(f, g) = ||f - g||$ is a metric function on $C[a, b]$.

Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is an isometric embedding if for all $x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$. That is, the distance between any two points in $X$ is the same as the distance between their images under $f$. An isometry is a surjective isometric embedding.

**Exercise 1.2.** a) Show that every isometric embedding is injective.

b) Show that every isometry is bijective and thus admits an inverse function.

c) Show that if $f : (X, d_X) \to (Y, d_Y)$ is an isometry, so is $f^{-1} : (Y, d_Y) \to (X, d_X)$.

For metric spaces $X$ and $Y$, let $\text{Iso}(X, Y)$ denote the set of all isometries from $X$ to $Y$. Put $\text{Iso}(X) = \text{Iso}(X, X)$, the isometries from $X$ to itself. According to more general mathematical usage we ought to call elements of $\text{Iso}(X)$ “automorphisms” of $X$...but almost no one does.

**Exercise 1.3.** a) Let $f : X \to Y$ and $g : Y \to Z$ be isometric embeddings. Show that $g \circ f : X \to Z$ is an isometric embedding.

b) Show that $\text{Iso} X$ forms a group under composition.

c) Let $X$ be a set endowed with the discrete metric. Show that $\text{Iso} X = \text{Sym} X$ is the group of all bijections $f : X \to X$.

d) Can you identify the isometry group of $\mathbb{R}$? Of Euclidean $N$-space?

**Exercise 1.4.** a) Let $X$ be a set with $N \geq 1$ elements endowed with the discrete metric. Find an isometric embedding $X \hookrightarrow \mathbb{R}^{N-1}$.

b) Show that there is no isometric embedding $X \hookrightarrow \mathbb{R}^{N-2}$.

c) Deduce that an infinite set endowed with the discrete metric is not isometric to any subset of a Euclidean space.

**Exercise 1.5.** a) Let $G$ be a finite group. Show that there is a finite metric space $X$ such that $\text{Iso} X \cong G$ (isomorphism of groups).

b) Prove or disprove: for every group $G$, there is a metric space $X$ with $\text{Iso} X \cong G$?
Let \( A \) be a nonempty subset of a metric space \( X \). The diameter of \( A \) is
\[
\text{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}.
\]

**Exercise 1.6.**

a) Show that \( \text{diam} A = 0 \) iff \( A \) consists of a single point.
b) Show that \( A \) is bounded iff \( \text{diam} A < \infty \).
c) Show that for any \( x \in X \) and \( \epsilon > 0 \), \( \text{diam} B(x, \epsilon) \leq 2\epsilon \).

1. Further Exercises.

**Exercise 1.7.** Recall: for sets \( X, Y \) we have the symmetric difference
\[
X \Delta Y = (X \setminus Y) \bigsqcup (Y \setminus X),
\]
the set of elements belonging to exactly one of \( X \) and \( Y \) (“exclusive or”). Let \( S \) be a finite set, and let \( 2^S \) be the set of all subsets of \( S \). Show that
\[
d : 2^S \times 2^S \to \mathbb{N}, \; d(X, Y) = X \Delta Y
\]
is a metric function on \( 2^S \), called the **Hamming metric**.

**Exercise 1.8.** Let \( X \) be a metric space.

a) Suppose \( \#X \leq 2 \). Show that there is an isometric embedding \( X \hookrightarrow \mathbb{R} \).
b) Let \( d \) be a metric function on the set \( X = \{a, b, c\} \). Show that up to relabelling the points we may assume
\[
d_1 = d(a, b) \leq d_2 = d(b, c) \leq d_3 = d(a, c).
\]
Find necessary and sufficient conditions on \( d_1, d_2, d_3 \) such that there is an isometric embedding \( X \hookrightarrow \mathbb{R} \). Show that there is always an isometric embedding \( X \hookrightarrow \mathbb{R}^2 \).
c) Let \( X = \{\bullet, a, b, c\} \) be a set with four elements. Show that
\[
d(\bullet, a) = d(\bullet, b) = d(\bullet, c) = 1, \; d(a, b) = d(a, c) = d(b, c) = 2
\]
gives a metric function on \( X \). Show that there is no isometric embedding of \( X \) into any Euclidean space.

**Exercise 1.9.** Let \( G = (V, E) \) be a connected graph. Define a function \( d : V \times V \to \mathbb{R} \) by taking \( d(P, Q) \) to be the length of the shortest path connecting \( P \) to \( Q \).

a) Show that \( d \) is a metric function on \( V \).
b) Show that the metric of Exercise ??c) arises in this way.
c) Find necessary and/or sufficient conditions for the metric induced by a finite connected graph to be isometric to a subspace of some Euclidean space.

**Exercise 1.10.** Let \( d_1, d_2 : X \times X \to \mathbb{R} \) be metric functions.

a) Show that \( d_1 + d_2 : X \times X \to \mathbb{R} \) is a metric function.
b) Show that \( \max(d_1, d_2) : X \times X \to \mathbb{R} \) is a metric function.

**Exercise 1.11.**

a) Show that for any \( x, y \in \mathbb{R} \) there is \( f \in \text{Iso} \mathbb{R} \) such that \( f(x) = y \).
b) Show that for any \( x \in \mathbb{R} \), there are exactly two isometries \( f \) of \( \mathbb{R} \) such that \( f(x) = x \).
c) Show that every isometric embedding \( f : \mathbb{R} \to \mathbb{R} \) is an isometry.
d) Find a metric space \( X \) and an isometric embedding \( f : X \to X \) which is not surjective.

**Exercise 1.12.** Consider the following property of a function \( d : X \times X \to [0, \infty) \):

(M1’) For all \( x \in X \), \( d(x, x) = 0 \).

A **pseudometric function** is a function \( d : X \times X \to [0, \infty) \) satisfying (M1’),
(M2) and (M3), and a **pseudometric space** is a pair \((X, d)\) consisting of a set \(X\) and a pseudometric function \(d\) on \(X\).

a) Show that every set \(X\) admits a pseudometric function.

b) Let \((X, d)\) be a pseudometric space. Define a relation \(\sim\) on \(X\) by \(x \sim y\) iff \(d(x, y) = 0\). Show that \(\sim\) is an equivalence relation.

c) Show that the pseudometric function is well-defined on the set \(X/\sim\) of \(\sim\)-equivalence classes: that is, if \(x \sim x'\) and \(y \sim y'\) then \(d(x, y) = d(x', y')\). Show that \(d\) is a metric function on \(X/\sim\).

1.2. Constructing Metrics.

**Lemma 1.** Let \((X, d)\) be a metric space, and let \(f : \mathbb{R}^+ \to \mathbb{R}^+\) be an increasing, concave function – i.e., \(-f\) is convex – with \(f(0) = 0\). Then \(d_f = f \circ d\) is a metric on \(X\).

**Proof.** The only nontrivial verification is the triangle inequality. Let \(x, y, z \in X\).

Since \(d\) is a metric, we have
\[
d(x, z) \leq d(x, y) + d(y, z).
\]

Since \(f\) is increasing, we have
\[
d_f(x, z) = f(d(x, z)) \leq f(d(x, y) + d(y, z)).
\]

Since \(-f\) is convex and \(f(0) = 0\), by the Generalized Two Secant Inequality or the Interlaced Secant Inequality, we have for all \(a \geq 0\) and all \(t > 0\) that
\[
\frac{f(a + t) - f(a)}{(a + t) - a} \leq \frac{f(t)}{t - 0}
\]
and thus
\[
f(a + t) \leq f(a) + f(t).
\]

Taking \(a = d(x, y)\) and \(t = d(y, z)\) and combining (??) and (??), we get
\[
d_f(x, z) \leq f(d(x, y) + d(y, z)) \leq d_f(x, y) + d_f(y, z).
\]

**Corollary 2.** Let \((X, d)\) be a metric space, and let \(\alpha > 0\). Let \(d_\alpha : X \times X \to \mathbb{R}\) be given by
\[
d_\alpha(x, y) = \frac{\alpha d(x, y)}{d(x, y) + 1}.
\]

Then \(d_\alpha\) is a metric on \(X\) and \(\text{diam}(X, d_\alpha) \leq \alpha\).

**Exercise 1.13.** Prove it.

**Proposition 3.** Let \(\{(X_i, d_i)\}_{i \in I}\) be an indexed family of metric spaces. Let \(X = \prod_{i \in I} X_i\). We define \(d : X \to \mathbb{R} \cup \{\infty\}\) by
\[
d(x, y) = \sup_{i \in I} d(x_i, y_i).
\]

If \(d(X \times X) \subseteq \mathbb{R}\), then \(d\) is a metric on \(X\).
2. Metric Topology

Let \( X \) be a metric space.

For \( x \in X \) and \( \epsilon \geq 0 \) we define the open ball
\[
B^o(x, \epsilon) = \{ y \in X \mid d(x, y) < \epsilon \}
\]
and the closed ball
\[
B^*(x, \epsilon) = \{ y \in X \mid d(x, y) \leq \epsilon \}.
\]
Notice that
\[
B^o(x, 0) = \emptyset, \quad B^*(x, 0) = \{ x \}.
\]
A subset \( Y \) of a metric space \( X \) is open if for all \( y \in Y \), there is \( \epsilon > 0 \) such that
\[
B^o(y, \epsilon) \subset Y.
\]
A subset \( Y \) of a metric space \( X \) is closed if its complement
\[
X \setminus Y = \{ x \in X \mid x \notin Y \}
\]
is open.

Exercise 2.1. Find a subset \( X \subset \mathbb{R} \) which is:
(i) both open and closed.
(ii) open and not closed.
(iii) closed and not open.
(iv) neither open nor closed.

Proposition 4. Let \( X \) be a metric space, and let \( \{ Y_i \}_{i \in I} \) be subsets of \( X \).

a) The union \( Y = \bigcup_{i \in I} Y_i \) is an open subset of \( X \).
b) If \( I \) is nonempty and finite, then the intersection \( Z = \bigcap_{i \in I} Y_i \) is an open subset of \( X \).

Proof. a) If \( y \in Y \), then \( y \in Y_i \) for at least one \( i \). Thus there is \( \epsilon > 0 \) such that
\[
B^o(y, \epsilon) \subset Y \subset Y_i.
\]
b) We may assume that \( I = \{ 1, \ldots, n \} \) for some \( n \in \mathbb{Z}^+ \). Let \( y \in Z \). Then for \( 1 \leq i \leq n \), there is \( \epsilon_i > 0 \) such that \( B^o(y, \epsilon_i) \subset Y_i \). Then \( \epsilon = \min_{1 \leq i \leq n} \epsilon_i > 0 \) and \( B^o(y, \epsilon) \subset B^o(y, \epsilon_i) \subset Y_i \) for all \( 1 \leq i \leq n \), so \( B^o(y, \epsilon) \subset \bigcap_{i=1}^n Y_i = Z \). \( \square \)

Let \( X \) be a set, and let \( \tau \subset 2^X \) be family of subsets of \( X \). We say that \( \tau \) is a topology if:
(\text{T1}) \( \emptyset, X \in \tau; \)
(\text{T2}) For any set \( I \), if \( Y_i \in \tau \) for all \( i \in I \) then \( \bigcup_{i \in I} Y_i \in \tau; \)
(\text{T3}) For any nonempty finite set \( I \), if \( Y_i \in \tau \) for all \( i \in I \), then \( \bigcap_{i \in I} Y_i \in \tau. \)

The axioms (T2) and (T3) are usually referred to as “arbitrary unions of open sets are open” and “finite intersections of open sets are open”, respectively.

In this language, Proposition 2.1 may be rephrased as follows.

Proposition 5. In a metric space \((X, d)\), the open sets form a topology on \( X \).
We say that two metrics $d_1$ and $d_2$ on the same set $X$ are topologically equivalent if they determine the same topology: that is, every set which is open with respect to $d_1$ is open with respect to $d_2$.

**Example 2.1.** In $\mathbb{R}$, for $n \in \mathbb{Z}^+$, let $Y_n = (\frac{1}{n}, \frac{1}{n})$. Then each $Y_n$ is open but $\bigcap_{n=1}^{\infty} Y_n = \{0\}$ is not. This shows that infinite intersections of open subsets need not be open.

**Exercise 2.2.**
(a) Show that finite unions of closed sets are closed.
(b) Show that arbitrary intersections of closed sets are closed.
(c) Exhibit an infinite union of closed subsets which is not closed.

**Exercise 2.3.** A metric space $X$ is discrete if every subset $Y \subset X$ is open.
(a) Show that any set endowed with the discrete metric is a discrete metric space.
(b) A metric space $X$ is uniformly discrete if there is $\epsilon > 0$ such that for all $x \neq y \in X$, $d(x, y) \geq \epsilon$. Show: every uniformly discrete metric space is discrete.
(c) Let $X = \{\frac{1}{n}\}_{n=1}^{\infty}$ as a subspace of $\mathbb{R}$. Show that $X$ is discrete but not uniformly discrete.

**Proposition 6.**
(a) Open balls are open sets.
(b) A subset $Y$ of a metric space $X$ is open if it is a union of open balls.

**Proof.**
(a) Let $x \in X$, let $\epsilon > 0$, and let $y \in B^\circ(x, \epsilon)$. We claim that $B^\circ(y, \epsilon - d(x, y)) \subset B^\circ(x, \epsilon)$. Indeed, if $z \in B^\circ(y, \epsilon - d(x, y))$, then $d(y, z) < \epsilon - d(x, y)$, so $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + (\epsilon - d(x, y)) = \epsilon$.

(b) If $Y$ is open, then for all $y \in Y$, there is $\epsilon_y > 0$ such that $B^\circ(y, \epsilon_y) \subset Y$. It follows that $Y = \bigcup_{y \in Y} B^\circ(y, \epsilon_y)$. The fact that a union of open balls is open follows from part a) and the previous result. \qed

**Lemma 7.** Let $Y$ be a subset of a metric space $X$. Then the map $U \mapsto U \cap Y$ is a surjective map from the open subsets of $X$ to the open subsets of $Y$.

**Exercise 2.4.** Prove it. (Hint: for any $y \in Y$ and $\epsilon > 0$, let $B_X^\circ(y, \epsilon) = \{x \in X \mid d(x, y) < \epsilon\}$ and let $B_Y^\circ(y, \epsilon) = \{x \in Y \mid d(x, y) < \epsilon\}$. Then $B_X^\circ(y, \epsilon) = B_Y^\circ(y, \epsilon)$.)

Let $X$ be a metric space, and let $Y \subset X$. We define the interior of $Y$ as

$$Y^\circ = \{y \in Y \mid \exists \epsilon > 0 \text{ such that } B^\circ(y, \epsilon) \subset Y\}.$$  

In words, the interior of a set is the collection of points that not only belong to the set, but for which some open ball around the point is entirely contained in the set.

**Lemma 8.** Let $Y, Z$ be subsets of a metric space $X$.
(a) All of the following hold:
(i) $Y^\circ \subset Y$.
(ii) If $Y \subset Z$, then $Y^\circ \subset Z^\circ$.
(iii) $(Y^\circ)^\circ = Y^\circ$.
(b) The interior $Y^\circ$ is the largest open subset of $Y$: that is, $Y^\circ$ is an open subset of $Y$ and if $U \subset Y$ is open, then $U \subset Y^\circ$.
(c) $Y$ is open iff $Y = Y^\circ$.

**Exercise 2.5.** Prove it.
We say that a subset $Y$ is a **neighborhood** of $x \in X$ if $x \in Y^\circ$. In particular, a subset is open precisely when it is a neighborhood of each of its points. (This terminology introduces nothing essentially new. Nevertheless the situation it encapsulates is ubiquitous in this subject, so we will find the term quite useful.)

Let $X$ be a metric space, and let $Y \subset X$. A point $x \in X$ is an **adherent point** of $Y$ if every neighborhood $\mathcal{N}$ of $x$ intersects $Y$: i.e., $\mathcal{N} \cap Y \neq \emptyset$. Equivalently, for all $\epsilon > 0$, $B(x, \epsilon) \cap Y \neq \emptyset$.

We follow up this definition with another, rather subtly different one, that we will fully explore later, but it seems helpful to point out the distinction now. For $Y \subset X$, a point $x \in X$ is a **limit point** of $Y$ if every neighborhood of $X$ contains infinitely many points of $Y$. Equivalently, for all $\epsilon > 0$, we have $(B^o(x, \epsilon) \setminus \{x\}) \cap Y \neq \emptyset$.

In particular, every $y \in Y$ is an adherent point of $Y$ but not necessarily a limit point. For instance, if $Y$ is finite then it has no limit points.

The following is the most basic and important result of the entire section.

**Proposition 9.** For a subset $Y$ of a metric space $X$, the following are equivalent:

(i) $Y$ is closed: i.e., $X \setminus Y$ is open.

(ii) $Y$ contains all of its adherent points.

(iii) $Y$ contains all of its limit points.

**Proof.** (i) $\implies$ (ii): Suppose that $X \setminus Y$ is open, and let $x \in X \setminus Y$. Then there is $\epsilon > 0$ such that $B^o(x, \epsilon) \subset X \setminus Y$, and thus $B^o(x, \epsilon)$ does not intersect $Y$, i.e., $x$ is not an adherent point of $Y$.

(ii) $\implies$ (iii): Since every limit point is an adherent point, this is immediate.

(iii) $\implies$ (i): Suppose $Y$ contains all its limit points, and let $x \in X \setminus Y$. Then $x$ is not a limit point of $Y$, so there is $\epsilon > 0$ such that $(B^o(x, \epsilon) \setminus \{x\}) \cap Y = \emptyset$.

Since $x \notin Y$ this implies $B^o(x, \epsilon) \cap Y = \emptyset$ and thus $B^o(x, \epsilon) \subset X \setminus Y$. Thus $X \setminus Y$ contains an open ball around each of its points, so is open, so $Y$ is closed.

For a subset $Y$ of a metric space $X$, we define its closure of $Y$ as

$\overline{Y} = Y \cup \{\text{all adherent points of } Y\} = Y \cup \{\text{all limit points of } Y\}$.

**Lemma 10.** Let $Y, Z$ be subsets of a metric space $X$.

a) All of the following hold:

(KC1) $Y \subset \overline{Y}$.

(KC2) If $Y \subset Z$, then $\overline{Y} \subset \overline{Z}$.

(KC3) $\overline{\overline{Y}} = \overline{Y}$.

b) The closure $\overline{Y}$ is the smallest closed set containing $Y$: that is, $\overline{Y}$ is closed, contains $Y$, and if $Y \subset Z$ is closed, then $\overline{Y} \subset \overline{Z}$.

**Exercise 2.6.** Prove it.

**Lemma 11.** Let $Y, Z$ be subsets of a metric space $X$. Then:

a) $\overline{Y \cup Z} = \overline{Y} \cup \overline{Z}$.

b) $(Y \cap Z)^\circ = Y^\circ \cap Z^\circ$. 

Proof. a) Since $Y \cup Z$ is a finite union of closed sets, it is closed. Clearly $Y \cup Z \supset Y \cup Z$. So

$$Y \cup Z \subset Y \cup Z.$$  

Conversely, since $Y \subset Y \cup Z$ we have $Y \subset Y \cup Z$; similarly $Z \subset Y \cup Z$. So

$$Y \cup Z \supset Y \cup Z.$$

b) $Y^o \cap Z^o$ is a finite intersection of open sets, hence open. Clearly $Y^o \cap Z^o \subset Y \cap Z$. So

$$Y^o \cap Z^o \subset (Y \cap Z)^o.$$

Conversely, since $Y \cap Z \subset Y$, we have $(Y \cap Z)^o \subset Y^o$; similarly $(Y \cap Z)^o \subset Z^o$. So

$$(Y \cap Z)^o \subset Y^o \cap Z^o.$$ \[\Box\]

The similarity between the proofs of parts a) and b) of the preceding result is meant to drive home the point that just as open and closed are “dual notions” – one gets from one to the other via taking complements – so are interiors and closures.

**Proposition 12.** Let $Y$ be a subset of a metric space $Z$. Then

$$Y^o = X \setminus X \setminus Y$$

and

$$Y = X \setminus (X \setminus Y)^o.$$  

Proof. We will prove the first identity and leave the second to the reader. Our strategy is to show that $X \setminus X \setminus Y$ is the largest open subset of $Y$ and apply AX. Since $X \setminus X \setminus Y$ is the complement of a closed set, it is open. Moreover, if $x \in X \setminus X \setminus Y$, then $x \notin X \setminus Y \supset X \setminus Y$, so $x \in Y$. Now let $U \subset Y$ be open. Then $X \setminus U$ is closed and contains $X \setminus Y$, so it contains $X \setminus X \setminus Y$. Taking complements again we get $U \subset X \setminus X \setminus Y$. \[\Box\]

**Proposition 13.** For a subset $Y$ of a metric space $X$, consider the following:

(i) $B_1(Y) = Y \setminus Y^o$.

(ii) $B_2(Y) = Y \cap X \setminus Y$.

(iii) $B_3(Y) = \{x \in X \mid \text{every neighborhood } N \text{ of } x \text{ intersects both } Y \text{ and } X \setminus Y\}$. Then $B_1(Y) = B_2(Y) = B_3(Y)$ is a closed subset of $X$, called the **boundary of $Y$** and denoted $\partial Y$.

**Exercise 2.7.** Prove it.

**Exercise 2.8.** Let $Y$ be a subset of a metric space $X$.

a) Show $X = X^o \bigsqcup \partial X$ (disjoint union).

b) Show $(\partial X)^o = \emptyset$.

c) Show that $\partial(\partial Y) = \partial Y$.

**Example 2.2.** Let $X = \mathbb{R}$, $A = (-\infty, 0)$ and $B = [0, \infty)$. Then $\partial A = \partial B = \{0\}$, and

$$\partial(A \cup B) = \partial \mathbb{R} = \emptyset \neq \{0\} = (\partial A) \cup (\partial B);$$

$$\partial(A \cap B) = \partial \emptyset = \emptyset \neq \{0\} = (\partial A) \cap (\partial B).$$

Thus the boundary is not as well-behaved as either the closure or interior.
A subset $Y$ of a metric space $X$ is **dense** if $\overline{Y} = X$: explicitly, if for all $x \in X$ and all $\epsilon > 0$, $B^\circ(x, \epsilon)$ intersects $Y$.

**Example 2.3.** Let $X$ be a discrete metric space. The only dense subset of $X$ is $X$ itself.

**Example 2.4.** The subset $\mathbb{Q}^N = (x_1, \ldots, x_N)$ is dense in $\mathbb{R}^N$.

The **weight** of a metric space is the least cardinality of a dense subspace.

**Exercise 2.9.**

a) Show that the weight of any discrete metric space is its cardinality.

b) Show that the weight of any finite metric space is its cardinality.

c) Show that every cardinal number arises as the weight of a metric space.

Explicit use of cardinal arithmetic is popular in some circles but not in others. Much more commonly used is the following special case: a metric space is **separable** if it admits a countable dense subspace. Thus the previous example shows that Euclidean $N$-space is separable, and a discrete space is separable iff it is countable.

2.1. Further Exercises.

**Exercise 2.10.** Let $Y$ be a subset of a metric space $X$. Show:

$$(Y^\circ)^\circ = Y^\circ$$

and

$$\overline{Y^\circ} = \overline{Y}.$$

**Exercise 2.11.** A subset $Y$ of a metric space $X$ is **regularly closed** if $Y = \overline{Y^\circ}$ and **regularly open** if $Y = (Y^\circ)^\circ$.

a) Show that every regularly closed set is closed, every regularly open set is open, and a set is regularly closed iff its complement is regularly open.

b) Show that a subset of $\mathbb{R}$ is regularly closed iff it is a disjoint union of closed intervals.

c) Show that for any subset $Y$ of a metric space $X$, $\overline{Y^\circ}$ is regularly closed and $Y^\circ$ is regularly open.

**Exercise 2.12.** A metric space is a **door space** if every subset is either open or closed (or both). In a topologically discrete space, every subset is both open and closed, so such spaces are door spaces, however of a rather uninteresting type. Show that there is a subset of $\mathbb{R}$ which, with the induced metric, is a door space which is not topologically discrete.

3. Convergence

In any set $X$, a sequence in $X$ is just a mapping $x : \mathbb{Z}^+ \to X$, $n \mapsto x_n$. If $X$ is endowed with a metric $d$, a sequence $x$ in $X$ is said to **converge** to an element $x$ of $X$ if for all $\epsilon > 0$, there exists an $N = N(\epsilon)$ such that for all $n \geq N$, $d(x, x_n) < \epsilon$. We denote this by $x \to x$ or $x_n \to x$.

**Exercise 3.1.** Let $x$ be a sequence in the metric space $X$, and let $L \in X$. Show that the following are equivalent.

a) The $x \to L$.

b) Every neighborhood $N$ of $x$ contains all but finitely many terms of the sequence. More formally, there is $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $x_n \in N$. 

Proposition 14. In any metric space, the limit of a convergent sequence is unique: if \( L, M \in X \) are such that \( x \to L \) and \( x \to M \), then \( L = M \).

Proof. Seeking a contradiction, we suppose \( L \neq M \) and put \( d = d(L, M) > 0 \).

Let \( B_1 = B^c(L, \frac{d}{2}) \) and \( B_2 = B^c(M, \frac{d}{2}) \), so \( B_1 \) and \( B_2 \) are disjoint. Let \( N_1 \) be such that if \( n \geq N_1 \), \( x_n \in B_1 \), let \( N_2 \) be such that if \( n \geq N_2 \), \( x_n \in B_2 \), and let \( N = \max(N_1, N_2) \). Then for all \( n \geq N \), \( x_n \in B_1 \cap B_2 = \emptyset \): contradiction! \( \square \)

A subsequence of \( x \) is obtained by choosing an infinite subset of \( \mathbb{Z}^+ \), writing the elements in increasing order as \( n_1, n_2, \ldots \) and then restricting the sequence to this subset, getting a new sequence \( y, k \mapsto y_k = x_{n_k} \).

Exercise 3.2. Let \( n : \mathbb{Z}^+ \to \mathbb{Z}^+ \) be strictly increasing: for all \( k_1 < k_2, n_{k_1} < n_{k_2} \).

Let \( x : \mathbb{X} \to X \) be a sequence in a set \( X \). Interpret the composite sequence \( x \circ n : \mathbb{Z}^+ \to X \) as a subsequence of \( x \). Show that every subsequence arises in this way, i.e., by precomposing the given sequence with a unique strictly increasing function \( n : \mathbb{Z}^+ \to \mathbb{Z}^+ \).

Exercise 3.3. Let \( x \) be a sequence in a metric space. 

a) Show that if \( x \) is convergent, so is every subsequence, and to the same limit.

b) Show that conversely, if every subsequence converges, then \( x \) converges. (Hint: in fact this is not a very interesting statement. Why?)

c) A more interesting converse would be: suppose that there is \( L \in X \) such that every subsequence of \( x \) which is convergent converges to \( L \). Then \( x \to L \). Show that this fails in \( \mathbb{R} \). Show however that it holds in \( [a, b] \subset \mathbb{R} \).

Let \( x \) be a sequence in a metric space \( X \). A point \( L \in X \) is a partial limit of \( x \) if every neighborhood \( \mathcal{N} \) of \( L \) contains infinitely many terms of the sequence: more formally, for all \( N \in \mathbb{Z}^+ \), there is \( n \geq N \) such that \( x_n \in \mathcal{N} \).

Lemma 15. For a sequence \( x \) in a metric space \( X \) and \( L \in X \), TFAE:

(i) \( L \) is a partial limit of \( x \).

(ii) There is a subsequence \( x_{n_k} \) converging to \( L \).

Proof. (i) Suppose \( L \) is a partial limit. Choose \( n_1 \in \mathbb{Z}^+ \) such that \( d(x_{n_1}, L) < 1 \). Having chosen \( n_k \in \mathbb{Z}^+ \), choose \( n_{k+1} > n_k \) such that \( d(x_{n_{k+1}}, L) < \frac{1}{k+1} \). Then \( x_{n_k} \to L \).

(ii) Let \( \mathcal{N} \) be any neighborhood of \( L \), so there is \( \epsilon > 0 \) such that \( L \subset B^\epsilon(L, \epsilon) \subset \mathcal{N} \). If \( x_{n_k} \to L \), then for every \( \epsilon > 0 \) and all sufficiently large \( k \), we have \( d(x_{n_k}, L) < \epsilon \), so infinitely many terms of the sequence lie in \( \mathcal{N} \). \( \square \)

The following basic result shows that closures in a metric space can be understood in terms of convergent sequences.

Proposition 16. Let \( Y \) be a subset of \( (X, d) \). For \( x \in X \), TFAE:

(i) \( x \in \overline{Y} \).

(ii) There exists a sequence \( x : \mathbb{Z}^+ \to Y \) such that \( x_n \to x \).

Proof. (i) \( \implies \) (ii): Suppose \( y \in \overline{Y} \), and let \( n \in \mathbb{Z}^+ \). There is \( x_n \in Y \) such that \( d(y, x_n) < \epsilon \). Then \( x_n \to y \).

\( \neg \) (i) \( \implies \) \( \neg \) (ii): Suppose \( y \notin \overline{Y} \); then there is \( \epsilon > 0 \) such that \( B^\epsilon(y, \epsilon) \cap Y = \emptyset \).

Then no sequence in \( Y \) can converge to \( y \). \( \square \)
Corollary 17. Let $X$ be a set, and let $d_1, d_2 : X \times X \to X$ be two metrics. Suppose that for every sequence $x \in X$ and every point $x \in X$, we have $x \overset{d_2}{\to} x \iff x \overset{d_1}{\to} x$: that is, the sequence $x$ converges to the point $x$ with respect to the metric $d_1$ if and only if it converges to the point $x$ with respect to the metric $d_2$. Then $d_1$ and $d_2$ are topologically equivalent: they have the same open sets.

Proof. Since the closed sets are precisely the complements of the open sets, it suffices to show that the closed sets with respect to $d_1$ are the same as the closed sets with respect to $d_2$. So let $Y \subset X$ and suppose that $Y$ is closed with respect to $d_1$. Then, still with respect to $d_1$, it is equal to its own closure, so by Proposition ?? for $x \in X$ we have that $x$ lies in $Y$ iff there is a sequence $y$ in $Y$ such that $y \to x$ with respect to $d_1$. But now by our assumption this latter characterization is also valid with respect to $d_2$, so $Y$ is closed with respect to $d_2$. And conversely, of course.

4. Continuity

Let $f : X \to Y$ be metric spaces, and let $x \in X$. We say $f$ is continuous at $x$ if for all $\epsilon > 0$, there is $\delta > 0$ such that for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$. We say $f$ is continuous if it is continuous at every $x \in X$.

Let $f : X \to Y$ be a map between metric spaces. A real number $C \geq 0$ is a Lipschitz constant for $f$ if for all $x, y \in X$, $d(f(x), f(y)) \leq Cd(x, y)$. A map $f$ is Lipschitz if some $C \geq 0$ is a Lipschitz constant for $f$.

Exercise 4.1. a) Show that a Lipschitz function is continuous.

b) Show that if $f$ is Lipschitz, the infimum of all Lipschitz constants for $f$ is a Lipschitz constant for $f$.

c) Show that an isometry is Lipschitz.

Lemma 18. For a map $f : X \to Y$ of metric spaces, TFAE:
(i) $f$ is continuous.
(ii) For every open subset $V \subset Y$, $f^{-1}(V)$ is open in $X$.

Proof. (i) $\implies$ (ii): Let $x \in f^{-1}(V)$, and choose $\epsilon > 0$ such that $B^o(f(x), \epsilon) \subset V$. Since $f$ is continuous at $x$, there is $\delta > 0$ such that for all $x' \in B^o(x, \delta)$, $f(x') \in B^o(f(x), \epsilon) \subset V$: that is, $B^o(x, \delta) \subset f^{-1}(V)$.

(ii) $\implies$ (i): Let $x \in X$, let $\epsilon > 0$, and let $V = B^o(f(x), \epsilon)$. Then $f^{-1}(V)$ is open and contains $x$, so there is $\delta > 0$ such that $B^o(x, \delta) \subset f^{-1}(V)$.

That is: for all $x'$ with $d(x, x') < \delta$, $d(f(x), f(x')) < \epsilon$.

A map $f : X \to Y$ between metric spaces is open if for all open subsets $U \subset X$, $f(U)$ is open in $Y$. A map $f : X \to Y$ is a homeomorphism if it is continuous, is bijective, and the inverse function $f^{-1} : Y \to X$ is continuous. A map $f : X \to Y$ is a topological embedding if it is continuous, injective and open.

Exercise 4.2. For a metric space $X$, let $X_D$ be the same underlying set endowed with the discrete metric.

a) Show that the identity map $1 : X_D \to X$ is continuous.

b) Show that the identity map $1 : X \to X_D$ is continuous iff $X$ is discrete (in the topological sense: every point of $x$ is an isolated point).
Example 4.1. a) Let $X$ be a metric space which is not discrete. Then (c.f. Exercise X.X) the identity map $1 : X_D \to X$ is bijective and continuous but not open. The identity map $1 : X \to X_D$ is bijective and open but not continuous.

b) The map $f : \mathbb{R} \to \mathbb{R}$ by $x \mapsto |x|$ is continuous – indeed, Lipschitz with $C = 1$ – but not open: $f(\mathbb{R}) = [0, \infty)$.

Exercise 4.3. Let $f : \mathbb{R} \to \mathbb{R}$.

a) Show that at least one of the following holds:

(i) $f$ is increasing: for all $x_1 \leq x_2$, $f(x_1) \leq f(x_2)$.
(ii) $f$ is decreasing: for all $x_1 \leq x_2$, $f(x_1) \geq f(x_2)$.
(iii) $f$ is of “$\Lambda$-type”: there are $a < b < c$ such that $f(a) < f(b) > f(c)$.
(iv) $f$ is of “$V$-type”: there are $a < b < c$ such that $f(a) > f(b) < f(c)$.

b) Suppose $f$ is a continuous injection. Show that $f$ is strictly increasing or strictly decreasing.

c) Let $f : \mathbb{R} \to \mathbb{R}$ be increasing. Show that for all $x \in \mathbb{R}$
\[
\sup_{y < x} f(y) \leq f(x) \leq \inf_{y > x} f(y).
\]
Show that
\[
\sup_{y < x} f(y) = f(x) = \inf_{y > x} f(y)
\]
iff $f$ is continuous at $x$.

d) Suppose $f$ is bijective and strictly increasing. Show that $f^{-1}$ is strictly increasing.

e) Show that if $f$ is strictly increasing and surjective, it is a homeomorphism. Deduce that every continuous bijection $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism.

Lemma 19. For a map $f : X \to Y$ between metric spaces, TFAE:

(i) $f$ is a homeomorphism.
(ii) $f$ is continuous, bijective and open.

Exercise 4.4. Prove it.

Proposition 20. Let $X, Y, Z$ be metric spaces and $f : X \to Y$, $g : Y \to Z$ be continuous maps. Then $g \circ f : X \to Z$ is continuous.

Proof. Let $W$ be open in $Z$. Since $g$ is continuous, $g^{-1}(W)$ is open in $Y$. Since $f$ is continuous, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is open in $X$.

Proposition 21. For a map $f : X \to Y$ of metric spaces, TFAE:

(i) $f$ is continuous.
(ii) If $x_n \to x$ in $X$, then $f(x_n) \to f(x)$ in $Y$.

Proof. (i) $\implies$ (ii) Let $\epsilon > 0$. Since $f$ is continuous, by Lemma ?? there is $\delta > 0$ such that if $x' \in B^\circ(x, \delta)$, $f(x') \in B^\circ(f(x), \epsilon)$. Since $x_n \to x$, there is $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $x_n \in B^\circ(x, \delta)$, and thus for all $n \geq N$, $f(x_n) \in B^\circ(f(x), \epsilon)$.

$\neg$ (i) $\implies \neg$ (ii): Suppose that $f$ is not continuous: then there is $x \in X$ and $\epsilon > 0$ such that for all $n \in \mathbb{Z}^+$, there is $x_n \in X$ with $d(x_n, x) < \frac{\epsilon}{n}$ and $d(f(x_n), f(x)) \geq \epsilon$.

Then $x_n \to x$ and $f(x_n)$ does not converge to $f(x)$.

In other words, continuous functions between metric spaces are precisely the functions which preserve limits of convergent sequences.

Exercise 4.5. a) Let $f : X \to Y$, $g : Y \to Z$ be maps of topological spaces. Let $x \in X$. Use $\epsilon$‘s and $\delta$‘s to show that if $f$ is continuous at $x$ and $g$ is continuous at
$f(x)$ then $g \circ f$ is continuous at $x$. Deduce another proof of Proposition ?? using the $(\varepsilon, \delta)$-definition of continuity.

b) Give (yet) another proof of Proposition ?? using Proposition ??.

In higher mathematics, one often meets the phenomenon of rival definitions which are equivalent in a given context (but may not be in other contexts of interest). Often a key part of learning a new subject is learning which versions of definitions give rise to the shortest, most transparent proofs of basic facts. When one definition makes a certain proposition harder to prove than another definition, it may be a sign that in some other context these definitions are not equivalent and the proposition is true using one but not the other definition. We will see this kind of phenomenon often in the transition from metric spaces to topological spaces. However, in the present context, all definitions in sight lead to immediate, straightforward proofs of “compositions of continuous functions are continuous”. And indeed, though the concept of a continuous function can be made in many different general contexts (we will meet some, but not all, of these later), to the best of my knowledge it is always clear that compositions of continuous functions are continuous.

**Lemma 22.** Let $(X, \rho_X)$ be a metric space, $(Y, \rho_Y)$ be a complete metric space, $Z \subset X$ a dense subset and $f : Z \rightarrow Y$ a continuous function.

a) There exists at most one extension of $f$ to a continuous function $F : X \rightarrow Y$.

(N.B.: This holds for any topological space $X$ and any Hausdorff space $Y$.)

b) $f$ is uniformly continuous $\implies$ $f$ extends to a uniformly continuous $F : X \rightarrow Y$.

c) If $f$ is an isometric embedding, then its extension $F$ is an isometric embedding.

4.1. Further Exercises.

**Exercise 4.6.** Let $X$ be a metric space, and let $f, g : X \rightarrow \mathbb{R}$ be continuous functions. Show that $\{x \in X \mid f(x) < g(x)\}$ is open and $\{x \in X \mid f(x) \leq g(x)\}$ is closed.

**Exercise 4.7.**

a) Let $X$ be a metric space, and let $Y \subset X$. Let $1_Y : X \rightarrow \mathbb{R}$ be the characteristic function of $Y$: for $x \in X$, $1_Y(x) = 1$ if $x \in Y$ and $0$ otherwise. Show that $1_Y$ is not continuous at $x \in X$ iff $x \in \partial Y$.

b) Let $Y \subset \mathbb{R}^N$ be a bounded subset. Deduce that $1_Y$ is Riemann integrable iff $\partial Y$ has measure zero. (Such sets $Y$ are called **Jordan measurable**.)

5. Equivalent Metrics

It often happens in geometry and analysis that there is more than one natural metric on a set $X$ and one wants to compare properties of these different metrics. Thus we are led to study equivalence relations on the class of metrics on a given set...but in fact it is part of the natural richness of the subject that there is more than one natural equivalence relation. We have already met the coarsest one we will consider here: two metrics $d_1$ and $d_2$ on $X$ are **topologically equivalent** if they determine the same topology; equivalently, in view of $X$, for all sequences $x$ in $X$ and points $x$ of $X$, we have $x \xrightarrow{d_1} x \iff x \xrightarrow{d_2} x$. Since continuity is characterized in terms of open sets, equivalent metrics on $X$ give rise to the same class of continuous functions on $X$ (with values in any metric space $Y$).

**Lemma 23.** Two metrics $d_1$ and $d_2$ on a set $X$ are topologically equivalent iff the identity function $1_X : (X, d_1) \rightarrow (X, d_2)$ is a homeomorphism.
To say that $1_X$ is a homeomorphism is to say that $1_X$ is continuous from $(X,d_1)$ to $(X,d_2)$ and that its inverse – which also happens to be $1_X$! – is continuous from $(X,d_2)$ to $(X,d_1)$. This means that every $d_2$-open set is $d_1$-open and every $d_1$-open set is $d_2$-open.

The above simple reformulation of topological equivalence suggests other, more stringent notions of equivalence of metrics $d_1$ and $d_2$, in terms of requiring $1_X : (X,d_1) \to (X,d_2)$ to have stronger continuity properties. Namely, we say that two metrics $d_1$ and $d_2$ are uniformly equivalent (resp. Lipschitz equivalent) if $1_X$ is uniformly continuous (resp. Lipschitz and with a Lipschitz inverse).

**Lemma 24.** Let $d_1$ and $d_2$ be metrics on a set $X$.

a) The metrics $d_1$ and $d_2$ are uniformly equivalent iff for all $\epsilon > 0$ there are $\delta_1, \delta_2 > 0$ such that for all $x_1,x_2 \in X$ we have

$$d_1(x_1,x_2) \leq \delta_1 \implies d_2(x_1,x_2) \leq \epsilon \quad \text{and} \quad d_2(x_1,x_2) \leq \delta_2 \implies d_1(x_1,x_2) \leq \epsilon.$$

b) The metrics $d_1$ and $d_2$ are Lipschitz equivalent iff there are constants $C_1, C_2 \in (0, \infty)$ such that for all $x_1,x_2 \in X$ we have

$$C_1d_2(x_1,x_2) \leq d_1(x_1,x_2) \leq C_2d_2(x_1,x_2).$$

**Exercise 5.1.** Prove it.

**Remark 25.** The typical textbook treatment of metric topology is not so careful on this point: one must read carefully to see which of these equivalence relations is meant by “equivalent metrics”.

**Exercise 5.2.**

a) Explain how the existence of a homeomorphism of metric spaces $f : X \to Y$ which is not uniformly continuous can be used to construct two topologically equivalent metrics on $X$ which are not uniformly equivalent. Then construct such an example, e.g. with $X = \mathbb{R}$ and $Y = (0,1)$.

b) Explain how the existence of a uniformeomorphism of metric spaces $f : X \to Y$ which is not a Lipschitzhomeomorphism can be used to construct two uniformly equivalent metrics on $X$ which are not Lipschitz equivalent.

c) Exhibit a uniformeomorphism $f : \mathbb{R} \to \mathbb{R}$ which is not a Lipschitzhomeomorphism.

d) Show that $\sqrt{x} : [0,1] \to [0,1]$ is a uniformeomorphism and not a Lipschitzhomeomorphism.\footnote{In particular, compactness does not force continuous maps to be Lipschitz!}

**Proposition 26.** Let $(X,d)$ be a metric space. Let $f : [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function with $f(0) = 0$, and suppose that $f \circ d : X \times X \to \mathbb{R}$ is a metric function. Then the metrics $d$ and $f \circ d$ are uniformly equivalent.

**Proof.** Let $A = f(1)$. The function $f : [0,1] \to [0,A]$ is continuous and strictly increasing, hence it has a continuous and strictly increasing inverse function $f^{-1} : [0,A] \to [0,1]$. Since $[0,1]$ and $[0,A]$ are compact metric spaces, $f$ and $f^{-1}$ are in fact uniformly continuous. The result follows easily from this, as we leave to the reader to check.}$\square$

In particular that for any metric $d$ on a set $X$ and any $\alpha > 0$, the metric $d_\alpha(x,y) = \frac{d(x,y)}{\alpha d(x,y)}$ of $X$.X is uniformly equivalent to $d$. In particular, every metric is uniformly equivalent to a metric with diameter at most $\alpha$. The following exercise gives a second, convexity-free approach to this.
Exercise 5.3. Let \((X, d)\) be a metric space, and let \(d_b : X \times X \to \mathbb{R}\) be given by \(d_b(x, y) = \min \{d(x, y), 1\}\). Show that \(d_b\) is a bounded metric on \(X\) which is uniformly equivalent to \(d\).

6. PRODUCT METRICS

6.1. Minkowski’s Inequality.

Theorem 27. (Jensen’s Inequality) Let \(f : I \to \mathbb{R}\) be continuous and convex. For any \(x_1, \ldots, x_n \in I\) and any \(\lambda_1, \ldots, \lambda_n \in [0, 1]\) with \(\lambda_1 + \cdots + \lambda_n = 1\), we have

\[
f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n).
\]

Proof. We go by induction on \(n\), the base case \(n = 1\) being trivial. So suppose Jensen’s Inequality holds for some \(n \in \mathbb{Z}^+\), and consider \(x_1, \ldots, x_{n+1} \in I\) and \(\lambda_1, \ldots, \lambda_{n+1} \in [0, 1]\) with \(\lambda_1 + \cdots + \lambda_{n+1} = 1\). If \(\lambda_{n+1} = 0\) we are reduced to the case of \(n\) variables which holds by induction. Similarly if \(\lambda_{n+1} = 1\) then \(\lambda_1 = \cdots = \lambda_n = 0\) and we have, trivially, equality. So we may assume \(\lambda_{n+1} \in (0, 1)\) and thus also that \(1 - \lambda_{n+1} \in (0, 1)\). Now for the big trick: we write

\[
\lambda_1 x_1 + \cdots + \lambda_n x_n = (1 - \lambda_{n+1}) \left( \frac{\lambda_1}{1 - \lambda_{n+1}} x_1 + \cdots + \frac{\lambda_n}{1 - \lambda_{n+1}} x_n \right) + \lambda_{n+1} x_{n+1},
\]

so that

\[
f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = f \left( (1 - \lambda_{n+1}) \left( \frac{\lambda_1}{1 - \lambda_{n+1}} x_1 + \cdots + \frac{\lambda_n}{1 - \lambda_{n+1}} x_n \right) + \lambda_{n+1} x_{n+1} \right)
\leq (1 - \lambda_{n+1}) f \left( \frac{\lambda_1}{1 - \lambda_{n+1}} x_1 + \cdots + \frac{\lambda_n}{1 - \lambda_{n+1}} x_n \right) + \lambda_{n+1} f(x_{n+1}).
\]

Since \(\frac{\lambda_1}{1 - \lambda_{n+1}}, \ldots, \frac{\lambda_n}{1 - \lambda_{n+1}}\) are non-negative numbers that sum to 1, by induction the \(n\) variable case of Jensen’s Inequality can be applied to give that the above expression is less than or equal to

\[
(1 - \lambda_{n+1}) \left( \frac{\lambda_1}{1 - \lambda_{n+1}} f(x_1) + \cdots + \frac{\lambda_n}{1 - \lambda_{n+1}} f(x_n) \right) + \lambda_{n+1} f(x_{n+1})
= \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) + \lambda_{n+1} f(x_{n+1}).
\]

\[\square\]

Theorem 28. (Weighted Arithmetic Geometric Mean Inequality) Let \(x_1, \ldots, x_n \in [0, \infty)\) and \(\lambda_1, \ldots, \lambda_n \in [0, 1]\) be such that \(\lambda_1 + \cdots + \lambda_n = 1\). Then:

\[(3) \quad x_1^{\lambda_1} \cdots x_n^{\lambda_n} \leq \lambda_1 x_1 + \cdots + \lambda_n x_n.
\]

Taking \(\lambda_1 = \cdots = \lambda_n = \frac{1}{n}\), we get the arithmetic geometric mean inequality:

\[
(x_1 \cdots x_n)^{\frac{1}{n}} \leq \frac{x_1 + \cdots + x_n}{n}.
\]

Proof. We may assume \(x_1, \ldots, x_n > 0\). For \(1 \leq i \leq n\), put \(y_i = \log x_i\). Then

\[
x_1^{\lambda_1} \cdots x_n^{\lambda_n} = e^{\lambda_1 y_1 + \cdots + \lambda_n y_n} \leq e^{\lambda_1 y_1 + \cdots + \lambda_n y_n} = \lambda_1 x_1 + \cdots + \lambda_n x_n.
\]

\[\square\]
Theorem 29. (Young’s Inequality)
Let \( x, y \in [0, \infty) \) and let \( p, q \in (1, \infty) \) satisfy \( \frac{1}{p} + \frac{1}{q} = 1 \). Then
\[
xy \leq \frac{x^p}{p} + \frac{y^q}{q}.
\]

Proof. When either \( x = 0 \) or \( y = 0 \) the left hand side is zero and the right hand side is non-negative, so the inequality holds and we may thus assume \( x, y > 0 \). Now apply the Weighted Arithmetic-Geometric Mean Inequality with \( n = 2, x_1 = x^p, x_2 = y^q, \lambda_1 = \frac{1}{p}, \lambda_2 = \frac{1}{q} \). We get
\[
xy = (x^p)^{\frac{1}{p}} (y^q)^{\frac{1}{q}} = x_1 y_2 \leq \lambda_1 x_1 + \lambda_2 x_2 = \frac{x^p}{p} + \frac{y^q}{q}.
\]

Theorem 30. (Hölder’s Inequality)
Let \( x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R} \) and let \( p, q \in (1, \infty) \) satisfy \( \frac{1}{p} + \frac{1}{q} = 1 \). Then
\[
|x_1 y_1| + \ldots + |x_n y_n| \leq (|x_1|^p + \ldots + |x_n|^p)^{\frac{1}{p}} (|y_1|^q + \ldots + |y_n|^q)^{\frac{1}{q}}.
\]

Proof. As above, the result is clear if either \( x_1 = \ldots = x_n = 0 \) or \( y_1 = \ldots = y_n = 0 \), so we may assume that neither of these is the case. For \( 1 \leq i \leq n \), apply Young’s Inequality with
\[
x = \frac{|x_i|}{(|x_1|^p + \ldots + |x_n|^p)^{\frac{1}{p}}}, y = \frac{|y_i|}{(|y_1|^q + \ldots + |y_n|^q)^{\frac{1}{q}}},
\]
and sum the resulting inequalities from \( i = 1 \) to \( n \), getting
\[
\sum_{i=1}^{n} |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1.
\]

Theorem 31. (Minkowski’s Inequality)
For \( x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R} \) and \( p \geq 1 \):
\[
(|x_1 + y_1|^p + \ldots + |x_n + y_n|^p)^{\frac{1}{p}} \leq (|x_1|^p + \ldots + |x_n|^p)^{\frac{1}{p}} + (|y_1|^p + \ldots + |y_n|^p)^{\frac{1}{p}}.
\]

Proof. When \( p = 1 \), the inequality reads
\[
|x_1 + y_1| + \ldots + |x_n + y_n| \leq |x_1| + |y_1| + \ldots + |x_n| + |y_n|
\]
and this holds just by applying the triangle inequality: for all \( 1 \leq i \leq n \), \( |x_i + y_i| \leq |x_i| + |y_i| \). So we may assume \( p > 1 \). Let \( q \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \), and note that then \((p-1)q = p\). We have
\[
|x_1 + y_1|^p + \ldots + |x_n + y_n|^p
\]
\[
\leq |x_1||x_1 + y_1|^{p-1} + \ldots + |x_n||x_n + y_n|^{p-1} + |y_1||x_1 + y_1|^{p-1} + \ldots + |y_n||x_n + y_n|^{p-1}
\]
\[
= |x_1|^{p+1} + \ldots + |x_n|^{p+1} + |y_1|^{p+1} + \ldots + |y_n|^{p+1}
\]
\[
= \left( |x_1|^p + \ldots + |x_n|^p \right)^{\frac{1}{p}} (|y_1|^p + \ldots + |y_n|^p)^{\frac{1}{q}}
\]
and using \( 1 - \frac{1}{q} = \frac{1}{p} \), we get the desired result.
For \( p \in [1, \infty) \) and \( x \in \mathbb{R}^N \), we put
\[
||x||_p = \left( \sum_{i=1}^{N} |x_i|^p \right)^{\frac{1}{p}}
\]
and
\[
d_p : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}, \quad d_p(x, y) = ||x - y||_p.
\]
We also put
\[
||x||_\infty = \max_{1 \leq i \leq N} |x_i|
\]
and
\[
d_\infty : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}, \quad d_\infty(x, y) = ||x - y||_\infty.
\]

**Lemma 32.** a) For each fixed nonzero \( x \in \mathbb{R}^N \), the function \( p \mapsto ||x||_p \) is decreasing and \( \lim_{p \to \infty} ||x||_p = ||x||_\infty. \)

b) For all \( 1 \leq p \leq \infty \) and \( x \in \mathbb{R}^N \) we have
\[
||x||_\infty \leq ||x||_p \leq ||x||_1 = |x_1| + \ldots + |x_N| \leq N||x||_\infty.
\]

**Proof.**
a) Let \( 1 \leq p \leq p' < \infty \), and let \( 0 \neq x = (x_1, \ldots, x_N) \in \mathbb{R}^N \). For any \( \alpha \geq 0 \) we have \( ||\alpha x||_p = ||\alpha||_p|x|_p \), so we are allowed to rescale: put \( y = (1/|x|_p)^{1/p}x \), so \( ||y||_{p'} \leq 1 \). Then \( |y_i| \leq 1 \) for all \( i \), so \( |y_i|^p \leq |y_i|^{p'} \) for all \( i \), so \( ||y||_{p'} \geq 1 \) and thus \( ||x||_p \geq ||x||_p' \).

Similarly, by scaling we reduce to the case in which the maximum of the \( |x_i|'s \) is equal to 1. Then in \( \lim_{p \to \infty} |x_1|^p + \ldots + |x_N|^p \), all of the terms \( |x_i|^p \) with \( |x_i| < 1 \) converge to 0 as \( p \to \infty \); the others converge to 1; so the given limit is the number of terms with absolute value 1, which lies between 1 and \( N \): that is, it is always at least one and it is bounded independently of \( p \). Raising this to the \( 1/p \) power and taking the limit we get 1.

b) The inequalities \( ||x||_\infty \leq ||x||_p \leq ||x||_1 \) follow from part a). For the latter inequality, let \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \) and suppose that \( i \) is such that \( |x_i| = \max_{1 \leq i \leq N} |x_i| \). Then
\[
|x_1| + \ldots + |x_N| \leq |x_1| + \ldots + |x_i| = N||x||_\infty.
\]

**Theorem 33.** For each \( p \in [1, \infty) \), \( d_p \) is a metric on \( \mathbb{R}^N \), and all of these metrics are Lipschitz equivalent.

**Proof.** For any \( 1 \leq p \leq \infty \) and \( x, y, z \in \mathbb{R}^N \), Minkowski’s Inequality gives
\[
d_p(x, z) = ||x - z||_p = ||(x - y) + (y - z)||_p \leq ||x - y||_p + ||y - z||_p = d_p(x, y) + d_p(y, z).
\]
Thus \( d_p \) satisfies the triangle inequality; that \( d_p(x, y) = d_p(y, x) \) and \( d_p(x, y) = 0 \iff x = y \) is immediate. So each \( d_p \) is a metric on \( \mathbb{R}^N \). Lemma ?? shows that for all \( 1 \leq p \leq \infty \), \( d_p \) is Lipschitz equivalent to \( d_\infty \). Since Lipschitz equivalence is indeed an equivalence relation, this implies that all the metrics \( d_p \) are Lipschitz equivalent.

The metric \( d_2 \) on \( \mathbb{R}^N \) is called the **Euclidean metric**. The topology that it generates is called the **Euclidean topology**. The point of the above discussion is that all metrics \( d_p \) are close enough to the Euclidean metric so as to generate the Euclidean topology.
6.2. Product Metrics.

Let \((X_i, d_i)_{i \in I}\) be an indexed family of metric spaces. Our task is to put a metric on the Cartesian product \(X = \prod_{i \in I} X_i\). Well, but that can’t be right: we have already put some metric on an arbitrary set, namely the discrete metric. Rather we want to put a metric on the product which usefully incorporates the metrics on the factors, in a way which generalizes the metrics \(d_p\) on \(\mathbb{R}^N\).

This is still not precise enough. We are lingering over this point a bit to emphasize the fundamental perspective of general topological spaces that we currently lack: eventually we will discuss the product topology, which is a canonically defined topology on any Cartesian product of topological spaces. With this perspective, the problem can then be gracefully phrased as that of finding a metric on a Cartesian product of metric spaces that induces the product topology. For now we bring out again our most treasured tool: sequences. Namely, convergence in the Euclidean topology on \(\mathbb{R}^N\) has the fundamental property that a sequence \(x\) in \(\mathbb{R}^N\) converges iff for all \(1 \leq i \leq N\), its \(i\)th component sequence \(x^{(i)}\) converges in \(\mathbb{R}\).

In general, let us say that a metric on \(X = \prod_{i \in I} X_i\) is good if for any sequence \(x\) in \(X\) and point \(x \in X\), we have \(x \to x\) in \(X\) iff for all \(i \in I\), the component sequence \(x^{(i)}\) converges to the \(i\)th component \(x^{(i)}\) of \(x\).

In the case of finite products, we have already done almost all of the work.

Lemma 34. For \(1 \leq i \leq N\), let \(\{x^{(i)}_n\}\) be a sequence of non-negative real numbers, and for \(n \in \mathbb{Z}^+\) let \(m_n = \max_{1 \leq i \leq N} x^{(i)}_n\). Then \(m_n \to 0 \iff x^{(i)}_n \to 0\) for all \(1 \leq i \leq N\).

Exercise 6.1. Prove it.

Theorem 35. Let \((X_1, d_1), \ldots, (X_N, d_N)\) be a finite sequence of metric spaces, and put \(X = \prod_{i=1}^N X_i\). Fix \(p \in [1, \infty]\), and consider the function

\[
d_p : X \times X \to \mathbb{R}, \quad d_p((x_1, \ldots, x_N), (y_1, \ldots, y_N)) = \left( \sum_{i=1}^N |d_i(x_i, y_i)|^p \right)^{\frac{1}{p}}.\]

a) The function \(d_p\) is a metric function on \(X\).

b) For \(p, p' \in [1, \infty]\), the metrics \(d_p\) and \(d_{p'}\) are Lipschitz equivalent.

c) The function \(d_p\) is a good metric on \(X\).

Proof. Notice first that if each \(X_i\) is equal to \(\mathbb{R}\) with the standard Euclidean metric, then parts a) and b) reduce to Theorem ?? and part c) is a familiar (and easy) fact from basic real analysis: a sequence in \(\mathbb{R}^N\) converges iff each of its component sequences converge. The proofs of parts a) and b) in the general case are almost identical and are left to the reader as a straightforward but important exercise.

In view of part b), it suffices to establish part c) for any one value of \(p\), and the easiest is probably \(p = \infty\), since \(d_\infty(x, y) = \max_i d_i(x_i, y_i)\). If \(x\) is a sequence in \(X\) and \(x\) is a point of \(X\), we are trying to show that

\[
d_\infty(x_n, x) = \max_i d_i(x^{(i)}_n, x^{(i)}) \to 0 \iff \forall 1 \leq i \leq N, \quad d_i(x^{(i)}_n, x^{(i)}) \to 0.\]

This follows from Lemma ??.
For infinite families of metric spaces, things get more interesting. The following is a variant of [?, Thm. IX.7.2].

**Theorem 36.** Let $I$ be an infinite set, let $(X_i, d_i)_{i \in I}$ be an indexed family of metric spaces, let $X = \prod_{i \in I} X_i$, and let

$$d : X \times X \to [0, \infty], \quad d(x, y) = \sup_{i \in I} d_i(x_i, y_i).$$

a) If there is a finite subset $J \subset I$ and $D < \infty$ such that for all $i \in I \setminus J$ we have $\text{diam} X_i \leq D$, then $d$ is a metric on $X$.

b) The following are equivalent:
- (i) $d$ is a good metric.
- (ii) For all $\delta > 0$, the set $\{ i \in I \mid \text{diam} X_i \geq \delta \}$ is finite.

**Proof.**

**Corollary 37.** Let $(X_n, d_n)_{n=1}^\infty$ be an infinite sequence of metric spaces. Then there is a good metric on the Cartesian product $X = \prod_{i=1}^\infty X_n$.

**Proof.** The idea is simple: the given sequence of metrics need not satisfy the hypotheses of Theorem ?? – e.g. they will not if each $X_n$ has infinite diameter – but we can replace each $d_n$ with a topologically equivalent metric so that the hypotheses hold. Indeed, the metric $d'_n = \frac{d_n^2}{d_n + 1}$ is topologically equivalent and has diameter at most $\frac{1}{2}$. The family $(X_n, d'_n)$ satisfies the hypotheses of Theorem ??b), so $d = \sup_n d'_n$ is a good metric on $X$.

Notice that Corollary ?? in particular shows that $\prod_{i=1}^\infty \mathbb{R}$ and $\prod_{i=1}^\infty [a, b]$ can be given metrics so that convergence amounts to convergence in each factor. These are highly interesting and important examples in the further study of analysis and topology. The latter space is often called the **Hilbert cube**.

There is a case left over: what happens when we have a family of metrics indexed by an uncountable set $I$? In this case the condition that all but finitely many factors have diameter less than any given positive constant turns out to be prohibitively strict.

**Exercise 6.2.** Let $(X_i, d_i)_{i \in I}$ be a family of metrics indexed by an uncountable set $I$. Suppose that $\text{diam} X_i > 0$ for uncountably many $i \in I$ – equivalently, uncountably many $X_i$ contains more than one point. Show that there is $\delta > 0$ such that $\{ i \in I \mid \text{diam} X_i \geq \delta \}$ is uncountable.

Thus Theorem ?? can never be used to put a good metric on an uncountable product except in the trivial case that all but countably many of the spaces $X_i$ consist of a single point. (Nothing is gained by taking Cartesian products with one-point sets: this is the multiplicative equivalent of repeatedly adding zero!) At the moment this seems like a weakness of the result. Later we will see that is is essential: the Cartesian product of an uncountable family of metric spaces each consisting of more than a single point cannot in fact be given any good metric. In later terminology, this is an instance of nonmetrizability of large Cartesian products.

6.3. **Further Exercises**.
Exercise 6.3. Let $X$ and $Y$ be metric spaces, and let $X \times Y$ be endowed with any good metric. Let $f : X \to Y$ be a function.

a) Show that if $f$ is continuous, its graph $G(f) = \{(x, f(x)) \mid x \in X\}$ is a closed subset of $X \times Y$.

b) Give an example of a function $f : [0, \infty) \to [0, \infty)$ which is discontinuous at 0 but for which $G(f)$ is closed in $[0, \infty) \times [0, \infty)$.

7. Compactness

7.1. Basic Properties of Compactness.

Let $X$ be a metric space. For a subset $A \subset X$, a family $\{Y_i\}_{i \in I}$ of subsets of $X$ is a covering of $A$ if $A \subset \bigcup_{i \in I} Y_i$. A subset $A \subset X$ is compact if for every open covering $\{U_i\}_{i \in I}$ of $A$ there is a finite subset $J \subset I$ such that $\{U_i\}_{i \in J}$ covers $A$.

Exercise 7.1. Show (directly) that $A = \{0\} \cup \{\frac{1}{n}\}_{n=1}^{\infty} \subset \mathbb{R}$ is compact.

Exercise 7.2. Let $X$ be a metric space, and let $A \subset X$ be a finite subset. Show that $A$ is compact.

Lemma 38. Let $X$ be a metric space, and let $K \subset Y \subset X$. Then $K$ is compact as a subset of $Y$ if and only if $K$ is compact as a subset of $X$.

Proof. Suppose $K$ is compact as a subset of $Y$, and let $\{U_i\}_{i \in I}$ be a family of open subsets of $X$ such that $K \subset \bigcup_{i \in I} U_i$. Then $\{U_i \cap Y\}_{i \in I}$ is a covering of $K$ by open subsets of $Y$, and since $K$ is compact as a subset of $Y$, there is a finite subset $J \subset I$ such that $K \subset \bigcup_{i \in J} U_i \cap Y \subset \bigcup_{i \in J} U_i$.

Suppose $K$ is compact as a subset of $X$, and let $\{V_i\}_{i \in I}$ be a family of open subsets of $Y$ such that $K \subset \bigcup_{i \in I} V_i$. By $X$, we may write $V_i = U_i \cap Y$ for some open subset of $X$. Then $K \subset \bigcup_{i \in I} V_i \subset \bigcup_{i \in J} U_i$, so there is a finite subset $J \subset I$ such that $K \subset \bigcup_{i \in J} U_i$. Intersecting with $Y$ gives

$$K = K \cap Y \subset \left( \bigcup_{i \in J} U_i \right) \cap Y = \bigcup_{i \in J} V_i.$$ 

A sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of $X$ is expanding if $A_n \subset A_{n+1}$ for all $n \geq 1$. We say the sequence is properly expanding if $A_n \subsetneq A_{n+1}$ for all $n \geq 1$. An expanding open cover is an expanding sequence of open subsets with $X = \bigcup_{n=1}^{\infty} A_n$; we define a properly expanding open covering similarly.

Exercise 7.3. Let $\{A_n\}_{n=1}^{\infty}$ be a properly expanding open covering of $X$.

a) Let $J \subset \mathbb{Z}^+$ be finite, with largest element $N$. Show that $\bigcup_{i \in J} A_i = A_N$.

b) Suppose that an expanding open covering $\{A_n\}_{n=1}^{\infty}$ admits a finite subcovering. Show that there is $N \in \mathbb{Z}^+$ such that $X = A_N$.

c) Show that a properly expanding open covering has no finite subcovering, and thus if $X$ admits a properly expanding open covering it is not compact.

An open covering $\{U_i\}_{i \in I}$ is disjoint if for all $i \neq j$, $U_i \cap U_j = \emptyset$.

Exercise 7.4. a) Let $\{U_i\}_{i \in I}$ be a disjoint open covering of $X$. Show that the covering admits no proper subcovering.

b) Show: if $X$ admits an infinite disjoint open covering, it is not compact.

c) Show: a discrete space is compact if and only if it is finite.
Any property of a metric space formulated in terms of open sets may, by taking complements, also be formulated in terms of closed sets. Doing this for compactness we get the following simple but useful criterion.

**Proposition 39.** For a metric space $X$, the following are equivalent:

(i) $X$ is compact.

(ii) $X$ satisfies the finite intersection property: if $\{A_i\}_{i \in I}$ is a family of closed subsets of $X$ such that for all finite subsets $J \subset I$ we have $\bigcap_{i \in J} A_i \neq \emptyset$, then $\bigcap_{i \in I} A_i \neq \emptyset$.

**Exercise 7.5.** Prove it.

Another easy but crucial observation is that compactness is somehow antithetical to discreteness. More precisely, we have the following result.

**Proposition 40.** For a metric space $X$, the following are equivalent:

(i) $X$ is both compact and topologically discrete.

(ii) $X$ is finite.

**Exercise 7.6.** Prove it.

**Lemma 41.** Let $X$ be a metric space and $A \subset X$.

a) If $X$ is compact and $A$ is closed in $X$, then $A$ is compact.

b) If $A$ is compact, then $A$ is closed in $X$.

c) If $X$ is compact, then $X$ is bounded.

**Proof.** a) Let $\{U_i\}_{i \in I}$ be a family of open subsets of $X$ which covers $A$: i.e., $A \subset \bigcup_{i \in I} U_i$. Then the family $\{U_i\}_{i \in I} \cup \{X \setminus A\}$ is an open covering of $X$. Since $X$ is compact, there is a finite subset $J \subset I$ such that $X = \bigcup_{i \in J} U_i \cup (X \setminus A)$, and it follows that $A \subset \bigcup_{i \in J} U_i$.

b) Let $U = X \setminus A$. For each $p \in U$ and $q \in A$, let $V_q = B(p, \frac{d(p,q)}{2})$ and $W_q = B(p, \frac{d(p,q)}{2})$, so $V_q \cap W_q = \emptyset$. Moreover, $\{W_q\}_{q \in A}$ is an open covering of the compact set $A$, so there are finitely many points $q_1, \ldots, q_n \in A$ such that

$$A \subset \bigcup_{i=1}^n W_i =: W,$$

say. Put $V = \bigcap_{i=1}^n V_i$. Then $V$ is a neighborhood of $p$ which does not intersect $W$, hence lies in $X \setminus A = U$. This shows that $U = X \setminus A$ is open, so $A$ is closed.

c) Let $x \in X$. Then $\{B^o(x, n)\}_{n=1}^\infty$ is an expanding open covering of $X$; since $X$ is compact, we have a finite subcovering. By Exercise 2, we have $X = B^o(x, N)$ for some $N \in \mathbb{Z}^+$, and thus $X$ is bounded. \qed

**Example 7.1.** Let $X = [0, 10] \cap \mathbb{Q}$ be the set of rational points on the unit interval. As a subset of itself, $X$ is closed and bounded. For $n \in \mathbb{Z}^+$, let

$$U_n = \{x \in X \mid d(x, \sqrt{2}) > \frac{1}{n}\}.$$

Then $\{U_n\}_{n=1}^\infty$ is a properly expanding open covering of $X$, so $X$ is not compact.

**Proposition 42.** Let $f : X \to Y$ be a surjective continuous map of topological spaces. If $X$ is compact, so is $Y$.

**Proof.** Let $\{V_i\}_{i \in I}$ be an open cover of $Y$. For $i \in I$, put $U_i = f^{-1}(V_i)$. Then $\{U_i\}_{i \in I}$ is an open cover of $X$. Since $X$ is compact, there is a finite $J \subset I$ such that $\bigcup_{i \in J} U_i = X$, and then $Y = f(X) = f(\bigcup_{i \in J} U_i) = \bigcup_{i \in J} f(U_i) = \bigcup_{i \in J} V_i$. \qed
Theorem 43. (Extreme Value Theorem) Let $X$ be a compact metric space. A continuous function $f : X \to \mathbb{R}$ is bounded and attains its maximum and minimum: there are $x_m, x_M \in X$ such that for all $x \in X$, $f(x_m) \leq f(x) \leq f(x_M)$.

Proof. Since $f(X) \subset \mathbb{R}$ is compact, it is closed and bounded. Thus $\inf f(X)$ is a finite limit point of $f(X)$, so it is the minimum; similarly $\sup f(X)$ is the maximum.

7.2. Heine-Borel.

When one meets a new metric space $X$, it is natural to ask: which subsets $A$ of $X$ are compact? Lemma ?? gives the necessary condition that $A$ must be closed and bounded. In an arbitrary metric space this is nowhere near sufficient, and one need look no farther than an infinite set endowed with the discrete metric: every subset is closed and bounded, but the only compact subsets are the finite subsets. In fact, compactness is a topological property whereas we saw in §6 that given any metric space there is a topologically equivalent bounded metric.

Nevertheless in some metric spaces it is indeed the case that every closed, bounded set is compact. In this section we give a concrete treatment that Euclidean space $\mathbb{R}^N$ has this property: this is meant to be a reminder of certain ideas from honors calculus / elementary real analysis that we will shortly want to abstract and generalize.

A sequence $\{A_n\}_{n=1}^\infty$ of subsets of $X$ is nested if $A_{n+1} \supset A_n$ for all $n \geq 1$.

Let $a_1 \leq b_1, a_2 \leq b_2, \ldots, a_n \leq b_n$ be real numbers. We put

$$\prod_{i=1}^n [a_i, b_i] = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \forall 1 \leq i \leq n, a_i \leq x_i \leq b_i\}.$$ 

We will call such sets closed boxes.

Exercise 7.7.

a) Show: a subset $A \subset \mathbb{R}^n$ is bounded iff it is contained in some closed box.

b) Show that

$$\text{diam} \left( \prod_{i=1}^n [a_i, b_i] \right) = \sqrt{\sum_{i=1}^n (b_i - a_i)^2}.$$ 

Lemma 44. (Lion-Hunting Lemma) Let $\{B_m\}_{m=1}^\infty$ be a nested sequence of closed boxes in $\mathbb{R}^n$. Then there is $x \in \bigcap_{m=1}^\infty B_m$.

b) If $\lim_{m \to \infty} \text{diam } B_m = 0$, then $\bigcap_{m=1}^\infty B_m$ consists of a single point.

Proof. Write $B_m = \prod_{i=1}^n [a_i(m), b_i(m)]$. Since the sequence is nested, we have

$$a_i(m) \leq a_i(m + 1) \leq b_i(m + 1) \leq b_i(m)$$

for all $i$ and $m$. Then $x_m = (x_m(1), \ldots, x_m(n)) \in \bigcap_{m=1}^\infty B_m$ iff for all $1 \leq i \leq n$ we have $a_m(i) \leq x_m(i) \leq b_m(i)$. For $1 \leq i \leq n$, put

$$A_i = \sup_m a_m(i), \quad B_i = \inf_m b_m(i).$$
It then follows that
\[ \bigcap_{m=1}^{\infty} B_m = \prod_{i=1}^{n}[A_i, B_i], \]
which is nonempty.

**Exercise 7.8.** In the above proof it is implicit that \( A_i \leq B_i \) for all \( 1 \leq i \leq n \). Convince yourself that you could write down a careful proof of this (e.g. by writing down a careful proof!).

**Exercise 7.9.** Under the hypotheses of the Lion-Hunting Lemma, show that the following are equivalent:
(i) \( \inf \{ \text{diam } B_m \}_{m=1}^{\infty} = 0 \).
(ii) \( \bigcap_{m=1}^{\infty} B_m \) consists of a single point.

**Theorem 45.** (Heine-Borel) Every closed, bounded subset \( A \subset \mathbb{R}^n \) is compact.

**Proof.** Because every closed bounded subset is a subset of a closed box and closed subsets of compact sets are compact, it is sufficient to show the compactness of every closed box \( B = \prod_{i=1}^{n}[a_i, b_i] \). Let \( \mathcal{U} = \{ U_i \}_{i \in I} \) be an open covering of \( B \).

Seeking a contradiction we suppose \( \mathcal{U} \) admits no finite subcovering. We bisect \( B \) into \( 2^n \) closed subboxes of equal size, so that e.g. the bottom leftmost one is \( \prod_{i=1}^{n}[a_i, \frac{a_i+b_i}{2}] \). It must be that at least one of the subboxes cannot be covered by any finite number of sets in \( \mathcal{U} \): if all \( 2^n \) of them have finite subcoverings, taking the union of \( 2^n \) finite subcoverings, we get a finite subcovering of \( B \). Identify one such subbox \( B_1 \), and notice that \( \text{diam } B_1 = \frac{1}{2} \text{diam } B \). Now bisect \( B_1 \) and repeat the argument: we get a nested sequence \( \{ B_m \}_{m=1}^{\infty} \) of closed boxes with
\[ \text{diam } B_m = \frac{\text{diam } B}{2^m}. \]

By the Lion-Hunting Lemma there is \( x \in \bigcap_{m=1}^{\infty} B_m \).\(^2\) Choose \( U_0 \in \mathcal{U} \) such that \( x \in U_0 \). Since \( U_0 \) is open, for some \( \epsilon > 0 \) we have
\[ x \in B^2(x, \epsilon) \subset U_0. \]

For sufficiently large \( m \) we have \( \text{diam } B_m < \epsilon \). Thus every point in \( B_m \) has distance less than \( \epsilon \) from \( x \) so
\[ B_m \subset B^2(x, \epsilon) \subset U_0. \]

This contradicts the heck out of the fact that \( B_m \) admits no finite subcovering.

**Proposition 46.** Let \( X \) be a compact metric space, and let \( A \subset X \) be an infinite subset. Then \( A \) has a limit point in \( X \).

**Proof.** Seeking a contradiction we suppose that \( A \) has no limit point in \( X \). Then also no subset \( A' \subset A \) has any limit points in \( X \). Since a set is closed if it contains all of its limit points, every subset of \( A \) is closed in \( X \). In particular \( A \) is closed in \( X \), hence \( A \) is compact. But since for all \( x \in A \), \( A \setminus \{ x \} \) is closed in \( A \), we have that \( \{ x \} \) is open in \( A \). (In other words, \( A \) is discrete.) Thus \( \{ \{ x \} \}_{x \in A} \) is an infinite cover of \( A \) without a finite subcover, so \( A \) is not compact: contradiction.

**Theorem 47.** (Bolzano-Weierstrass for Sequences) Every bounded sequence in \( \mathbb{R}^n \) admits a convergent subsequence.

\(^2\)Though we don’t need it, it follows from Exercise 1.9 that the intersection point \( x \) is unique.
Proof. Step 1: Let $N = 1$. I leave it to you to carry over the proof of Bolzano-Weierstrass in $\mathbb{R}$ given in § 2.2 to our current sequential situation: replacing the Monotonicity Lemma with the Rising Sun Lemma, the endgame is almost identical. Step 2: Let $N \geq 2$, and let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}^N$. Then each coordinate sequence $\{x_n(i)\}_{n=1}^{\infty}$ is bounded, so Step 1 applies to each of them.

However, if we just extract subsequences for each component separately, we will have $N$ different subsequences, and it will in general not be possible to get one subsequence out of all of them. So we proceed in order: first we extract a subsequence such that the first coordinates converge. Then we extract a subsequence of the subsequence such that the second coordinates converge. This does not disturb what we’ve already done, since every subsequence of a convergent sequence is convergent (we’re applying this in the familiar context of real sequences, but it is equally true in any metric space). Thus we extract a sub-sub-sub...subsequence ($N$ “subs” altogether) which converges in every coordinate and thus converges. But a sub-sub....subsequence is just a subsequence, so we’re done.

A metric space is **sequentially compact** if every sequence admits a convergent subsequence.

A metric space $X$ is **limit point compact** if every infinite subset $A \subset X$ has a limit point in $X$.

8. **Completeness**

8.1. Lion Hunting In a Metric Space.

Recall the Lion-Hunting Lemma: any nested sequence of closed boxes in $\mathbb{R}^N$ has a common intersection point; if the diameters approach zero, then there is a unique intersection point. This was the key to the proof of the Heine-Borel Theorem.

Suppose we want to hunt lions in an arbitrary metric space: what should we replace “closed box” with? The following exercise shows that we should at least keep the “closed” part in order to get something interesting.

**Exercise 8.1.** Find a nested sequence $A_1 \supset A_2 \supset \ldots \supset A_n \ldots$ of nonempty subsets of $[0, 1]$ with $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

So perhaps we should replace “closed box” with “closed subset”? Well...we could. However, even in $\mathbb{R}$, if we replace “closed box” with “closed set”, then lion hunting need not succeed: for $n \in \mathbb{Z}^+$, let $A_n = [n, \infty)$. Then $\{A_n\}_{n=1}^{\infty}$ is a nested sequence of closed subsets with $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Suppose however that we consider nested covers of nonempty closed subsets with the additional property that $\text{diam } A_n \to 0$. In particular, all but finitely many $A_n$’s are bounded, so the previous problem is solved. Indeed, Lion-Hunting works under these hypothesis in $\mathbb{R}^N$ because of Heine-Borel: some $A_n$ is closed and bounded, hence compact, so we revisit the previous case.

A metric space is **complete** if for every nested sequence $\{A_n\}$ of nonempty closed subsets with diameter tending to 0 we have $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$. 
The following result shows that completeness, like compactness, is a kind of intrinsic closedness property.

**Lemma 48.** Let $Y$ be a subset of a metric space $X$.

a) If $X$ is complete and $Y$ is closed, then $Y$ is complete.

b) If $Y$ is complete, then $Y$ is closed.

*Proof.* a) If $Y$ is closed in $X$, then a nested sequence $\{A_n\}_{n=1}^{\infty}$ of nonempty closed subsets of $Y$ with diameter approaching 0 is also a nested sequence of nonempty closed subsets of $X$ with diameter approaching 0. Since $X$ is complete, there is $x \in \bigcap_n A_n$.

b) If $Y$ is not closed, let $y$ be a sequence in $Y$ converging to an element $x \in X \setminus Y$. Put $A_n = \{y_k \mid k \geq n\}$. Then $\{A_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty closed subsets of $Y$ of diameter approaching 0 and with empty intersection. \(\square\)

### 8.2. Cauchy Sequences

Our Lion Hunting definition of completeness is conceptually pleasant, but it seems like it could be a lot of work to check in practice. It is also – we now admit – not the standard one. We now make the transition to the standard definition.

The proof of (ii) \(\implies\) (i) in Theorem X.X suggests that it would be sufficient to lion hunt using nested sequences of closed balls. This is not hard to show.

**Lemma 49.** A metric space in which each sequence of closed balls with diameters tending to zero has nonempty intersection is complete.

*Proof.* Let $\{A_n\}_{n=1}^{\infty}$ be a nested sequence of nonempty closed subsets with diameter tending to zero. We may assume without loss of generality that each $A_n$ has finite diameter, and we may choose for all $n \in \mathbb{Z}^+$, $x_n \in A_n$ and a positive real number $r_n$ such that $A_n \subset B^*(x_n, r_n)$ and $r_n \to 0$. By assumption, there is a unique point $x \in \bigcap_n B^*(x_n, r_n)$. Then $x_n \to x$. Fix $n \in \mathbb{Z}^+$. Then $x$ is the limit of the sequence $x_n, x_{n+1}, \ldots$ in $A_n$, and since $A_n$ is closed, $x \in A_n$. \(\square\)

Let us nail down which sequences of closed balls we can use for lion hunting.

**Lemma 50.** Let $\{B^*(x_n, r_n)\}_{n=1}^{\infty}$ be a nested sequence of closed balls in a metric space $X$ with $r_n \to 0$. Then for all $\epsilon > 0$, there is $N = N(\epsilon)$ such that for all $m, n \geq N$, $d(x_m, x_n) \leq \epsilon$.

*Proof.* Fix $\epsilon > 0$, and choose $N$ such that $r_N \leq \frac{\epsilon}{2}$. Then if $m, n \geq N$ we have $x_n, x_m \in B^*(x_N, r_N)$, so $d(x_n, x_m) \leq 2r_N \leq \epsilon$. \(\square\)

At last, we have motivated the following definition. A sequence $\{x_n\}$ in an metric space $X$ is **Cauchy** if for all $\epsilon > 0$, there is $N = N(\epsilon)$ such that for all $m, n \geq N$, $d(x_m, x_n) < \epsilon$. Thus in a nested sequence of closed balls with diameter tending to zero, the centers of the balls form a Cauchy sequence. Moreover:

**Lemma 51.** Let $\{x_n\}$ be a sequence in a metric space $X$, and for $n \in \mathbb{Z}^+$ put $A_n = \{x_k \mid k \geq n\}$. The following are equivalent:

(i) The sequence $\{x_n\}$ is Cauchy.

(ii) $\lim_{n \to \infty} \text{diam } A_n = 0$.

**Exercise 8.2.** Prove it.
Exercise 8.3. Show that every convergent sequence is Cauchy.

Lemma 52. Every partial limit of a Cauchy sequence is a limit.

Proof. Let \( \{x_n\} \) be a Cauchy sequence, and let \( x \in X \) be such that some subsequence \( x_{n_k} \to x \). Fix \( \epsilon > 0 \), and choose \( N \) such that for all \( m,n \geq N \), \( d(x_m, x_n) < \frac{\epsilon}{2} \). Choose \( K \) such that \( n_K \geq N \) and for all \( k \geq K \), \( d(x_{n_k}, x) < \frac{\epsilon}{2} \).

Then for all \( n \geq N \),
\[
d(x_n, x) \leq d(x_n, x_{n_K}) + d(x_{n_K}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
\( \square \)

Proposition 53. For a metric space \( X \), the following are equivalent:
(i) \( X \) is complete.
(ii) Every Cauchy sequence in \( X \) is convergent.

Proof. \( \neg (ii) \implies \neg (i) \): Suppose \( \{x_n\} \) is a Cauchy sequence which does not converge. By Lemma ??, the sequence \( \{x_n\} \) has no partial limit, so \( A_n = \{x_k \mid k \geq n\} \) is a nested sequence of closed subsets with diameter tending to 0 and \( \bigcap_n A_n = \emptyset \), so \( X \) is not complete. \( (ii) \implies (i) \): By Lemma ??, it is enough to show that any nested sequence of closed balls with diameters tending to zero has nonempty intersection. By Lemma ??, the sequence of centers \( \{x_n\} \) is Cauchy, hence converge to \( x \in X \) by assumption. For each \( n \in \mathbb{Z}^+ \), the sequence \( x_n, x_{n+1}, \ldots \) lies in \( B^*(x_n, r_n) \), hence the limit, \( x \), lies in \( B^*(x_n, r_n) \).
\( \square \)

8.3. Baire’s Theorem.

A subset \( A \) of a metric space \( X \) is nowhere dense if \( \overline{A} \) contains no nonempty open subset, or in other words if \( \overline{A} = \emptyset \).

Exercise 8.4. Let \( x \) be a point of a metric space \( X \). Show that \( x \) is a limit point of \( X \) iff \( \{x\} \) is nowhere dense.

Theorem 54. (Baire) Let \( X \) be a complete metric space.

a) Let \( \{U_n\}_{n=1}^\infty \) be a sequence of dense open subsets of \( X \). Then \( U = \bigcap_{n=1}^\infty U_n \) is also dense in \( X \).

b) Let \( \{A_n\}_{n=1}^\infty \) be a countable collection of nowhere dense subsets of \( X \). Then \( A = \bigcup_{n=1}^\infty A_n \) has empty interior.

Proof. a) We must show that for every nonempty open subset \( W \) of \( X \) we have \( W \cap U \neq \emptyset \). Since \( U_1 \) is open and dense, \( W \cap U_1 \) is nonempty and open and thus contains some closed ball \( B^*(x_1, r_1) \) with \( 0 < r_1 \leq 1 \). For \( n \geq 1 \), having chosen \( x_n \) and \( r_n \leq \frac{1}{2^n} \), since \( U_{n+1} \) is open and dense, \( B(x_n, r_n) \cap U_{n+1} \) is nonempty and open and thus contains some closed ball \( B^*(x_{n+1}, r_{n+1}) \) with \( 0 < r_{n+1} \leq \frac{1}{2^{n+1}} \). Since \( X \) is complete, there is a (unique)
\[
x \in \bigcap_{n=2}^\infty B^*(x_n, r_n) \subset \bigcap_{n=1}^\infty B(x_n, r_n) \cap U_n \subset \bigcup_{n=1}^\infty U_n = U.
\]
Moreover
\[
x \in B^*(x_1, r_1) \subset W \cap U_1 \subset W,
\]
so
\[
x \in U \cap W.
\]
b) Without loss of generality we may assume that each \( A_n \) is closed, because \( A_n \) is nowhere dense iff \( \overline{A_n} \) is nowhere dense, and a subset of a nowhere dense set is certainly nowhere dense. For \( n \in \mathbb{Z}^+ \), let \( U_n = X \setminus A_n \). Each \( U_n \) is open; moreover, since \( \overline{A_n} \) contains no nonempty open subset, every nonempty open subset must intersect \( U_n \) and thus \( U_n \) is dense. By part a), \( \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} X \setminus A_n = X \setminus A \) is dense. Again, this means that every nonempty open subset of \( X \) meets the complement of \( A \) so no nonempty open subset of \( X \) is contained in \( A \).

**Corollary 55.** Let \( X \) be a nonempty complete metric space in which every point is a limit point. Then \( X \) is uncountable.

*Proof.* Let \( A = \{a_n \mid n \in \mathbb{Z}^+\} \) be a countably infinite subset of \( X \). Then each \( \{a_n\} \) is nowhere dense, so by Theorem ??, \( A = \bigcup_{n=1}^{\infty} \{a_n\} \) has empty interior. In particular, \( A \subset X \).

Observe that Corollary ?? applies to \( \mathbb{R} \) and gives a purely topological proof of its uncountability!

**Corollary 56.** A countably infinite complete metric space has infinitely many isolated points.

*Proof.* Let \( X \) be a complete metric space with only finitely many isolated points, say \( A = \{a_1, \ldots, a_n\} \). We will show that \( X \) is uncountable. Let \( Y = X \setminus A \), let \( y \in Y \), and let \( V \) be an open neighborhood of \( y \) in \( Y \), so \( V = U \cap Y \) for some open neighborhood of \( y \) in \( X \). By definition of \( Y \), \( V \) is infinite. However, intersecting with \( Y \) only involves removing finitely many points, so \( V \) must also be infinite! It follows that every point of the metric space \( Y \) is a limit point. As in any metric space, the subset of all isolated points is open, so its complement \( Y \) is closed in the complete space \( X \), so it too is complete. Thus by Corollary ?? \( Y \) must be uncountably infinite, hence so is \( X \).

One interesting consequence of these results is that we can deduce purely topological consequences of the metric condition of completeness.

**Example 8.1.** Let \( \mathbb{Q} \) be the rational numbers, equipped with the usual Euclidean metric \( d(x, y) = |x - y| \). As we well know, \( (\mathbb{Q}, d) \) is not complete. But here is a more profound question: is there some topologically equivalent metric \( d' \) on \( \mathbb{Q} \) which is complete? Now in general a complete metric can be topologically equivalent to an incomplete metric: e.g. this happens on \( \mathbb{R} \). But that does not happen here: any topologically equivalent metric is a metric on a countable set in which no point is isolated (the key observation being that the latter depends only on the topology), so by Corollary ?? cannot be complete.

This example motivates the following definition: a metric space \( (X, d) \) is **topologically complete** if there is a complete metric \( d' \) on \( X \) which is topologically equivalent to \( d \).

**Exercise 8.5.** Show that the space of irrational numbers \( \mathbb{R} \setminus \mathbb{Q} \) (still with the standard Euclidean metric \( d(x, y) = |x - y| \)) is topologically complete.

9. **Total Boundedness**

We saw above that the property of boundedness is not only not preserved by homeomorphisms of metric spaces, it is not even preserved by uniform homeomorphisms of
metric spaces (and also that it is preserved by Lipschitz-homeomorphisms). Though this was as simple as replacing any unbounded metric by the standard bounded metric \(d_0(x, y) = \min d(x, y), 1\), intuitively it is still a bit strange: e.g. playing around a bit with examples, one soon suspects that for subspaces of Euclidean space \(\mathbb{R}^N\), the property of boundedness is preserved by uniform-homeomorphisms.

The answer to this puzzle lies in identifying a property of metric spaces: perhaps the most important property that does not get “compactness level PR”.

A metric space \(X\) is **totally bounded** if for all \(\epsilon > 0\), it admits a finite cover by open \(\epsilon\)-balls: there is \(N \in \mathbb{Z}^+\) and \(x_1, \ldots, x_N \in X\) such that \(X = \bigcup_{i=1}^N B(x_i, \epsilon)\). Since any finite union of bounded sets is bounded, certainly total boundedness implies boundedness (thank goodness).

Notice that we could require the balls to be closed without changing the definition: just slightly increase or decrease \(\epsilon\). (And indeed, sometimes we will want to use one form of the definition and sometimes the other.) In fact we don’t really need balls at all; the following is useful reformulation.

**Lemma 57.** For a metric space \(X\), the following are equivalent:

(i) For all \(\epsilon > 0\), there exists a finite family \(S_1, \ldots, S_N\) of subsets of \(X\) such that \(\text{diam}(S_i) \leq \epsilon\) for all \(i\) and \(X = \bigcup_{i=1}^N S_i\).

(ii) \(X\) is totally bounded.

**Proof.** (i) \(\Rightarrow\) (ii): We may assume each \(S_i\) is nonempty, and choose \(x_i \in S_i\). Since \(\text{diam}(S_i) \leq \epsilon\), \(S_i \subset B^*(x_i, \epsilon)\) and thus \(X = \bigcup_{i=1}^N B^*(x_i, \epsilon)\).

(ii) \(\Rightarrow\) (i): For every \(\epsilon > 0\), choose \(x_1, \ldots, x_N\) such that \(\bigcup_{i=1}^N B^*(x_i, \frac{\epsilon}{2}) = X\). We have covered \(X\) by finitely many sets each of diameter at most \(\epsilon\). \(\square\)

**Corollary 58.**

a) Every subset of a totally bounded metric space is totally bounded.

b) Let \(f : X \to Y\) be a uniform-homeomorphism of metric spaces. Then \(X\) is totally bounded iff \(Y\) is totally bounded.

**Proof.** a) Suppose that \(X\) is totally bounded, and let \(Y \subset X\). Since \(X\) is totally bounded, for each \(\epsilon > 0\) there exist \(S_1, \ldots, S_N \subset X\) such that \(\text{diam}(S_i) < \epsilon\) for all \(i\) and \(X = \bigcup_{i=1}^N S_i\). Then \(\text{diam}(S_i \cap Y) < \epsilon\) for all \(i\) and \(Y = \bigcup_{i=1}^N (S_i \cap Y)\).

b) Suppose \(X\) is totally bounded. Let \(\epsilon > 0\), and choose \(\delta > 0\) such that \(f\) is \((\epsilon, \delta)\)-uniformly continuous. Since \(X\) is totally bounded there are finitely many sets \(S_1, \ldots, S_N \subset X\) with \(\text{diam}(S_i) \leq \delta\) for all \(1 \leq i \leq N\) and \(X = \bigcup_{i=1}^N S_i\). For \(1 \leq i \leq N\), let \(T_i = f(S_i)\). Then \(\text{diam}(T_i) \leq \epsilon\) for all \(i\) and \(Y = \bigcup_{i=1}^N T_i\). It follows that \(Y\) is uniformly bounded. Using the uniformly continuous function \(f^{-1} : Y \to X\) gives the converse implication. \(\square\)

**Lemma 59.** (Archimedes) A subset of \(\mathbb{R}^N\) is bounded iff it is totally bounded.

**Proof.** Total boundedness always implies boundedness. Moreover any bounded subset of \(\mathbb{R}^N\) lies in some cube \(C_n = [-n, n]^N\) for some \(n \in \mathbb{Z}^+\), so by Corollary ?? it is enough to show that \(C_n\) is totally bounded. But \(C_n\) can be written as the union
Let \( \epsilon > 0 \). An \( \epsilon \)-net in a metric space \( X \) is a subset \( N \subset X \) such that for all \( x \in X \), there is \( n \in N \) with \( d(x, n) < \epsilon \). An \( \epsilon \)-packing in a metric space \( X \) is a subset \( P \subset X \) such that \( d(p, p') \geq \epsilon \) for all \( p, p' \in P \).

These concepts give rise to a deep duality in discrete geometry between packing – namely, placing objects in a space without overlap – and covering – namely, placing objects in a space so as to cover the entire space. Notice that already we can cover the plane with closed unit balls or we can pack the plane with closed unit balls but we cannot do both at once. The following is surely the simplest possible duality principle along these lines.

**Proposition 60.** Let \( X \) be a metric space, and let \( \epsilon > 0 \).

\( a) \) The space \( X \) admits either a finite \( \epsilon \)-net or an infinite \( \epsilon \)-packing.

\( b) \) If \( X \) admits a finite \( \epsilon \)-net then it does not admit an infinite \((2\epsilon)\)-packing.

\( c) \) Thus \( X \) is totally bounded iff for all \( \epsilon > 0 \), there is no infinite \( \epsilon \)-packing.

**Proof.**

\( a) \) First suppose that we do not have a finite \( \epsilon \)-net in \( X \). Then \( X \) is nonempty, so we may choose \( p_1 \in X \). Since \( X \neq B(p_1, \epsilon) \), there is \( p_2 \in X \) with \( d(p_1, p_2) \geq \epsilon \).

Inductively, having constructed an \( n \) element \( \epsilon \)-packing \( P_n = \{p_1, \ldots, p_n\} \), since \( P_n \) is not a finite \( \epsilon \)-net there is \( p_{n+1} \in X \) such that \( d(p_i, p_{n+1}) \geq \epsilon \) for all \( 1 \leq i \leq n \), so \( P_{n+1} = P_n \cup \{p_{n+1}\} \) is an \( n+1 \) element \( \epsilon \)-packing. Then \( P = \bigcup_{n \in \mathbb{Z}^+} P_n \) is an infinite \( \epsilon \)-packing.

\( b) \) Seeking a contradiction, suppose that we have both an infinite \((2\epsilon)\)-packing \( P \) and a finite \( \epsilon \)-net \( N \). Since \( P \) is infinite, \( N \) is finite and \( X = \bigcup_{n \in N} B(n, \epsilon) \), there must be distinct points \( p \neq p' \in P \) each lying in \( B(n, \epsilon) \) for some \( n \in N \), and then by the triangle inequality \( d(p, p') \leq d(p, n) + d(n, n) < 2\epsilon \).

\( c) \) To say that \( N \subset X \) is an \( \epsilon \)-net means precisely that if we place an open ball of radius \( \epsilon \) centered at each point of \( N \), then the union of these balls covers \( X \). Thus total boundedness means precisely the existence of a finite \( \epsilon \)-net for all \( \epsilon > 0 \). The result then follows immediately from part \( a) \).

**Theorem 61.** A metric space \( X \) is totally bounded iff each sequence \( x \) in \( X \) admits a Cauchy subsequence.

**Proof.** If \( X \) is not totally bounded, then by Proposition 60 there is an infinite \( \epsilon \)-packing for some \( \epsilon > 0 \). Passing to a countably infinite subset \( P = \{p_n\}_{n=1}^\infty \), we get a sequence such that for all \( m \neq n \), \( d(p_m, p_n) \geq \epsilon \). This sequence has no Cauchy subsequence.

Now suppose that \( X \) is totally bounded, and let \( x \) be a sequence in \( X \). By total boundedness, for all \( n \in \mathbb{Z}^+ \), we can write \( X \) as a union of finitely many closed subsets \( Y_1, \ldots, Y_N \) each of diameter at most \( \frac{1}{n} \) (here \( N \) is of course allowed to depend on \( n \)). An application of the Pigeonhole Principle gives us a subsequence all of whose terms lie in \( Y_i \) for some \( i \), and thus we get a subsequence each of whose terms have distance at most \( \epsilon \). Unfortunately this is not quite what we want: we need one subsequence each of whose sufficiently large terms differ by at most \( \frac{1}{n} \). We attain this via a diagonal construction: namely, let

\[ x_{1,1}, x_{1,2}, \ldots, x_{1,n}, \ldots \]
be a subsequence each of whose terms have distance at most 1. Since subspaces of totally bounded spaces are totally bounded, we can apply the argument again inside the smaller metric space $Y_i$ to get a subsubsequence

$$x_{2,1}, x_{2,2}, \ldots, x_{2,n}, \ldots$$

each of whose terms differ by at most $\frac{1}{n}$ and each $x_{2,n}$ is selected from the subsequence $\{x_{1,n}\}$; and so on; for all $m \in \mathbb{Z}^+$ we get a subsub...subsequence

$$x_{m,1}, x_{m,2}, \ldots, x_{m,n}, \ldots$$

each of whose terms differ by at most $\frac{1}{n}$. Now we choose the diagonal subsequence: put $y_n = x_{n,n}$ for all $n \in \mathbb{Z}^+$. We allow the reader to check that this is a subsequence of the original sequence $x$. This sequence satisfies $d(y_n, y_{n+k}) \leq \frac{1}{n}$ for all $k \geq 0$, so we get a Cauchy subsequence.

10. Separability

We remind the reader that we are an ardent fan of [?]. The flattery becomes especially sincere at this point: c.f. [?, §5.2].

Recall that a metric space is separable if it admits a countable dense subset.

Exercise 10.1. Let $f : X \to Y$ be a continuous surjective map between metric spaces. Show that if $X$ is separable, so is $Y$.

We want to compare this property with two others that we have not yet introduced.

A base $\mathcal{B} = \{B_i\}$ for the topology of a metric space $X$ is a collection of open subsets of $X$ such that every open subset $U$ of $X$ is a union of elements of $\mathcal{B}$: precisely, there is a subset $J \subset \mathcal{B}$ such that $\bigcup_{i \in J} B_i = U$. (We remark that taking $J = \emptyset$ we get the empty union and thus the empty set.)

The example par excellence of a base for the topology of a metric space $X$ is to take $\mathcal{B}$ to be the family of all open balls in $X$. In this case, the fact that $\mathcal{B}$ is a base for the topology is in fact the very definition of the metric topology: the open sets are precisely the unions of open balls.

A countable base is just what it sounds like: a base which, as a set, is countable (either finite or countably infinite).

Proposition 62. a) Let $X$ be a metric space, let $\mathcal{B} = \{B_i\}$ be a base for the topology of $X$, and let $Y \subset X$ be a subset. Then $\mathcal{B} \cap Y := \{B_i \cap Y\}$ is a base for the topology of $Y$.

b) If $X$ admits a countable base, then so does all of its subsets.

Proof. a) This is almost immediate from the fact that the open subsets of $Y$ are precisely those of the form $U \cap Y$ for $U$ open in $X$. We leave the details to the reader.

b) This is truly immediate. □

Theorem 63. For a metric space $X$, the following are equivalent:

(i) $X$ is separable.

(ii) $X$ admits a countable base.

(iii) $X$ is Lindelöf.
Proof. (i) \implies (ii): Let \( Z \) be a countable dense subset. The family of open balls with center at some point of \( Z \) and radius \( \frac{1}{n} \) is then also countable (because a product of two countable sets is countable). So there is a sequence \( \{U_n\}_{n=1}^{\infty} \) in which every such ball appears at least once. I claim that every open set of \( X \) is a union of such balls. Indeed, let \( U \) be a nonempty subset (we are allowed to take the empty union to get the empty set!), let \( p \in U \), and let \( \epsilon > 0 \) be such that \( B(p, \epsilon) \subset U \). Choose \( n \) sufficiently large such that \( \frac{1}{n} < \frac{\epsilon}{2} \) and choose \( z \in Z \) such that \( d(z, p) < \frac{\epsilon}{2} \). Then \( p \in B(z, \frac{\epsilon}{2}) \subset B(p, \epsilon) \subset U \). It follows that \( U \) is a union of balls as claimed.

(ii) \implies (iii): Let \( B = \{B_n\}_{n=1}^{\infty} \) be a countable base for \( X \), and let \( \{U_i\}_{i \in I} \) be an open covering of \( X \). For each \( p \in X \), we have \( p \in U_i \) for some \( i \). Since \( U_i \) is a union of elements of \( B \) and \( p \in U_i \), we must have \( p \in B_{n(p)} \subset U_i \) for some \( n(p) \) depending on \( p \). Thus we have all the essential content for a countable subcovering, and we formalize this as follows: let \( J \) be the set of all positive integers \( n \) such that \( B_n \) lies in \( U_i \) for some \( i \); notice that \( J \) is countable! For each \( n \in J \), choose \( i_n \in I \) such that \( B_n \subset U_{i_n} \). It then follows that \( X = \bigcup_{n \in J} U_{i_n} \).

(iii) \implies (i): For each \( n \in \mathbb{Z}^+ \), the collection \( \{B(p, \frac{1}{n})\}_{p \in X} \) certainly covers \( X \). Since \( X \) is Lindelöf, there is a countable subcover. Let \( Z_n \) be the set of centers of the elements of this countable subcover, so \( Z_n \) is a countable \( \frac{1}{n} \)-net. Put \( Z = \bigcup_{n \in \mathbb{Z}^+} Z_n \). Then \( Z \) is a countable dense subset. \( \Box \)

As usual, we now get to play the good properties of separability, existence of countable bases, and Lindelöfness off against one another. For instance, we get:

**Corollary 64.**

a) Every subset of a separable metric space is separable.

b) Every subset of a Lindelöf metric space is Lindelöf.

c) If \( f : X \to Y \) is a continuous surjective map of metric spaces and \( X \) has a countable base, so does \( Y \).

We suggest that the reader pause and try to give a proof of Corollary ?? directly from the definition: it is really not straightforward to do so.

**Exercise 10.2.**

Let \( X \) be a separable metric space, and let \( E \subset X \) be a discrete subset: every point of \( E \) is an isolated point. Show that \( E \) is countable.

Recall that point \( p \) in a metric space is isolated if \( \{p\} \) is an open set. If we like, we can rephrase this by saying that \( p \) admits a neighborhood of cardinality 1. Otherwise \( p \) is a limit point: every neighborhood of \( p \) contains points other than \( p \). Because finite metric spaces are discrete, we can rephrase this by saying that every neighborhood of \( p \) is infinite. This little discussion perhaps prepares us for the following more technical definition.

A point \( p \) of a metric space \( X \) is an \( \omega \)-limit point if every neighborhood of \( p \) in \( X \) is uncountable.

**Theorem 65.**

A separable metric space has at most continuum cardinality.

**Exercise 10.3.**

Prove it. (Hint: think about limits of sequences.)

**Theorem 66.**

Let \( X \) be an uncountable separable metric space. Then all but countably many points of \( X \) are \( \omega \)-limit points.

**Theorem 67.**

Let \( X \) be an uncountable, complete separable metric space. Then \( X \) has continuum cardinality.
Proof. By Theorem 11.5, X has at most continuum cardinality, so it will suffice to exhibit continuum-many points of X.

Step 1: We claim that for all \( \delta > 0 \), there is 0 \( \leq \epsilon \leq \delta \) and \( x, y \in X \) such that the closed \( \epsilon \)-balls \( B^*(x, \epsilon) \) and \( B^*(y, \epsilon) \) are disjoint and each contain uncountably many points. Indeed, by Theorem 11.5, X has uncountably many \( \omega \)-limit points. Choose two of them \( x \neq y \) and take any \( \epsilon < d(x, y) \).

Step 2: Applying the above construction with \( \delta = 1 \) we get uncountable disjoint closed subsets \( A_0 \) and \( A_1 \) each of diameter at most 1. Each of \( A_0 \) and \( A_1 \) is itself uncountable, complete and separable, so we can run the construction in \( A_0 \) and in \( A_1 \) to get uncountable disjoint closed subsets \( A_{00}, A_{01} \) in \( A_1 \) and \( A_{10}, A_{11} \) in \( A_2 \), each of diameter at most \( \frac{1}{2} \). Continuing in this way we get for each \( n \in \mathbb{Z}^+ \) a pairwise disjoint family of \( 2^n \) uncountable closed subsets \( A_{i_1 \ldots i_n} \) (with \( i_1, \ldots, i_n \in \{0, 1\} \)) each of diameter at most \( 2^{-n} \). Any infinite binary sequence \( \epsilon \in \{0, 1\}^{\mathbb{Z}^+} \) yields a nested sequence of nonempty closed subsets of diameter approaching zero, so by completeness each such sequence has a unique intersection point \( p_\epsilon \). If \( \epsilon \neq \epsilon' \) are distinct binary sequences, then for some \( n, \epsilon_n \neq \epsilon'_n \), so \( p_\epsilon \) and \( p_{\epsilon'} \) are contained in disjoint subsets and are thus distinct. This gives \( 2^{\#\mathbb{Z}^+} = \#\mathbb{R} \) points of X. \( \square \)

The following exercise gives a subtle sharpening of Corollary 11.5 (which does not directly use Theorem 11.5).

Exercise 10.4. a) (S. Ivanov) Let X be a complete metric space without isolated points. Show that X has at least continuum cardinality. (Suggestion: the lack of isolated points implies that every closed ball of positive radius is infinite. This allows one to run the argument of Step 2 of the proof of Theorem 11.5 above.)

b) Explain why the assertion that every uncountable complete metric space has at least continuum cardinality is equivalent to the Continuum Hypothesis: i.e., that every uncountable set has at least continuum cardinality.\(^3\)

11. Compactness Revisited

The following is perhaps the single most important theorem in metric topology.

Theorem 68. Let X be a metric space. The following are equivalent:

(i) X is compact: every open covering of X has a finite subcovering.
(ii) X is sequentially compact: every sequence in X has a convergent subsequence.
(iii) X is limit point compact: every infinite subset of X has a limit point in X.
(iv) X is complete and totally bounded.

Proof. We will show (i) \( \implies \) (iii) \( \iff \) (ii) \( \iff \) (iv) \( \implies \) (i).

(i) \( \implies \) (iii): Suppose X is compact, and let \( A \subset X \) have no limit point in X. We must show that A is finite. Recall that \( \overline{A} \) is obtained by adjoining the set \( A' \) of limit points of A, so in our case we have \( \overline{A} = A \cup A' = A \cup \emptyset = A \), i.e., A is closed in the compact space X, so A is itself compact. On the other hand, no point of A is a limit point, so A is discrete. Thus \( \{\{a\}_{a \in A}\} \) is an open covering of A, which certainly has no proper subcovering: we need all the points of A to cover A! So the given covering must itself be finite: i.e., A is finite.

(iii) \( \implies \) (ii): Let \( x \) be a sequence in X; we must find a convergent subsequence.

\(^3\)Curiously, Exercise 8 in §5.2 of [?] reads “Prove that every uncountable complete metric space has at least the cardinal number \( \mathfrak{c}' \). Is Kaplansky asking for a proof of the Continuum Hypothesis??
If some element occurs infinitely many times in the sequence, we have a constant subsequence, which is convergent. Otherwise \( A = \{x_n \mid n \in \mathbb{Z}^+\} \) is infinite, so it has a limit point \( x \in X \) and thus we get a subsequence of \( x \) converging to \( x \).

(ii) \( \Rightarrow \) (iii): Let \( A \subset X \) be infinite; we must show that \( A \) has a limit point in \( X \). The infinite set \( A \) contains a countably infinite subset; enumerating these elements gives us a sequence \( \{a_n\}_{n=1}^{\infty} \). By assumption, we have a subsequence converging to some \( x \in X \), and this \( x \) is a limit point of \( A \).

(ii) \( \Rightarrow \) (iv): Let \( x \) be a Cauchy sequence in \( X \). By assumption \( x \) has a convergent subsequence, which by Lemma ?? implies that \( x \) converges: \( X \) is complete. Let \( x \) be a sequence in \( X \). Then \( x \) has a convergent, hence Cauchy, subsequence. By Theorem ??, the space \( X \) is totally bounded.

(iv) \( \Rightarrow \) (i): Let \( x \) be a sequence in \( X \). By total boundedness \( x \) admits a Cauchy subsequence, which by completeness is convergent. So \( X \) is sequentially compact.

(iv) \( \Rightarrow \) (i): Seeking a contradiction, we suppose that there is an open covering \( \{U_i\}_{i \in I} \) of \( X \) without a finite subcovering. Since \( X \) is totally bounded, it admits a finite covering by closed balls of radius 1. It must be the case that for at least one of these balls, say \( A_1 \), the open covering \( \{U_i \cap A_1\}_{i \in I} \) of \( A_1 \) does not have a finite subcovering – for if each had a finite subcovering, by taking the finite union of these finite subcoverings we would get a finite subcovering of \( \{U_i\}_{i \in I} \). Since \( A_1 \) is a closed subset of a complete, totally bounded space, it is itself complete and totally bounded. So we can cover \( A_1 \) by finitely many closed balls of radius \( \frac{1}{2} \) and run the same argument, getting at least one such ball, say \( A_2 \subset A_1 \), for which the open covering \( \{U_i \cap A_2\}_{i \in I} \) has no finite subcovering. Continuing in this way we build a nested sequence of closed balls \( \{A_n\}_{n=1}^{\infty} \) of radii tending to 0, and thus also \( \text{diam } A_n \to 0 \). By completeness there is a point \( p \in \bigcap_{n=1}^{\infty} A_n \). Since \( \bigcup_{i \in I} U_i = X \), certainly we have \( p \in U_i \) for at least one \( i \in I \). Since \( U_i \) is open, there is some \( \epsilon > 0 \) such that \( B(p, \epsilon) \subset U_i \). Choose \( N \in \mathbb{Z}^+ \) such that \( \text{diam } A_N < \epsilon \). Then since \( p \in A_N \), we have \( A_N \subset B(p, \epsilon) \subset U_i \). But this means that \( A_N = U_i \cap A_N \) is a one element subcovering of \( A_N \): contradiction. \( \Box \)

**Exercise 11.1.** A metric space is **countably compact** if every countable open cover admits a finite subcover. a) Show that for a metric space \( X \), the following are equivalent:

(i) \( X \) is countably compact.

(ii) For any sequence \( \{A_n\}_{n=1}^{\infty} \) of closed subsets, if for all finite nonempty subsets \( J \subset \mathbb{Z}^+ \) we have \( \bigcap_{n \in J} A_n \neq \emptyset \), then \( \bigcap_{n=1}^{\infty} A_n \neq \emptyset \).

(iii) For any nested sequence \( A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots \) of nonempty closed subsets of \( X \), we have \( \bigcap_{n=1}^{\infty} A_n \neq \emptyset \).

b) Show that a metric space is compact if and only if it is countably compact. (Suggestion: use the assumption that \( X \) is not limit-point compact to build a countable open covering without a finite subcovering.)

**11.1. Partial Limits.**

Let \( x \) be a sequence in a metric space \( X \). Recall that a \( p \in X \) is a **partial limit** of \( x \) if some subsequence of \( x \) converges to \( p \).

Though this concept has come up before, we have not given it much attention. This section is devoted to a more detailed analysis.
Exercise 11.2. Show that the partial limits of $\{(-1)^n\}_{n=1}^\infty$ are precisely $-1$ and $1$.

Exercise 11.3. Let $\{x_n\}$ be a real sequence which diverges to $\infty$ or to $-\infty$. Show that there are no partial limits.

Exercise 11.4. In $\mathbb{R}^2$, let $x_n = (n \cos n, n \sin n)$. Show that there are no partial limits.

Exercise 11.5. In $\mathbb{R}$, consider the sequence $0, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{3}{2}, -2, -\frac{5}{3}, -\frac{4}{3}, \ldots, 3, \frac{11}{4}, \ldots$ Show that every real number is a partial limit.

Exercise 11.6. a) Let $\{x_n\}$ be a sequence in a metric space such that every bounded subset of the space contains only finitely many terms of the sequence. Then there are no partial limits.

b) Show that a metric space admits a sequence as in part a) if and only if it is unbounded.

Proposition 69. In any compact metric space, every sequence has at least one partial limit.

Proof. Indeed, this is just a rephrasing of “compact metric spaces are sequentially compact”.

Exercise 11.7. Show that a convergent sequence in a metric space has a unique partial limit: namely, the limit of the sequence.

In general, the converse is not true: e.g. the sequence $\frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \ldots, \frac{1}{n}, n, \ldots$ has $0$ as its only partial limit, but it does not converge.

Proposition 70. In a compact metric space $X$, a sequence with exactly one partial limit converges.

Proof. Let $L$ be a partial limit of a sequence $\{x_n\}$, and suppose that the sequence does not converge to $L$. Then there is some $\epsilon > 0$ such that $B^c(L, \epsilon)$ misses infinitely many terms of the sequence. Therefore some subsequence lies in $Y = X \setminus B^c(L, \epsilon)$. This is a closed subset of a compact space, so it is compact, and therefore this subsequence has a partial limit $L' \in Y$, which is then a partial limit of the original sequence. Since $L \notin Y$, $L' \neq L$.

Proposition 71. Let $\{x_n\}$ be a sequence in a metric space $X$. Then the set $\mathcal{L}$ of partial limits of $\{x_n\}$ is a closed subset.

Proof. We will show that the complement of $\mathcal{L}$ is open: let $y \in X \setminus \mathcal{L}$. Then there is $\epsilon > 0$ such that $B^c(y, \epsilon)$ contains only finitely many terms of the sequence. Now for any $z \in B^c(y, \epsilon)$, $B^c(z, \epsilon - d(y, z)) \subset B^c(y, \epsilon)$ so $B^c(z, \epsilon - d(y, z))$ also contains only finitely many points of the sequence and thus $z$ is not a partial limit of the sequence. It follows that $B^c(y, \epsilon) \subset X \setminus \mathcal{L}$.

Now let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$: say $x_n \in [a, b]$ for all $n$. Since $[a, b]$ is closed, the set $\mathcal{L}$ of partial limits is contained in $[a, b]$, so it is bounded. By the previous result, $\mathcal{L}$ is closed. So $\mathcal{L}$ has a minimum and maximum element, say $\underline{L}$ and $\overline{L}$. The sequence converges iff $\underline{L} = \overline{L}$. 

We claim that $\mathcal{L}$ can be characterized as follows: for any $\epsilon > 0$, only finitely many terms of the sequence lie in $[\mathcal{L} + \epsilon, \mathcal{L}]$; and for any $\epsilon > 0$, infinitely many terms of the sequence lie in $[\mathcal{L} - \epsilon, \mathcal{L}]$. Indeed, if infinitely many terms of the sequence lay in $[\mathcal{L} + \epsilon, \mathcal{L}]$, then by Bolzano-Weierstrass there would be a partial limit in this interval, contradicting the definition of $\mathcal{L}$. The second implication is even easier: since $\mathcal{L}$ is a partial limit, then for all $\epsilon > 0$, the interval $[\mathcal{L} - \epsilon, \mathcal{L} + \epsilon]$ contains infinitely many terms of the sequence.

We can now relate $\mathcal{L}$ to the limit supremum. Namely, put

$$X_n = \{x_k \mid k \geq n\}$$

and put

$$\limsup s_n = \lim_{n \to \infty} \sup X_n.$$ 

Let us first observe that this limit exists: indeed, each $X_n$ is a subset of $[a, b]$, hence bounded, hence $\sup X_n \in [a, b]$. Since $X_{n+1} \subset X_n$, $\sup X_{n+1} \leq \sup X_n$, so $\{\sup X_n\}$ forms a bounded decreasing sequence and thus converges to its least upper bound, which we call the limit superior of the sequence $x_n$.

We claim that $\limsup x_n = \mathcal{L}$. We will show this by showing that $\limsup x_n$ has the characteristic property of $\mathcal{L}$. Let $\epsilon > 0$. Then since $(\limsup x_n) + \epsilon > \limsup x_n$, then for some (and indeed all sufficiently large) $N$ we have

$$x_n \leq \sup X_N < (\limsup x_n) + \epsilon,$$

showing the first part of the property: there are only finitely many terms of the sequence to the right of $(\limsup x_n) + \epsilon$. For the second part, fix $N \in \mathbb{Z}^+$; then

$$(\limsup x_n) - \epsilon < (\limsup x_n) \leq \sup X_N,$$

so that $(\limsup x_n - \epsilon)$ is not an upper bound for $X_N$: there is some $n \geq N$ with $(\limsup x_n - \epsilon) < x_n$. Since $N$ is arbitrary, this shows that there are infinitely many terms to the right of $(\limsup x_n - \epsilon)$.

We deduce that $\mathcal{L} = \limsup x_n$.

**Theorem 72.** Let $X$ be a metric space. For a nonempty subset $Y \subset X$, the following are equivalent:

(i) There is a sequence $\{x_n\}$ in $X$ whose set of partial limits is precisely $Y$.

(ii) There is a countable subset $Z \subset Y$ such that $Y = \mathcal{Z}$.

**Proof.** Step 1: First suppose $Y = \mathcal{Z}$ for a countable, nonempty subset $Z$. If $Z$ is finite then it is closed and $Y = Z$. In this case suppose the elements of $Z$ are $z_1, \ldots, z_N$, and take the sequence $z_1, \ldots, z_N, z_1, \ldots, z_N, \ldots$. On the other hand, if $Z$ is countably infinite then we may enumerate its elements $\{z_n\}_{n=1}^{\infty}$. We take the sequence $z_1, z_2, z_1, z_2, z_3, \ldots, z_1, \ldots, z_N, \ldots$. 


In either case: since each element $z \in Z$ appears infinitely many times as a term of the sequence, there is a constant subsequence converging to $z \in Z$. Since the set $L$ of partial limits is closed and contains $Z$, we must have

$$L \supseteq Z = Y.$$  

Finally, every term of the sequence lies in the closed set $Y$, hence so does every term of every subsequence, and so the limit of any convergent subsequence must also lie in $Y$. Thus $L = Y$.

**Step 2:** Now let $\{x_n\}$ be any sequence in $X$ and consider the set $L$ of partial limits of the sequence. We may assume that $L \neq \emptyset$. We know that $L$ is closed, so it remains to show that there is a countable subset $Z \subseteq L$ such that $L = Z$: in other words, we must show that $L$ is a separable metric space. Let $W = \{x_n \mid n \in \mathbb{Z}^+\}$ be the set of terms of the sequence. Then $W$ is countable, and arguing as above we find $L \subseteq W$. Therefore $L$ is a subset of a separable metric space, so by Corollary ??, $L$ is itself separable. \qed

Though Theorem ?? must have been well known for many years, I have not been able to find it in print (in either texts or articles). In fact two recent articles address the collection of partial limits of a sequence in a metric space: [?] and [?]. The results that they prove are along the lines of Theorem ?? but not quite as general: the main result of the latter article is that in a separable metric space every nonempty closed subset is the set of partial limits of a sequence. Moreover the proof that they give is significantly more complicated.

### 11.2. Lebesgue Numbers.

**Lemma 73.** Let $(X, d)$ be a compact metric space, and let $U = \{U_i\}_{i \in I}$ be an open cover of $X$. Then $U$ admits a Lebesgue number: a $\delta > 0$ such that for every nonempty $A \subset X$ of diameter less than $\delta$, $A \subset U_i$ for at least one $i \in I$.

**Proof.** If $\delta$ is a Lebesgue number, so is any $0 < \delta' < \delta$. It follows that if Lebesgue numbers exist, then $\frac{1}{n}$ is a Lebesgue number for some $n \in \mathbb{Z}^+$. So, seeking a contradiction we suppose that for all $n \in \mathbb{Z}^+$, $\frac{1}{n}$ is not a Lebesgue number. This implies that for all $n \in \mathbb{Z}^+$ there is $x_n \in X$ such that $B^*(x_n, \frac{1}{n})$ is not contained in $U_i$ for any $i \in I$. Since $X$ is sequentially compact, there is a subsequence $x_{n_k} \to x \in X$. Choose $i \in I$ such that $x \in U_i$. Then for some $\epsilon > 0$, $B^*(x, \epsilon) \subset U_i$, and when $\frac{1}{n_k} < \frac{\epsilon}{2}$ we get

$$B^*(x_{n_k}, \frac{1}{n_k}) \subset B^*(x, \epsilon) \subset U_i,$$

a contradiction. \qed

**Proposition 74.** Let $f : X \to Y$ be a continuous map between metric spaces. For $\epsilon > 0$, suppose that the open cover $U_\epsilon = \{f^{-1}(B(y, \frac{\epsilon}{2})) \mid y \in Y\}$ of $X$ admits a Lebesgue number $\delta$. Then $f$ is $(\epsilon, \delta)$-UC.

**Proof.** If $d(x, x') < \delta$, $x, x' \in B(x, \delta)$. Since $\delta$ is a Lebesgue number for $U_\epsilon$, there is $y \in Y$ such that $f(B(x, \delta)) \subset B(y, \frac{\epsilon}{2})$ and thus

$$d(f(x), f(x')) < d(f(x), y) + d(y, f(x')) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

\[\square\]

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\(^4\)You need not agree that six years ago is “recent”, but some day you probably will!
Theorem 75. Let $X$ be a compact metric space and $f : X \to \mathbb{R}$ a continuous function. Then $f$ is uniformly continuous.

Proof. Let $\epsilon > 0$. By Lemma ??, the covering $\{f^{-1}(B(y, \frac{\epsilon}{2}))\}_{y \in \mathbb{R}}$ of $[a, b]$ has a Lebesgue number $\delta > 0$, and then by Proposition ??, $f$ is $(\epsilon, \delta)$-UC. Thus $f$ is uniformly continuous. \hfill \Box

12. Extension Theorems

Let $X$ and $Y$ be metric spaces, let $A \subset X$ be a subset, and let

$$f : A \to Y$$

be a continuous function. We say $f$ extends to $X$ if there is a continuous map

$$F : X \to Y$$

such that

$$\forall x \in A, F(x) = f(x).$$

We also say that $F$ extends $f$ and write $F|_A = f$. (That $F$ must be continuous is suppressed from the terminology: this is supposed to be understood.) We are interested in both the uniqueness and the existence of the extension.

Proposition 76. Let $X$ and $Y$ be metric spaces, let $A \subset X$ and let $f : A \to Y$ be a continuous function. If $A$ is dense in $X$, then there is at most one continuous function $F : X \to Y$ such that $F|_A = f$.

Proof. Suppose $F_1, F_2 : X \to Y$ both extend $f : A \to Y$, and let $x \in X$. Since $A$ is dense, there is a sequence $\mathbf{a}$ in $A$ which converges to $x$. Then

$$F_1(x) = F_1(\lim_{n \to \infty} a_n) = \lim_{n \to \infty} F_1(a_n) = \lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} F_2(a_n) = F_2(\lim_{n \to \infty} a_n) = F_2(x).$$

Exercise 12.1. Let $A \subset X$, and let $f : A \to Y$ be a continuous map. Show that $f$ has at most one continuous extension to $F : \overline{A} \to Y$.

Proposition 77. Let $f : X \to Y$ be a uniformly continuous map of metric spaces. Let $\mathbf{x}$ be a Cauchy sequence in $X$. Then $f(\mathbf{x})$ is a Cauchy sequence in $Y$.

Proof. Let $\epsilon > 0$. By uniform continuity, there is $\delta > 0$ such that for all $y, z \in X$, if $d(y, z) \leq \delta$ then $d(f(y), f(z)) \leq \epsilon$. Since $\mathbf{x}$ is Cauchy, there is $N \in \mathbb{Z}^+$ such that if $m, n \geq N$ then $d(x_m, x_n) \leq \delta$. For all $m, n \geq N$, $d(f(x_m), f(x_n)) \leq \epsilon$. \hfill \Box

Theorem 78. Let $X$ be a metric space, $Y$ a complete metric space, $A \subset X$ a dense subset, and let $f : A \to Y$ be uniformly continuous.

a) There is a unique continuous map $F : \overline{A} \to Y$.

b) The unique extension $F : X \to Y$ is uniformly continuous.

c) If $f$ is an isometric embedding, then so is $F$.

Proof. a) Exercise ?? shows that if $F : \overline{A} \to Y$ is any continuous extension, then $F(x)$ must be $\lim_{n \to \infty} f(a_n)$ for any sequence $\mathbf{a} \to x$. It remains to show that this limit actually exists and does not depend upon the choice of sequence $\mathbf{a}$ which converges to $x$. But we are well prepared for this: since $\mathbf{a} \to x$ in $X$, as a sequence in $A$, $\mathbf{a}$ is Cauchy. Since $f$ is uniformly continuous, $f(\mathbf{a})$ is Cauchy. Since $Y$ is complete,
Let \( f(a) \) converges. If \( b \) is another sequence in \( A \) converging to \( x \), then \( d(a_n, b_n) \to 0 \), so by uniform continuity, \( d(f(a_n), f(b_n)) \to 0 \).

b) Fix \( \epsilon > 0 \), and choose \( \delta > 0 \) such that \( f \) is \((\frac{\epsilon}{2}, \delta)\)-uniformly continuous. We claim that \( F \) is \((\epsilon, \delta)\)-uniformly continuous. Let \( x, y \in X \) with \( d(x, y) \leq \delta \). Choose sequences \( a \) and \( b \) in \( A \) converging to \( x \) and \( y \) respectively. Then \( d(x, y) = \lim_{n \to \infty} d(a_n, b_n) \), so by our choice of \( \delta \) for all sufficiently large \( n \) we have \( d(a_n, b_n) \leq \delta \). For such \( n \) we have \( d(f(a_n), f(b_n)) \leq \epsilon \), so

\[
d(f(x), f(y)) = \lim_{n \to \infty} d(f(a_n), f(b_n)) \leq \epsilon.
\]

c) Suppose \( f \) is an isometric embedding, let \( x, y \in X \) and choose sequences \( a, b \) in \( A \) converging to \( x \) and \( y \) respectively. Then

\[
d(f(x), f(y)) = d(f(\lim_{n \to \infty} a_n), f(\lim_{n \to \infty} b_n)) = \lim_{n \to \infty} d(f(a_n), f(b_n)) = \lim_{n \to \infty} d(a_n, b_n) = d(x, y).
\]

Exercise 12.2. The proof of Theorem ?? does not quite show the simpler-looking statement that if \( f : A \to Y \) is \((\epsilon, \delta)\)-uniformly continuous then so is the extended function \( F : X \to Y \). Show that this is in fact true. (Suggestion: this follows from what was proved via a limiting argument.)

Exercise 12.3. Maintain the setting of Theorem ??.

a) Show: if \( f : A \to Y \) is contractive, so is \( F \).

b) Show: if \( f : A \to Y \) is Lipschitz, so is \( F \). Show in fact that the optimal Lipschitz constants are equal: \( L(F) = L(f) \).

Exercise 12.4. Let \( P : \mathbb{R} \to \mathbb{R} \) be a polynomial function, i.e., there are \( a_0, \ldots, a_d \in \mathbb{R} \) such that \( P(x) = a_d x^d + \cdots + a_1 x + a_0 \).

a) Show that \( P \) is uniformly continuous iff its degree \( d \) is at most 1.

b) Taking \( A = \mathbb{Q} \), \( X = Y = \mathbb{R} \), use part a) to show that uniform continuity is not a necessary condition for the existence of a continuous extension.

Exercise 12.5. Say that a function \( f : X \to Y \) between metric spaces is \textbf{Cauchy continuous} if for every Cauchy sequence \( x \) in \( X \), \( f(x) \) is Cauchy in \( Y \).

a) Show: uniform continuity implies Cauchy continuity implies continuity.

b) Show: Theorem ?? holds if “uniform continuity” is replaced everywhere by “Cauchy continuity”.

c) Let \( X \) be totally bounded. Show: Cauchy continuity implies uniform continuity.

Theorem 79. (Tietze Extension Theorem) Let \( X \) be a metric space, \( Y \subset X \) a closed subset, and let \( f : Y \to \mathbb{R} \) be a continuous function. Then there is a continuous function \( F : X \to \mathbb{R} \) with \( F|_Y = f \).

Proof. We will give a proof of a more general version of this result later on in these notes.

Corollary 80. For a metric space \( X \), the following are equivalent:

(i) \( X \) is compact.

(ii) Every continuous function \( f : X \to \mathbb{R} \) is bounded.

Proof. (i) \( \implies \) (ii): this is the Extreme Value Theorem.

(ii) \( \implies \) (i): By contraposition and using Theorem ?? it suffices to assume that \( X \) is not limit point compact – thus admits a countably infinite, discrete closed
subset $Y$ – and from this build an unbounded continuous real-valued function. We did this by hand when $X = [a, b]$ in $X$. Here we do something similar but higher tech. Namely, write $Y = \{x_n\}_{n=1}^\infty$ and define $f$ on $A$ by $f(n) = x_n$. By the Tietze Extension Theorem, there is a continuous function $F : X \to \mathbb{R}$ with $F|_Y = f$. Since $F$ takes on all positive integer values, it is unbounded.

12.1. $C(X, Y)$.

For metric spaces $X$ and $Y$, let $C_b(X, Y)$ denote the set of bounded continuous functions $f : X \to Y$. For $f, g \in C_b(X, Y)$, we put

$$d(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

**Lemma 81.** a) The function $d : C_b(X, Y) \times C_b(X, Y) \to \mathbb{R}$ is a metric function.

b) A sequence $\{f_n\}$ in $C_b(X, Y)$ converges to $f$ iff $f_n$ converges uniformly to $f$.

**Exercise 12.6.** Prove it.

**Theorem 82.** Let $X$ and $Y$ be metric spaces. The following are equivalent:

(i) The space $Y$ is complete.

(ii) The space $C_b(X, Y)$ is complete.

13. Completion

Completeness is such a desirable property that given a metric space which is not complete we would like to fix it by adding in the missing limits of Cauchy sequences. Of course, the above description is purely intuitive: although we may visualize $\mathbb{R}$ as being constructed from $\mathbb{Q}$ by “filling in the irrational holes”, it is much less clear that something like this can be done for an arbitrary metric space.

The matter of the problem is this: given a metric space $X$, we want to find a complete metric space $Y$ and an isometric embedding $\iota : X \hookrightarrow Y$.

However this can clearly be done in many ways: e.g. we can isometrically embed $\mathbb{Q}$ in $\mathbb{R}$ but also in $\mathbb{R}^N$ for any $N$ (in many ways, but e.g. as $r \mapsto (r, 0, \ldots, 0)$). Intuitively, the embedding $\mathbb{Q} \hookrightarrow \mathbb{R}$ feels natural while (e.g.) the embedding $\mathbb{Q} \hookrightarrow \mathbb{R}^{17}$ feels wasteful. If we reflect on this for a bit, we see that we can essentially recover the good case from the bad case by passing from $Y$ to the closure of $\iota(X)$ in $Y$. We then get $\mathbb{R} \times 0^{16}$, which is evidently isometric to $\mathbb{R}$ (and even compatibly with the embedding of $\mathbb{Q}$: more on this shortly).

In general: if $\iota : X \to Y$ is an isometric embedding into a complete metric space, then (because closed subsets of complete metric spaces are complete), $\iota : X \to \iota(X)$ is an isometric embedding into a complete metric space with dense image, or for short a dense isometric embedding. Remarkably, adding the density condition gives us a uniqueness result.

**Lemma 83.** Let $X$ be a metric space, and for $i = 1, 2$, let $\iota_i : X \to Y_i$ be dense isometric embeddings into a complete metric space. Then there is a unique isometry $\Phi : Y_1 \to Y_2$ such that $\iota_2 = \Phi \circ \iota_1$. 
Applying Theorem ?? with \( A = X, X = Y_1 \) and \( Y_2 = Y \) we get an isometric embedding \( \Phi : Y_1 \to Y_2 \). Similarly, we get an isometric embedding \( \Phi' : Y_2 \to Y_1 \). The compositions \( \Phi' \circ \Phi \) and \( \Phi' \circ \Phi \) are continuous maps restricting to \( 1_X \) on the dense subspace \( X \), so by Proposition ?? we must have
\[
\Phi' \circ \Phi = 1_{X_1}, \quad \Phi \circ \Phi' = 1_{X_2}.
\]
So \( \Phi \) is an isometry and \( \Phi' = \Phi^{-1} \). Proposition ?? gives the uniqueness of \( \Phi \). \( \square \)

This motivates the following key definition: a completion of a metric space \( X \) is a complete metric space \( \hat{X} \) and a dense isometric embedding \( \iota : X \hookrightarrow \hat{X} \). It follows from Lemma ?? that if a metric space admits a completion then any two completions are isometric (and even more: the embedding into the completion is essentially unique). Thus for any metric space \( X \) we have associated a new metric space \( \hat{X} \). Well, not quite: there is the small matter of proving the existence of \( \hat{X} \)!

To know “everything but existence” perhaps seems bizarre (even Anselmian?). In fact it is quite common in modern mathematics to define an object by a characteristic property and then be left with the task of “constructing” the object, which can generally be done in several different ways. In this particular instance there are two standard constructions of “the” completion \( \hat{X} \) of a metric space \( X \).

**First Construction of the Completion:**

Let \( X^\infty = \prod_{n=1}^\infty X \) be the set of all sequences in \( X \). Inside \( X^\infty \), we define \( \mathcal{X} \) to be the set of all Cauchy sequences. We introduce an equivalence relation on \( \mathcal{X} \) by \( x_\bullet \sim y_\bullet \) if \( \rho(x_n, y_n) \to 0 \). Put \( \hat{X} = \mathcal{X} / \sim \). For any \( x \in X \), define \( \iota(x) = (x, x, \ldots) \), the constant sequence based on \( x \). This converges (to \( x \)), so is Cauchy and hence lies in \( \mathcal{X} \). The composite map \( X \xrightarrow{\iota} \mathcal{X} \xrightarrow{\sim} \hat{X} \) (which we continue to denote by \( \iota \)) is injective, since \( \rho(x_n, y_n) = \rho(x, y) \) does not approach zero. We define a map \( \hat{\rho}: \mathcal{X} \times \mathcal{X} \to \mathbb{R} \) by
\[
\hat{\rho}(x_\bullet, y_\bullet) = \lim_{n \to \infty} \rho(x_n, y_n).
\]

To see that this limit exists, we may reason as follows: the sequence \( x_\bullet \times y_\bullet \) is Cauchy in \( X \times X \), so its image under the uniformly continuous function \( \rho \) is Cauchy in the complete metric space \( \mathbb{R} \), so it is convergent. It is easy to see that \( \hat{\rho} \) factors through to a map \( \hat{\rho}: \hat{X} \to \mathbb{R} \). The verification that \( \hat{\rho} \) is a metric on \( \hat{X} \) and that \( \iota : X \to \hat{X} \) is an isometric embedding is straightforward and left to the reader. Moreover, if \( x_\bullet = \{x_n\} \) is a Cauchy sequence in \( X \), then the sequence of constant sequences \( \{\iota(x_n)\} \) is easily seen to converge to \( x_\bullet \) in \( \hat{X} \).

The first construction is satisfying in that it supplies the “missing points” in the most direct possible way: essentially we add the Cauchy sequence itself in as the missing point. This doesn’t quite work because in general many different sequences will have the same limit, so we mod out by a natural equivalence relation. (FIXME: explain this in terms of the associated metric of a pseudometric.)

**Second Construction of the Completion:**

By \( X \), the set \( C_b(X, \mathbb{R}) \) of bounded continuous functions \( f: X \to \mathbb{R} \) is a complete metric space under \( d(f, g) = \sup_{x \in X} d(f(x), g(x)) \). Fix a point \( \bullet \in X \). For \( x \in X \),
let \( D_x : X \to \mathbb{R} \) be given by
\[
D_x(y) = d(\bullet, y) - d(x, y).
\]

We claim that \( D_x : X \to \mathbb{R} \) is bounded and continuous, and thus we get a map
\[
\mathcal{D} : X \to C_b(\mathbb{X}, \mathbb{R}), \ x \mapsto D_x.
\]
We further claim that \( D \) is an isometric embedding. Assuming both: we’re done!

As above, we get \( \hat{\mathbb{X}} \) by taking the closure of \( \mathcal{D}(X) \) in the complete space \( C_b(\mathbb{X}, \mathbb{R}) \).

**Corollary 84.** (Functoriality of completion)
a) Let \( f : X \to Y \) be a uniformly continuous map between metric spaces. Then there exists a unique map \( F : \hat{\mathbb{X}} \to \hat{\mathbb{Y}} \) making the following diagram commute:
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\hat{X} & \xrightarrow{F} & \hat{Y}.
\end{array}
\]
b) If \( f \) is an isometric embedding, so is \( F \).
c) If \( f \) is an isometry, so is \( F \).

**Proof.** a) The map \( f' : X \to Y \to \hat{Y} \), being a composition of uniformly continuous maps, is uniformly continuous. Applying the universal property of completion to \( f' \) gives a unique extension \( \hat{X} \to \hat{Y} \).

Part b) follows immediate from Lemma ??b). As for part c), if \( f \) is an isometry, so is its inverse \( f^{-1} \). The extension of \( f^{-1} \) to a mapping from \( \hat{Y} \) to \( \hat{X} \) is easily seen to be the inverse function of \( F \).


**Lemma 85.** Let \( Y \) be a dense subspace of a metric space \( X \). Then \( X \) is totally bounded iff \( Y \) is totally bounded.

**Proof.** If \( X \) is totally bounded, then every subspace of \( X \) is totally bounded, so we do not need the density of \( Y \) for this direction. Conversely, suppose \( Y \) is totally bounded, and let \( \epsilon > 0 \). Then there is a finite \( \epsilon \)-net \( N \) in \( Y \). I claim that for any \( \epsilon' > 0 \), \( N \) is a finite \( \epsilon' \)-net in \( X \). Indeed, let \( x \in X \). Since \( Y \) is dense in \( X \), there is \( y \in Y \) with \( d(x, y) < \epsilon' - \epsilon \), and there is \( n \in N \) with \( d(y, n) < \epsilon \), so \( d(x, n) < \epsilon' \). It follows that \( X \) is totally bounded.

**Theorem 86.** For a metric space \( X \), the following are equivalent:
(i) \( X \) is totally bounded.
(ii) The completion of \( X \) is compact.

**Proof.** Let \( \iota : X \to \hat{X} \) be “the” isometric embedding of \( X \) into its completion.

(i) \( \implies \) (ii): By Lemma ??, since \( X \) is totally bounded and dense in \( \hat{X} \), also \( \hat{X} \) is totally bounded. Of course \( \hat{X} \) is complete, so by Theorem ?? \( \hat{X} \) is compact.

(ii) \( \implies \) (i): If \( \hat{X} \) is compact, then \( \hat{X} \) is totally bounded by Theorem ??, hence so is its subspace \( X \).

We easily deduce the following interesting characterization of total boundedness.

**Corollary 87.** A metric space \( X \) can be isometrically embedded in a compact metric space iff it is totally bounded.
Exercise 13.1. a) Prove it.
b) Let $X$ be a metric space. Suppose there is a compact metric space $C$ and a uniform embedding $f : X \to C$—i.e., the map $f : X \to f(X)$ is a uniform homeomorphism. Show that $X$ is totally bounded.

The previous exercise shows that “isometric embedding” can be weakened to “uniform embedding” without changing the result. What about topological embeddings? This time the answer must be different, as e.g. $\mathbb{R}$ can be topologically embedded in a compact space: e.g. the arctangent function is a homeomorphism from $\mathbb{R}$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$ and thus a topological embedding from $\mathbb{R}$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Here is something in the other direction.

Lemma 88. A metric space which can be topologically embedded in a compact metric space is separable.

Proof. Indeed, let $f : X \hookrightarrow C$ be a topological embedding into a compact metric space $C$. In particular $C$ is separable. Moreover $X$ is homeomorphic to $f(X)$, which is a subspace of $C$, hence also separable by $X.X$. \hfill $\square$

Much more interestingly, the converse of Lemma 88 holds: every separable metric space can be topologically embedded in a compact metric space. This is quite a striking result. In particular implies that separability is precisely the topologically invariant part of the metrically stronger property of total boundedness, in the sense that for a metric space $(X, d)$, there is a topologically equivalent totally bounded metric $d'$ on $X$ iff $X$ is separable.

Unfortunately this result is beyond our present means to prove. Well, truth be told it is not really so unfortunate: we take it as a motivation to develop more purely topological tools. In fact we will later quickly deduce this result from one of the most important theorems in all of general topology, the Urysohn Embedding Theorem.

14. Cantor Space

14.1. Defining the Cantor Set.
14.2. Characterizing the Cantor Set.

Theorem 89. Let $X$ be a metric space which is nonempty, compact, totally disconnected and perfect (i.e., without isolated points). Then $X$ is homeomorphic to the Cantor set.

14.3. The Alexandroff-Hausdorff Theorem.

Theorem 90. (Alexandroff-Hausdorff)
For a metric space $X$, the following are equivalent:
(i) There is a continuous surjective map $f : C \to X$.
(ii) $X$ is nonempty and compact.

Proof. In this proof, for a metric space $X$, we denote $\prod_{i=1}^{\infty} X$ by $X^\infty$.

(i) $\implies$ (ii): Since $C$ is compact, if $f$ is continuous then $X = f(C)$ is compact. And just to be sure: since $C \neq \emptyset$, $X = f(C) \neq \emptyset$.

(ii) $\implies$ (i): Here lies the content, of course. Our proof closely follows a lovely
short note of I. Rosenholtz [?]. Note first that the result is actually topological rather than metric: i.e., it depends only on the underlying topological spaces. Without changing the underlying topology on $X$ we may (and shall) assume that $\text{diam } X \leq 1$. We break the argument up into several steps.

Step 1: We claim there is a continuous injection $f : X \to [0, 1]^\infty$. Since $X$ is a compact metric space it is separable: let $\{x_n\}_{n=1}^\infty$ be a countable dense subset, and put $f(x) = \{d(x, x_n)\}_{n=1}^\infty$: that is, the $n$th component is the distance $d(x, x_n)$. We know that each distance function $d(\cdot, x_n)$ is continuous, so by $X \times \mathbb{R}$ $f$ is continuous. Now suppose that $x, y \in X$ are such that $f(x) = f(y)$. We may choose a subsequence $\{x_{n_k}\}$ converging to $x$, so that

$$0 = \lim_{k \to \infty} d(x, x_{n_k}) = \lim_{k \to \infty} d(y, x_{n_k})$$

and thus $\{x_{n_k}\}$ also converges to $y$. Since the limit of a sequence in a metric space is unique, we conclude $x = y$.

In fact this only uses the assumption that $X$ is separable!

Step 2: There is a continuous surjection $f : C \to [0, 1]$.

Use the Middle Thirds definition of $C$.

Step 3: There is a homeomorphism $C \cong C^\infty$.

Step 4: There is a continuous surjection $C \to [0, 1]^\infty$.

Step 5: If $K \subset C$ is closed, there is a continuous surjection $C \to K$.

Step 6: By Step 1, there is a continuous injection $\iota : X \hookrightarrow [0, 1]^\infty$, and by Step 4 there is a continuous surjection $F : C \to [0, 1]^\infty$. Let $K = F^{-1}(X)$, which is a closed subset of $C$. By Step 5, there is a continuous surjection $f : C \to K$. Then $F \circ f : C \to [0, 1]^\infty$ is continuous and

$$(F \circ f)(C) = F(f(C)) = F(K) = X.$$

15. Contraction and Attraction

15.1. Banach’s Fixed Point Theorem.

Lemma 91. Let $X$ be a metric space, let $f : X \to X$ be a continuous function, let $x_0 \in X$, and let $x_0, x_1 = f(x_0), x_2 = f(x_1), \ldots$ be the sequence of iterates of $x_0$ under $f$. Suppose that $x_n$ converges to $x \in X$. Then $L$ is a fixed point of $f$: $f(x) = x$.

Proof. Since $x_n \to x$ and $f$ is continuous, $x_{n+1} = f(x_n) \to f(x)$.

Theorem 92. (Banach Fixed Point Theorem [?]) Let $X$ be a complete metric space, and let $f : X \to X$ be a contraction mapping: that is, there is $C \in (0, 1)$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq C d(x, y).$$

Then:

a) There is a unique $L \in X$ with $f(L) = L$, i.e., a fixed point of $f$.

b) For every $x_0 \in X$, define the sequence of iterates of $x_0$ under $f$ by $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$,

$$d(x_n, L) \leq \frac{C^n}{1-C} d(x_0, L).$$
Proof. Step 0: By Theorem X.X, \( f \) has at most one fixed point.

Step 1: Let \( x_0 \in I \), fix \( \epsilon > 0 \), let \( N \) be a large positive integer to be chosen (rather sooner than) later, let \( n \geq N \) and let \( k \geq 0 \). Then
\[
|a_{n+k} - a_n| \leq |a_{n+k} - a_{n+k-1}| + |a_{n+k-1} - a_{n+k-2}| + \ldots + |a_{n+1} - a_n| \\
\leq C^{n+k-1}|a_1 - a_0| + C^{n+k-2}|a_1 - a_0| + \ldots + C^n|a_1 - a_0| \\
= |a_1 - a_0|C^n \left( 1 + C + \ldots + C^{k-1} \right) = |a_1 - a_0|C^n \left( \frac{1 - C^k}{1 - C} \right) < \left( \frac{|a_1 - a_0|}{1 - C} \right) C^n.
\]
Since \( |C| < 1 \), \( C^n \to 0 \), so we may choose \( N \) such that for all \( n \geq N \) and all \( k \in \mathbb{N} \), \( \left( \frac{|a_1 - a_0|}{1 - C} \right) C^n < \epsilon \). So \( \{x_n\} \) is Cauchy and hence, since \( X \) is complete, convergent.

Step 2: By Step 1 and Lemma ??, for each \( x_0 \in I \) the sequence of iterates of \( x_0 \) under \( f \) converges to a fixed point. By Step 0, \( f \) has at most one fixed point. So there must be a unique fixed point \( L \in I \), which is the limit of every sequence of iterates: i.e., \( L \) is an attracting point for \( f \). This completes the proof of part a).

Step 3: . . .

For the last century, Banach’s Fixed Point Theorem has been one of the most important and useful results in mathematical analysis: it gives a very general condition for the existence of fixed points, and a remarkable number of “existence theorems” can be reduced to the existence of a fixed point of some function on some metric space. For instance, if you continue on in your study of mathematics you will surely learn about systems of differential equations, and the most important result in this area is that – with suitable hypotheses and precisions, of course – every system of differential equations has a unique solution. The now standard proof of this seminal result uses Banach’s Fixed Point Theorem\(^5\).

Theorem 93. (Edelstein [?]) Let \( X \) be a compact metric space, and let \( f : X \to X \) be a weakly contractive mapping. Then \( f \) is attractive:

a) There is a unique fixed point \( L \) of \( f \).

b) For all \( x_0 \in X \), the sequence of iterates of \( x_0 \) under \( f \) converges to \( f \).

Proof. We follow [?].

Step 0: The Extreme Value Theorem has the following generalization to compact metric spaces: if \( X \) is a compact metric space, then any continuous function \( f : X \to \mathbb{R} \) is bounded and attains its maximum and minimum values. Recall that we gave two proofs of the Extreme Value Theorem for \( X = [a, b] \): one using Real Induction and one using the fact that every sequence in \( [a, b] \) admits a convergent subsequence. Since by definition this latter property holds in a compact metric space, it is the second proof that we wish to carry over here, and we ask the interested reader to check that it does carry over with no new difficulties.

Step 1: We claim \( f \) has a fixed point. Here we need a new argument: the one we gave for \( X = [a, b] \) used the Intermediate Value Theorem, which is not available in our present context. So here goes: let \( g : X \to \mathbb{R} \) by \( g(x) = d(x, f(x)) \). Since \( f \) is continuous, so is \( g \) and thus the Extreme Value Theorem applies and in particular \( g \) attains a minimum value: there is \( L \in X \) such that for all \( y \in X \), \( d(L, f(L)) \leq d(y, f(y)) \). But if \( f(L) \neq L \), then by weak contractivity we have \( d(f(L), f(f(L))) < \)

\(^5\)In fact the title of [?] indicates that applications to integral equations are being explicitly considered. An “integral equation” is very similar in spirit to a differential equation: it is an equating relating an unknown function to its integral(s).
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d(L, f(L)), i.e., g(f(L)) < g(L), contradiction. So L is a fixed point for f.

Step 2: The argument of Step 2 of the proof of Theorem ?? carries over directly to show that L is an attracting point for f. □

References


