MATH 4200 HW: PROBLEM SET SEVEN: MORE SEPARATION, CONNECTEDNESS, LOCAL COMPACTNESS

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Directions: Please do five problems, including at least one from each of the three sections.

A topological space $X$ is completely regular if for every closed subset $A \subset X$ and every point $p \in X \setminus A$, there is a continuous function $f : X \to [0,1]$ with $f(A) = \{0\}$ and $f(p) = 1$. A topological space is Tychonoff if it is Hausdorff and completely regular.

Let $P$ be a property of topological spaces. A topological space is weakly locally $P$ if every point of the space admits a neighborhood with property $P$ (in the subspace topology). A topological space is locally $P$ if every point of the space admits a neighborhood base of neighborhoods with property $P$. (Note: here it is important that we say “neighborhood” and not “open neighborhood”.)

WARNING: Against the above convention, we define a locally compact space to be a Hausdorff space in which each point has a base of compact neighborhoods. (See Exercise 6.14 for an example which shows that “Hausdorff” does not follow automatically.)

A topological space $X$ is path-connected if for all $p, q \in X$ there is a continuous function $\gamma : [0,1] \to X$ with $\gamma(0) = p$, $\gamma(1) = q$.


Exercise 7.1. a) Show: a completely regular space is quasi-regular. Deduce: a Tychonoff space is regular.

b) Show: a completely regular Kolmogorov space is Tychonoff.

c) Show: a space is completely regular iff its Kolmogorov quotient is Tychonoff.

Exercise 7.2. a) Show: a topological space is quasi-regular iff every point admits a neighborhood base of closed neighborhoods.

b) Show: a topological space $X$ is quasi-normal iff for all subsets $B \subset U \subset X$ with $B$ closed and $U$ open, there is an open subset $V$ with $B \subset V \subset \overline{V} \subset U$.

Exercise 7.3.

a) Show: a subspace of a completely regular space is completely regular.

(That is: complete regularity is a hereditary property of topological spaces.)

b) Let $\{X_i\}_{i \in I}$ be a family of nonempty topological spaces. Show: $\prod_{i \in I} X_i$ is completely regular iff each $X_i$ is completely regular.

7.2. Connectedness.
Exercise 7.4. (Caging Connected Subsets) Let $Y$ be a connected subset of a topological space $X$. Let $X = U \coprod V$ be a separation of $X$. Show: $Y \subset U$ or $Y \subset V$.

Exercise 7.5. Let $X$ be a topological space, and let $\{Y_i\}_{i \in I}$ be a family of connected subsets of $X$.

a) Show: if $\bigcap_{i \in I} Y_i \neq \emptyset$, then $\bigcup_{i \in I} Y_i$ is connected.

b) Show: if $I$ is an ordered set and $Y_i \subset Y_j$ for all $i \leq j$ then $\bigcup_{i \in I} Y_i$ is connected.

c) Show: if $X$ is arc-connected, then for all $x, y \in X$ there is a continuous function $\gamma : [0, 1] \to X$ with $\gamma(0) = x, \gamma(1) = y$. Show (again?): there is a subspace of $\mathbb{R}$ with this property.

d) Show: if $X$ is homeomorphic to the coproduct of its connected components.

e) Find a subset of $\Omega$ such that $C_p((0, 0)) = \{0, y, y \in [0, 1]\}$.

Exercise 7.6. Let $X$ be a topological space.

a) Show: if $Y \subset X$ is connected, so is $Y$.

b) Let $p \in X$. Let $C_p(p)$ be the union of all connected subsets of $X$ which contain $p$.

Show: $\{C_p(p)\}_{p \in X}$ gives a partition of $X$ into closed, connected sets. The sets $C_p(p)$ are called the connected components of $X$. A topological space is totally disconnected if $C_p(p) = \{p\}$ for all $p \in X$.

c) Show: the following are equivalent:

(i) Each connected component $C_p(p)$ is open.

(ii) $X$ is homeomorphic to the coproduct of its connected components.

d) Show: the conditions of part c) are not satisfied for a totally disconnected space which is not discrete. Show (again?): there is a subspace of $\mathbb{R}$ with this property.

e) Let $X$ be a topological space, and consider the following binary relation $R$ on $X$: we say that $xRy$ if there is no separation $X = U \coprod V$ with $x \in U, y \in V$.

a) Show: $R$ is an equivalence relation on $X$. We denote the equivalence class of $p \in X$ by $Q \{p\}$ and call it the quasi-component at $p$.

c) Define a relation $R_P$ on $X$ by $xR_P y$ if there is a continuous function $\gamma : [0, 1] \to X$ with $\gamma(0) = x, \gamma(1) = y$. Show: $R_P$ is an equivalence relation on $X$. We denote the equivalence class of $p \in X$ by $C_p(p)$ and call it the path component at $p$.

d) Show: for all $p \in X$ we have

$$C_p(p) \subset C_p(p) \subset C_p(p).$$

Exercise 7.8. Show: if $X$ is compact, then $C_p = C_Q(p)$ for all $p \in P$.

Exercise 7.9. A topological space is arc-connected if for all $x, y \in X$ there is an injective continuous function $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

a) Show: every arc-connected space is path-connected.

b) Let $X$ be a set with $1 < \#X < \#\mathbb{R}$. Show: the indiscrete topology on $X$ is path-connected but not arc-connected.

c) Show: A connected, locally path-connected space is path-connected.
7.3. Local Compactness.

Exercise 7.11. Show: a Hausdorff topological space in which each point admits a compact neighborhood is locally compact.

Exercise 7.12.

a) Show: an open subset of a locally compact space is locally compact.
b) Show: a finite product of locally compact spaces is locally compact.
c) Show: \( \prod_{n=1}^{\infty} \mathbb{R} \) is not locally compact.

The next two exercises closely follow the wikipedia article http://en.wikipedia.org/wiki/Alexandroff_extension, which I wrote some years ago.

Exercise 7.13. Let \( c : X \hookrightarrow Y \) be an embedding from a topological space \( X \) to a compact space \( Y \), with dense image and such that \( Y \setminus c(X) \) consists of a single point \( \{\infty\} \). We say that \( c \) is a one-point compactification of \( X \).

a) Show: \( X \) is locally compact (in particular Hausdorff!) and not compact.
b) Show: the open subsets of \( Y \) containing \( \infty \) are precisely those of the form \( \infty \cup c(U) \), where \( U \subset X \) and \( X \setminus U \) is compact.
c) Find a one-point compactification for: one open interval; the coproduct of \( N \) open intervals for \( N \) finite; the coproduct of a countably infinite set of open intervals.

Exercise 7.14. Let \( X \) be a topological space, and let \( \infty \) be anything which is not an element of \( X \). Let \( X^* = X \cup \infty \). We topologize \( X^* \) as follows: the open subsets are the open subsets \( U \subset X \) and all subsets \( V \) which contain \( \infty \) and such that \( X \setminus V \) is closed and compact. Write \( c : X \hookrightarrow X^* \) for the inclusion map. We call \( c : X \hookrightarrow X^* \) the Alexandroff extension of \( X \).

(In practice one usually calls \( X^* \) the Alexandroff extension, and the map \( c \) is intended to be understood.)

a) Show: the inclusion map \( c : X \hookrightarrow X^* \) is an open embedding.
b) Show: \( X^* \) is quasi-compact.
c) Show: \( c(X) \) is dense in \( X^* \) iff \( X \) is not compact.
d) Show: \( X \) is locally compact iff \( X^* \) is compact.
e) Show: every locally compact space is Tychonoff.

(Hint: You may use the Tychonoff Embedding Theorem.)

Combining the previous exercises, we see that a space \( X \) admits a one-point compactification iff it is locally compact. In this case one also speaks of the Alexandroff compactification. It ought to be clear that by any measure of the size of a compactification, the one-point compactification is the smallest. It is also an admirably simple compactification when it exists: its drawback is that it exists precisely for locally compact spaces, whereas the Tychonoff embedding theorem shows that some compactifications exist for all Tychonoff spaces.