6. Products, Quotients and Manifolds

A topological space $X$ is **locally Euclidean** if every $x$ in $X$ has an open neighborhood which is homeomorphic to $\mathbb{R}^N$. (The value of $N$ is allowed to depend on $x$: e.g. the disjoint union of a line and a plane is locally Euclidean.)

A **manifold** is a topological space which is locally Euclidean, second countable and Hausdorff. (Most people agree that manifolds should be Hausdorff, but not everyone agrees that manifolds should be second countable. See Exercise 6.14 for an example of the weirdness prevented by the Hausdorff axiom. An advantage of our definition is that any subspace of Euclidean space is Hausdorff and second countable, so we need these conditions in order to embed manifolds in Euclidean space. Conversely, it is a classic theorem of Whitney that every manifold can be embedded in Euclidean space. I hope we will prove it by the end of the course.)

All products of topological spaces are assumed to be endowed with the product topology unless otherwise specified.

A property of topological spaces is **hereditary** if every subspace of a space with that property possesses that property.

6.1. **Mandatorium.**

**Exercise 6.1.** Let $X$ be a topological space. Let $\Delta = \{(x, x) \in X \times X \mid x \in X\}$ be the **diagonal** in $X \times X$.

a) Show: the map $X \to \Delta$ given by $x \mapsto (x, x)$ is a homeomorphism.

b) Show: $X$ is Hausdorff iff $\Delta$ is closed in $X \times X$.

**Exercise 6.2.** Let $X$ be a topological space, and let $Y \subset X$ be a subspace.

a) Show: if $X$ is quasi-compact and $Y$ is closed then $Y$ is quasi-compact.

b) Show: if $X$ is Hausdorff and $Y$ is quasi-compact then $Y$ is closed.

c) Show: if $X$ is quasi-compact, $Y$ is Hausdorff and $f : X \to Y$ is a continuous surjection, then $f$ is a closed map.

**Exercise 6.3.** a) Show: a subspace of a Kolmogorov space is Kolmogorov.

(I.e., being Kolmogorov is hereditary.)

b) Show: a subspace of a separated space is separated.
c) Show: a subspace of a Hausdorff space is Hausdorff.
   (I.e., being Hausdorff is hereditary.)
d) Show: a subspace of a quasi-regular space is quasi-regular.
   (I.e., being quasi-regular is hereditary.)

Comment: In contrast, quasi-normality and normality are not hereditary. This is too hard for an exercise: more on it later.

6.2. On Products.

Exercise 6.4. Let \( \{X_i\}_{i \in I} \) be a family of nonempty spaces, and put \( X = \prod_{i \in I} X_i \).
Show: \( X \) is Hausdorff if and only if \( X_i \) is Hausdorff for all \( i \in I \).

Exercise 6.5. a) Suppose a topological space \( Y \) satisfies the conclusion of the Tube Lemma. Show: for all topological spaces \( X \), \( \pi_1 : X \times Y \to X \) is a closed map.
b) If \( Y \) satisfies the conclusion of the Tube Lemma, must it be quasi-compact?
(This is very hard at the moment, perhaps unfairly so. But it’s interesting...)

Exercise 6.6. Let \( \{X_i\}_{i \in I} \) be a family of nonempty topological spaces. Let \( X = \prod_{i \in I} X_i \) (product topology!) and let \( X_B \) be the same Cartesian product endowed with the box topology.

a) Show: if \( \{i \in I \mid X_i \) is not indiscrete \} \) is infinite, then the box topology is strictly finer than the product topology.
b) Show: \( X_B \) is discrete if and only if \( X_i \) is discrete for all \( i \in I \).
c) Show: \( X \) is discrete if and only if \( X_i \) is discrete for all \( i \in I \) and \( \{i \in I \mid \#X_i \geq 2 \} \) is finite.

Exercise 6.7. For \( n \in \mathbb{Z}^+ \), let \( X_n \) be a finite discrete space, and let \( X = \prod_{n=1}^{\infty} X_n \).
Our goal is to show \( X \) is compact without using Tychonoff’s Theorem. Here are two methods for this. You can do either part a) or part b) for full credit.

a) Show: \( X \) is metrizable and sequentially compact (c.f. Exercise 4.22), hence compact.
b) Show: \( X \) is homeomorphic to a closed, bounded subset of \( \mathbb{R} \) (c.f. the proof given in class that \( \prod_{n=1}^{\infty} \{0, 2\} \) is homeomorphic to the Cantor set), hence compact.
c) Invent your own method.

6.3. On Quotient Maps.

Exercise 6.8. Let \( f : X \to Y \) be a surjective continuous map of topological spaces. Show: \( f \) is a quotient map iff: for all \( A \subset Y \), \( A \) is closed in \( Y \) iff \( f^{-1}(A) \) is closed in \( X \).

Exercise 6.9. a) Let \( f : X \to Y \) be a surjective map. Show: if \( A \subset Y \), then \( f(f^{-1}(A)) = A \).
b) Let \( f : X \to Y \) be continuous, surjective and open. Show: \( f \) is a quotient map.
c) Let \( f : X \to Y \) be continuous, surjective and closed. Show: \( f \) is a quotient map.

Exercise 6.10. a) Let \( \pi_i : \prod_{i \in I} X_i \to X_i \) be a projection map on a product of topological spaces. Show: \( \pi_i \) is a quotient map.
b) Show: \( \pi_1 : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x \) is not closed.
c) Let \( S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \). Let \( f : [0, 2\pi] \to S^1 \) by \( f(\theta) = (\cos \theta, \sin \theta) \).
Show: \( f \) is a closed quotient map which is not open.
Exercise 6.11. Let $f : X \to Y$ be a continuous map. A section of $f$ is a continuous map $\sigma : Y \to X$ such that $f \circ \sigma = 1_Y$. Show that if $f$ admits a section, it is a quotient map.

Exercise 6.12. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection onto the first coordinate: $(x, y) \mapsto x$. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ or } y = 0\}$. Let $f = \pi_1|_A : A \to \mathbb{R}$.

a) Show: $f$ is quotient map. (Suggestion: use the previous exercise.)

b) Show: $f$ is neither open nor closed.

Exercise 6.13. Let $q : X \to Y$ be a quotient map with associated equivalence relation $\sim$ on $X$. Notice that we may view $\sim$ as a subset of $X \times X$.

a) Show: if $Y$ is Hausdorff, then $\sim$ is a closed subset of $X \times X$.

b) Suppose $q$ is an open map. Show: if $\sim$ is a closed subset of $X \times X$, then $Y$ is Hausdorff.

(Hint: this is a generalization of Exercise 6.1).

6.4. Manifolds.

Exercise 6.14. Let $X = \{(x, y) \in \mathbb{R}^2 \mid y \in \{0, 1\}\}$: thus $X$ is a disjoint union of two horizontal lines. Let $\sim$ be the following equivalence relation on $X$ generated by: for all $x \neq 0$ we have $(x, 0) \sim (x, 1)$. (I.e., we identify these pairs of points and no other pairs of distinct points.) Let $Y = X/\sim$.

a) Draw a picture of $Y$.

b) Show: $Y$ is locally Euclidean and second countable.

c) Show: a locally Euclidean space is separated.

d) Show: $Y$ is not Hausdorff and thus not a manifold (according to us).

e) Is the quotient map $q : X \to Y$ open? Is it closed?

Exercise 6.15. For $N \in \mathbb{Z}^+$, let

$$S^N = \{x = (x_1, \ldots, x_{N+1}) \in \mathbb{R}^{N+1} \mid x_1^2 + \ldots + x_{N+1}^2 = 1\}.$$  

We give $\mathbb{R}^{N+1}$ the Euclidean topology and $S^N$ the subspace topology.

a) Show that for all $p, q \in S^N$ there is a homeomorphism $\varphi : S^N \to S^N$ such that $\varphi(p) = q$. (Hint: linear maps are continuous.)

b) Show: for all $p \in S^N$, $S^N \setminus \{p\}$ is homeomorphic to $\mathbb{R}^N$. (I want a real proof, not a handwave argument! You might consider googling stereographic projection.)

c) Show that $S^N$ is a manifold.

Exercise 6.16. a) Show that an open subset of a manifold is a manifold.

b) Show that a finite product of manifolds is a manifold.

c) On $S^N$, let $\sim$ be the equivalence relation obtained by identifying $x$ with $-x$ for all $x$. We denote $S^N/\sim$ by $\mathbb{R}P^N$. Show: $\mathbb{R}P^N$ is a manifold.

6.5. The Kolmogorov Quotient.

Exercise 6.17. Two points $x$ and $y$ in a topological space are topologically indistinguishable if for all open subsets $U$ of $X$ we have $x \in U \iff y \in U$. We write $x \sim y$ if $x$ and $y$ are topologically indistinguishable.

a) Show: $\sim$ is an equivalence relation on $X$.

b) Let $f : X \to Y$ be continuous. Show: if $x_1$ and $x_2$ are topologically indistinguishable in $X$, then $f(x_1)$ and $f(x_2)$ are topologically indistinguishable in $Y$.

Exercise 6.18. Let $X_K = X/\sim$ be the identification space induced by the equivalence relation of topological indistinguishability, and let $q : X \to X_K$ be the natural
map (i.e., which maps $x$ to its $\sim$-equivalence class $[x]$). We call $q$ the **Kolmogorov quotient** of $X$.

- Show that $q$ is open, so we get an induced map $q : \tau_X \to \tau_{X_K}$ by $U \mapsto q(U)$. Show that this map is a bijection.

- Show: $X_K$ is Kolmogorov.

- Show: the map $q$ is **universal** for continuous maps from $X$ into a Kolmogorov space $Y$ in the following sense: for every Kolmogorov space $Y$ and continuous map $f : X \to Y$, there is a unique continuous map $F : X_K \to Y$ such that $f = F \circ q$.

- Show: $X$ is quasi-compact iff $X_K$ is quasi-compact.

**Exercise 6.19.** Let $\prec$ be the specialization preordering on $X$: c.f. Exercise 5.15. Let $\approx$ be the relation on $X$ defined by $x \approx y$ if $x \prec y$ and $y \prec x$. By Exercise 0.9, $\approx$ is an equivalence relation on $X$.

- Show: $\approx$ is the relation of topological indistinguishability.

- Reinterpret the Kolmogorov quotient $X_K$ as $X/\approx$, i.e., as a quotient on which the specialization quasi-ordering defines a partial ordering.

- Formulate and prove a universal property of the partial ordering associated to a quasi-ordering.