3. Ordered Spaces

Directions: Please solve Exercise 3.6 and any five other problems. Starred parts of problems are extra credit.

General Comments:

Recall that a partially ordered set is a set $X$ together with a binary relation $\leq$ which is reflexive, anti-symmetric and transitive. One says that two elements $x, y$ in a partially ordered set $X$ are comparable if either $x \leq y$ or $y \leq x$. In general, elements of a partially ordered set need not be comparable: perhaps the most natural example is the set of all subsets $2^X$ of a set $X$ (with more than one element!), partially ordered by inclusion: one certainly need not have either $A \subset B$ or $B \subset A$. A partial ordering is called total or linear if any two elements are comparable. In this course we will say ordered set for a set $X$ equipped with a total ordering.

If $(X, \leq)$ and $(Y, \leq)$ are partially ordered sets, a function $f : X \to Y$ is increasing if for all $x_1 \leq x_2$ in $X$, we have $f(x_1) \leq f(x_2)$ in $Y$; strictly increasing if for all $x_1 < x_2$ in $X$, we have $f(x_1) < f(x_2)$; decreasing if for all $x_1 \leq x_2$ in $X$, we have $f(x_1) \geq f(x_2)$ in $Y$; strictly increasing if for all $x_1 < x_2$ in $X$, we have $f(x_1) > f(x_2)$. We say $f$ is monotone if it is increasing or decreasing.

An order isomorphism $f : X \to Y$ is an increasing function which admits an increasing inverse function $g : Y \to X$. We say that two partially ordered sets are order-isomorphic if there is an order isomorphism between them.

In a totally ordered set $X$, we can define intervals just as we did for subsets of the real line. Well, almost: $\mathbb{R}$ has no least or greatest element; in the presence of these the definition must be modified slightly, in a way which should be familiar from our study of $[a, b]$. If $X$ has a least element $\mathbb{B}$, then we regard the intervals $X = [\mathbb{B}, \infty)$ and $[\mathbb{B}, x)$ for $\mathbb{B} < x$ as open; if $X$ has a greatest element $\mathbb{T}$, then we regard the intervals $X = (-\infty, \mathbb{T}]$ and $(x, \mathbb{T}]$ for $x < \mathbb{T}$ as open.

Exercise 3.1. Let $X$ and $Y$ be ordered sets, and let $f : X \to Y$ be an increasing bijection. Show: $f$ is an order isomorphism.

Exercise 3.2. a) Which linear functions $f : \mathbb{R} \to \mathbb{R}$ are order isomorphisms? b) Show: for $d \in \mathbb{Z}^+$, there is a degree $d$ polynomial function $P : \mathbb{R} \to \mathbb{R}$ which is
an order isomorphism iff $d$ is odd.

c) Show: the arctangent function gives an order isomorphism from $\mathbb{R}$ to $(\frac{-\pi}{2}, \frac{\pi}{2})$.

Exercise 3.3. (A Special Case of Tarski’s Fixed Point Theorem)$^1$

a) Let $f : [a, b] \to [a, b]$ be increasing. Show: $f$ has a fixed point: there is $c \in [a, b]$ such that $f(c) = c$.

b) Give a second proof. More precisely, if your first proof used Real Induction, give a proof that does not use Real Induction, and vice versa.

Exercise 3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

a) Show: if $f$ is a homeomorphism (for the order topology = the usual one), then $f$ is either strictly increasing or strictly decreasing.

b) Show: if $f$ is monotone and bijective, then it is continuous.

c) Show by example that $f$ can be strictly increasing and not continuous.$^2$

Exercise 3.5.

a) Show: any two nonempty open intervals on the real line – including $\mathbb{R} = (-\infty, \infty)$ – are order-isomorphic.

b) Show: any two closed, bounded intervals on the real line are order-isomorphic.

c) Classify all intervals in $\mathbb{R}$ up to order-isomorphism.

d) Classify all intervals in $\mathbb{R}$ up to homeomorphism, and show in particular that there are homeomorphic intervals which are not order-isomorphic (albeit for a rather shallow reason).

Exercise 3.6. Let $X$ be an ordered set.

a) Let $\tau_X$ be the collection of all subsets of $X$ which are unions of open intervals. Show: $\tau_X$ is a topology on $X$, called the order topology.

b) Convince yourself that the order topologies on $\mathbb{R}$ and $[a, b]$ are the familiar ones (i.e., coming from the standard metric $d(x, y) = |x - y|$).

c) Let $f : X \to Y$ be an order isomorphism of ordered sets. Show: $f : (X, \tau_X) \to (Y, \tau_Y)$ is a homeomorphism of topological spaces.

d) Show that any order topology is Hausdorff: if $x \neq y \in X$, there are open sets $U, V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Exercise 3.7. Let $X$ be an ordered set.

a) Show: $X$ satisfies (LUB) If $\emptyset \neq S \subset X$ is bounded above, then $S$ has a least upper bound.

b) Show: $X$ satisfies (GLB) if $\emptyset \neq S \subset X$ is bounded below, then $S$ has a greatest lower bound.

An ordered set satisfying these equivalent properties is called Dedekind complete.

b) An ordered set is complete if every subset $S \subset X$ has a least upper bound and a greatest lower bound (including $S = \emptyset$!). Show: $X$ is complete iff it is Dedekind complete and has top and bottom elements.

c) Show: every well-ordered subset is Dedekind complete.

d) Show: every interval in $\mathbb{R}$ is Dedekind complete.

e) Show: no interval in $\mathbb{Q}$ consisting of more than one point is Dedekind complete.

Exercise 3.8.

a) Apply the Principle of Ordered Induction in the order set $\mathbb{N}$ of natural numbers and deduce the Principle of Mathematical Induction.

b) Apply the Principle of Ordered Induction in any well-ordered set and deduce the Principle of Transfinite Induction.

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$^1$The idea for this exercise was told to me by James Propp.

$^2$This is easy. But from the abstract perspective, it is a bit distressing.
Exercise 3.9. a) Let \((X; \leq_X)\) and \((Y; \leq_Y)\) be ordered sets. On the Cartesian product \(X \times Y\) we define the following binary relation: \((x_1, y_1) \leq (x_2, y_2)\) if \(x_1 < x_2\) or \((x_1 = x_2\) and \(y_1 < y_2)\). Show: this gives an ordering on \(X \times Y\) and explain why it is called the lexicographic or dictionary ordering.

b) Let \(X_1, \ldots, X_n\) be ordered sets. Define an analogous lexicographic ordering on the Cartesian product \(X_1 \times \ldots \times X_n\) and show that it is a total ordering.

c) Let \(X\) be an ordered set, let \(Y\) be a well-ordered set, and let \(Y^X\) be the set of all functions \(f : X \rightarrow Y\). We define a binary relation \(\leq\) on \(Y^X\) by: we write \(f_1 \leq f_2\) if either \(f_1 = f_2\) or; for the least \(x \in X\) at which \(f_1(x) \neq f_2(x)\) we have \(f_1(x) < f_2(x)\). Show: this gives an ordering on \(Y^X\). Explain what this has to do with parts a) and b).

d) Lexicographic orderings make for some very interesting examples of topological spaces. For instance, consider \([0, 1] \times [0, 1]\) with the dictionary ordering. Show: it is complete.

Exercise 3.10. An ordered set \(X\) is order-dense if for all \(a < c\) in \(X\), there is \(b \in X\) with \(a < b < c\).

a) Show: \(\mathbb{Q}\) and \(\mathbb{R}\) are order-dense but \(\mathbb{N}\) and \(\mathbb{Z}\) are not.

b) Suppose that an ordered set \(X\) consists of more than one point and is connected in the order topology. Show: \(X\) is order-dense.

Each of the following results is more difficult. You don’t need to do any of them, and if you want to do any of them, you can turn them in directly to me at any point in the semester.

Exercise 3.11. a) Let \(X\) be a complete ordered set, and let \(Y \subseteq X\). Show: \(Y\) is complete (with respect to the ordering it inherits as a subset of \(X\)) iff it is a closed subset of \(X\) for the order topology.

b) Because all intervals in \(\mathbb{R}\) are Dedekind complete – be they closed or not – part a) does not hold with “complete” replaced by “Dedekind complete”. But the problem can be easily fixed: let \(X\) be a Dedekind complete ordered set, and let \(Y\) be a subset of \(X\). Let \(\tilde{Y}\) be obtained from \(Y\) by including \(\text{sup} Y\) if it exists in \(X\) (i.e., \(Y\) is nonempty and bounded above) and \(\text{inf} Y\) if it exists in \(X\) (i.e., \(Y\) is nonempty and bounded below). Show that \(Y\) is Dedekind complete iff \(\tilde{Y}\) is a closed subset of \(X\) for the order topology.

Exercise 3.12. Let \(X\) be an ordered set of cardinality at least 2. Show: \(X\) is connected in the order-topology iff \(X\) is Dedekind-complete and order-dense.

Exercise 3.13. Let \(X\) be a nonempty ordered set. Show: \(X\) is compact in the order topology iff \(X\) is complete.

Exercise 3.14. (Generalized Heine-Borel Theorem) Show: for an ordered set \(X\), the following are equivalent:

(i) \(X\) is Dedekind complete.

(ii) A subset \(S\) of \(X\) is compact in the order topology iff it is closed and bounded.

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3 If the notation here surprises you, suppose that \(X\) and \(Y\) are finite, that \(X\) has \(x\) elements and \(Y\) has \(y\) elements, and count the number of elements of \(Y^X\).

4 We will see later that the order topology on this space does not come from a metric!