4. Metric Spaces (no more lulz)

Directions: This week, please solve any seven problems. Next week, please solve seven more. Starred parts of problems are extra credit.

General Comments:

Every metric space \((X, d)\) has an associated topology \(\tau_d\), the metric topology. When we use topological concepts with regard to a metric space, you should always assume that the topology in question is the metric topology \(\tau_d\).

A topological space \(X\) is separable if there exists a countable (note: this includes the empty set and finite sets) dense subset, i.e., a subset \(Y \subset X\) with \(\overline{Y} = X\).

A metric space \(X\) is totally bounded if for all \(\epsilon > 0\) there is a finite subset \(\{x_1, \ldots, x_N\}\) of \(X\) such that \(\bigcup_{i=1}^{N} B(x_i, \epsilon) = X\): that is, \(X\) admits a finite covering by \(\epsilon\) balls for all positive \(\epsilon\).

A Cauchy sequence in a metric space \(X\) is a sequence \(x\) in \(X\) such that for all \(\epsilon > 0\), there is \(N = N(\epsilon)\) such that for all \(m, n \geq N\) we have \(d(x_m, x_n) < \epsilon\). A metric space is complete if every Cauchy sequence converges.

Two metrics \(d_1\) and \(d_2\) on the same set \(X\) are uniformly equivalent if the identity function on \(X\) is uniformly continuous as a function from \((X, d_1)\) to \((X, d_2)\) and also as a function from \((X, d_2)\) to \((X, d_1)\).

A metric function \(d : X \times X \rightarrow \mathbb{R}\) is ultrametric if for all \(x, y, z \in X\) we have \(d(x, z) \leq \max(d(x, y), d(y, z))\).

Exercise 4.1. a) Show that a metric space is totally bounded iff for every \(\epsilon > 0\) it admits a finite covering by sets of diameter at most \(\epsilon\).

b) Show that a subspace of a totally bounded metric space is totally bounded.

Exercise 4.2. a) Show that any discrete metric is an ultrametric.

b) (p-adic metric) Let \(p\) be a prime number. We define a function \(|\cdot|_p : \mathbb{Q} \rightarrow \mathbb{R}\) as follows: we put \(|0|_p = 0\). For \(0 \neq n \in \mathbb{Q}\), write \(n = p^a \frac{c}{d}\) with \(\gcd(p, cd) = 1\) and put \(|n|_p = p^{-a}\). Show that the function \(d_p : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{R}\) given by \(d_p(x, y) = |x - y|_p\) is an ultrametric function.

c) Show that if \(d : X \times X \rightarrow \mathbb{R}\) is an ultrametric function then for all \(\alpha \geq 0\), \(d^\alpha : (x, y) \rightarrow d(x, y)^\alpha\) is also a ultrametric function.
Exercise 4.3. a) Let \((X,d)\) be an ultrametric space, let \(x \in X\) and let \(\delta > 0\). Show that the closed ball \(B^*(x,\delta) = \{y \in X \mid d(x,y) \leq \delta\}\) is also open.
b) Let \(x \neq y\) be distinct points in an ultrametric space \((X,d)\). Show that there is a separation \(X = U \coprod V\) with \(x \in U\), \(y \in V\).
c) A topological space \(X\) is totally disconnected if the only connected (in the subspace topology) subsets \(Y\) of \(X\) are of the form \(Y = \{y\}\). Show that every ultrametric space is totally disconnected.

Exercise 4.4. For each \(n \in \mathbb{Z}^+\), let \(X_n\) be the set \([0,1]\) equipped with the metric 
\[d_n(0,1) = \frac{1}{2^n}.\]
Let \(X = \prod_{n=1}^{\infty} X_n = [0,1]^\infty\) with the metric 
\[d(x,y) = \sup_n d_n(x_n, y_n).\]
a) Show that \(d\) is an ultrametric on \(X\).
b) Show that a sequence \(x\) in \((X,d)\) converges iff each of its component sequences is eventually constant.
c) Show that \(X\) is homeomorphic to the classical Cantor set of Exercise 2.9. Deduce that \(X\) is compact and that the classical Cantor set is totally disconnected.
d) More generally, for each \(n \in \mathbb{Z}^+\) let \(X_n\) be any nonempty set endowed with the metric \(d_n\) for which any two distinct points have distance \(\frac{1}{2^n}\). Show that the metric 
\[d = \sup_n d_n\] on \(X = \prod_{n=1}^{\infty} X_n\) is an ultrametric.
e) In the setting of part d), suppose that each \(X_n\) consists of more than a single point. Show that the topology on \(X\) is not discrete. (Note well that this is stronger than saying that \(d\) is not the discrete metric: you have to show that not all sets are open.)

Exercise 4.5. a) Show that any metric space is Hausdorff.
b) Let \(X\) be any set endowed with the discrete metric. Find all convergent sequences in \(X\) and all Cauchy sequences in \(X\) and deduce that \(X\) is complete. (In particular there are complete metric spaces of all cardinalities.)

Exercise 4.6. a) Let \((X,d)\) be a metric space, and let \(f : [0,\infty) \rightarrow [0,\infty)\) be a continuous, strictly increasing function such that \(f(0) = 0\) and \(d_f = f \circ d\) is also a metric function on \(f\). (Recall that we proved that this holds if \(f\) is convex.) Show that the metrics \(d\) and \(d_f\) are uniformly equivalent.
b) Let \((X,d)\) be a metric space, and define \(d_{\min} : X \times X \rightarrow \mathbb{R}\) be \(d_{\min}(x,y) = \min(d(x,y),1)\). Show: \(d_{\min}\) is a metric on \(X\) which is uniformly equivalent to \(d\).
c) Deduce that any metric is uniformly equivalent to a bounded metric.
d) Show that if \(d_1\) and \(d_2\) are Lipschitz equivalent metric functions on a set \(X\), then \((X,d_1)\) is bounded iff \((X,d_2)\) is bounded.
e) Show that if \(d_1\) and \(d_2\) are uniformly equivalent metric functions on a set \(X\), then \((X,d_1)\) is totally bounded iff \((X,d_2)\) is totally bounded.
f) Deduce that any unbounded metric space is uniformly equivalent to a metric space which is bounded but not totally bounded.

Exercise 4.7. a) Show that in any metric space \(X\), every convergent sequence is a Cauchy sequence.
b) Show that in any metric space, a Cauchy sequence is bounded.
c) Give an explicit example of a Cauchy sequence in \(\mathbb{Q}\) which is not convergent. (Thus \(\mathbb{Q}\) is not complete.)
d) Show that in any metric space, a Cauchy sequence which admits a convergent subsequence is itself convergent.
e) Use the Bolzano-Weierstrass Theorem to show that \(\mathbb{R}\) is complete.
Exercise 4.8. Let \( \iota : X \to Y \) be an isometric embedding of metric spaces.  
\( \text{a) Suppose } x \text{ is a sequence in } X \text{ such that } \iota(x) \text{ is convergent in } Y. \text{ Show that } x \text{ is Cauchy in } X. \)  
(Comment: in a nutshell, the idea of \textbf{metric completion} is to give a converse to this simple fact: every Cauchy sequence in a metric space \( X \) ought to be convergent in some larger space in which \( X \) is isometrically embedded.)

\( \text{b) Suppose } Y \text{ is a subset of a metric space which is complete (with respect to the restricted metric: c.f. Exercise 4.11). Show that } Y \text{ is closed in } X. \)

\( \text{c) Suppose } X \text{ is a complete metric space and } Y \text{ is a closed subset of } X \). Show that \( Y \) is complete.

Exercise 4.9. Let \( d_1 \) and \( d_2 \) be metrics on a set \( X \). Show: \( d_1 \) and \( d_2 \) are uniformly equivalent iff: for all \( \epsilon > 0 \) there are \( \delta_1, \delta_2 > 0 \) such that for all \( x_1, x_2 \in X \),

\[
d_1(x, y) < \delta_1 \implies d_2(x, y) < \epsilon \quad \text{and} \quad d_2(x, y) < \delta_2 \implies d_1(x, y) < \epsilon.
\]

Exercise 4.10. a) Show that Lipschitz equivalent metrics are uniformly equivalent and uniformly equivalent metrics are topologically equivalent.

\( \text{b) Let } d_1 \text{ and } d_2 \text{ be uniformly equivalent metrics on a set } X, \text{ and let } x \text{ be a sequence in } X. \text{ Show that } x \text{ is Cauchy with respect to } d_1 \text{ iff it is Cauchy with respect to } d_2. \)

Deduce that \( (X, d_1) \) is complete iff \( (X, d_2) \) is complete.

\( \text{c) Find a metric } d \text{ on } \mathbb{R} \text{ which is topologically equivalent to the standard Euclidean metric but which is not complete.} \)

\( \text{(Suggestion: use the fact that } \mathbb{R} \text{ is homeomorphic to (0, 1),)} \)

Exercise 4.11. Let \( (X_1, d_1), \ldots, (X_n, d_N) \) be nonempty metric spaces. On \( X = \prod_{i=1}^{N} X_i \) we put the \( d_p \)-metric for some \( p \in [1, \infty] \).

\( \text{a) Let } x \text{ be a sequence in } X. \text{ Show that } x \text{ is Cauchy iff for each } 1 \leq i \leq N, \text{ its component sequence } x^{(i)} \text{ is Cauchy in } (X_i, d_i). \) (Suggestion: all the \( d_p \)-metrics are Lipschitz equivalent, so by the previous exercise it suffices to work with \( d_\infty \).

\( \text{b) Deduce that } (X, d_p) \text{ is complete iff } (X_i, d_i) \text{ is complete for all } i. \)

\( \text{c) Deduce in particular that } \mathbb{R}^N \text{ with the Euclidean metric is complete.} \)

Exercise 4.12. a) Show that every compact metric space is totally bounded.

\( \text{b) Show that every totally bounded metric space is separable.} \)

\( \text{c) Show that no separable metric space } X \text{ has cardinality larger than that of } \mathbb{R}. \) (Hint: if \( Z \) is a countable dense subset, then any element of \( X \) is the limit of a sequence in \( Z \), i.e., can be given by a function \( f : \mathbb{Z}^+ \to Z. \) How many such functions are there?!?)

Exercise 4.13. Let \( X \) be a finite metric space.

\( \text{a) Show that every subset of } X \text{ is open: i.e., the associated topology on } X \text{ is discrete.} \)

\( \text{b) Show that any two metric spaces with 0 points are isometric and that any two metric spaces with 1 point are isometric.} \)

\( \text{c) Classify all two point metric spaces up to isometry, and show that they can all be isometrically embedded in } \mathbb{R}. \)

\( \text{d) Classify all three point metric space up to isometry, and show that they can all be isometrically embedded in } \mathbb{R}^2. \)

\footnote{This problem has a star because it uses a bit more about cardinalities of sets than the finite/countably infinite/uncountably infinite trichotomy. If you are familiar with cardinalities of sets, this problem is not especially difficult.}
e) Let \( n \in \mathbb{Z}^+ \), and let \( S_n = \{1, \ldots, n\} \) endowed with the discrete metric. Show that \( S_n \) can be isometrically embedded in \( \mathbb{R}^{n-1} \).

f)* Show that \( S_n \) cannot be isometrically embedded in \( \mathbb{R}^{n-2} \).

**Exercise 4.14.** (The Metric Associated to a Connected Graph)

a) Let \( X \) be a four point set and let us name the elements \( \bullet, a, b, c \). We define a function \( d : X \times X \to \mathbb{R} \) by: \( d(x, x) = 0 \) for all \( x \in X \), \( d(\bullet, x) = 1 \) if \( x \neq \bullet \) and \( d(x, y) = 2 \) if \( x, y, \bullet \) are all distinct. Show that \( d \) is a metric function on \( X \). Show that \( (X, d_X) \) cannot be isometrically embedded in \( \mathbb{R}^N \) for any \( N \in \mathbb{Z}^+ \). In particular, this is the smallest example of a metric space which cannot be isometrically embedded in Euclidean space.

b) Let \( G = (V, E) \) be a simple, undirected, connected graph. Define \( d : V \times V \to \mathbb{R} \) by taking \( d(x, y) \) to be the minimal length of a path from \( x \) to \( y \). Show that \( d \) is a metric function on \( V \).

c) Explain why the metric of part a) is a special case of the metric of part b).

d)** Give necessary and/or sufficient conditions on a finite graph for its associated metric to be isometrically embeddable in \( \mathbb{R}^n \) for some \( n \in \mathbb{Z}^+ \).

**Exercise 4.15.** a) (This is hard.) Let \( G \) be a finite group. Show that there is a finite subset \( X \) of Euclidean space \( \mathbb{R}^n \) (for some \( n \) depending on \( G \)) such that the isometry group \( \text{Iso}X \) is isomorphic to \( G \).

b) (This is probably too hard, but you asked.) For any group \( G \), find a metric space \( X \) such that \( \text{Iso}X \cong G \).

**Exercise 4.16.** Let \( (X, \tau) \) be a topological space, and let \( Y \) be a subset of \( X \).

a) Let \( \tau_Y = \{U \cap Y \mid U \in \tau_X\} \): that is, we take each open set in \( X \) and intersect it with \( Y \). Show that \( \tau_Y \) is a topology on \( Y \), called the **subspace topology**.

b) Let \( \iota : Y \hookrightarrow X \) be the inclusion map from \( Y \) to \( X \). Show that \( \iota : (Y, \tau_Y) \to (X, \tau_X) \) is continuous.

**Exercise 4.17.** Let \( (X, d) \) be a metric space, and let \( Y \) be a subset of \( Y \).

a) Show that the restriction \( d_Y \) of \( d \) to \( Y \times Y \) is a metric function on \( Y \).

b) Show that if \( B \) is an open ball in \( X \), then \( B \cap Y \) need not be an open ball in \( Y \).

c) (Nonembarrassment of Riches) Notice that we have two topologies on \( Y \): let \( \tau_1 \) be the topology associated to the restricted metric \( (Y, d_Y) \), and let \( \tau_2 \) be the subspace topology (c.f. the previous exercise). Show that in fact \( \tau_1 = \tau_2 \).

**Exercise 4.18.** (The Trouble With Order Topologies) Let \( X \) be an ordered set, and let \( Y \) be a subset of \( X \). In analogy to the previous exercise, we can now define two topologies on \( Y \): on the one hand, restricting \( \leq \) to a relation on \( Y \) makes \( Y \) into an ordered set so it has the order topology, say \( \tau_1 \). On the other hand \( X \) has an order topology \( \tau_X \) and we can give \( Y \) the subspace topology, say \( \tau_2 \).

a) Give an example in which these two topologies \( \tau_1 \) and \( \tau_2 \) on \( Y \) do not coincide.

(Suggestion: take \( X = \mathbb{R} \).)

b)* Give an example in which the subspace topology \( \tau_2 \) on \( Y \) is not the topology associated to any ordering on \( Y \).\(^2\)

\(^2\)This is entirely straightforward and has been used in class several times. But why not check it?

\(^3\)Comment: this exercise is probably the reason why order topologies are not as popular as metric topologies.
Exercise 4.19. (Absoluteness of Quasi-Compactness) Let $X$ be a topological space, and let $Y$ be a subset of $X$. Show that the following are equivalent:

(i) $Y$ is quasi-compact as a subset of $X$: that is, for any family $\{U_i\}_{i \in I}$ of open subsets of $X$ with $Y \subseteq \bigcup_{i \in I} U_i$, there is a finite subset $J \subseteq I$ with $Y \subseteq \bigcup_{i \in J} U_i$.

(ii) $Y$ is quasi-compact in the subspace topology: that is, for any family $\{V_i\}_{i \in I}$ of open subsets of $Y$ with $\bigcup_{i \in I} V_i = Y$, there is a finite subset $J \subseteq I$ with $Y = \bigcup_{i \in J} V_i$.

Exercise 4.20. Let $X$ be a quasi-compact topological space, and let $Y \subseteq X$.

(a) Suppose $Y$ is closed. Show that $Y$ is quasi-compact.

(b) Suppose $X$ is Hausdorff and $Y$ is quasi-compact. Show that $Y$ is closed.

(c) Deduce that a subset of a compact metric space is compact if it is closed.

Exercise 4.21. Let $X$ be a metric space.

(a) Let $Y$ be a subset without any limit points in $X$. Show that $Y$ is discrete (in the subspace topology) and closed.

(b) Let $x$ be a sequence in $X$ without any convergent subsequence. Show that the term set $\{x_n \mid n \in \mathbb{Z}^+\}$ has no limit points in $X$.

(c) Show that any compact metric space is sequentially compact.

(d) We will see (much) later that there are topological spaces which are quasi-compact but not sequentially quasi-compact. It follows that somewhere above we must have used that $X$ is a metric space in a critical way. Where?

Exercise 4.22. a) Let $(X_1,d_1), \ldots, (X_n,d_n)$ be nonempty metric spaces. Let $d$ be any good metric on $X = \prod_{i=1}^n X_i$. (If you want to be concrete, you can take $d(x,y) = \sup_{i=1}^n d_i(x_i,y_i)$.) Show that $X$ is sequentially compact iff each $X_i$ is sequentially compact. (Hint: this is actually quite easy: just repeatedly extract subsequences.)

b) Let $\{(X_n,d_n)\}_{n=1}^\infty$ be a sequence of nonempty metric spaces, and let $d$ be any good metric on $X = \prod_{n=1}^\infty X_n$. Show that $X$ is sequentially compact iff each $X_n$ is sequentially compact. (Hint: this is a classic diagonalization argument.)

Exercise 4.23. Let $V$ be an $\mathbb{R}$-vector space. A norm on $V$ is a function $\| \cdot \| : V \to \mathbb{R}$ such that:

(N1) For all $x \in V$, $\|x\| \geq 0$, and $\|x\| = 0 \iff x = 0$.

(N2) (Homogeneity) For all $\lambda \in \mathbb{R}$ and $x \in V$, $\|\lambda x\| = |\lambda|\|x\|$.

(N3) (Triangle Inequality) For all $x, y \in V$, $\|x + y\| \leq \|x\| + \|y\|$.

A normed linear space is $(V, \| \cdot \|)$ with $\| \cdot \|$ a norm on the $\mathbb{R}$-vector space $V$.

a) Let $(V, \| \cdot \|)$ be a normed linear space. Show that $d_{\| \cdot \|} : V \times V \to \mathbb{R}$ given by $d_{\| \cdot \|}(x,y) = \|x - y\|$ is a metric function on $V$.

b) For $p \in [1,\infty]$ we defined a $p$-norm $\| \cdot \|_p$ on $\mathbb{R}^N$. Check that this is a norm in the sense of this exercise.

c) Convince yourself that there are many other norms on $\mathbb{R}^N$ besides the ones from part b). (Suggestions: e.g. finite sums and positive scalings of norms are norms.)

Exercise 4.24. Show that any two norms on $\mathbb{R}^N$ are Lipschitz equivalent. (Some hints: it suffices to compare an arbitrary norm $\| \cdot \|$ to $\| \cdot \|_1$. Use the homogeneity property to reduce to the unit balls, and use the compactness of the unit ball in $(\mathbb{R}^N, \| \cdot \|_1)$ to get the Lipschitz constants.)

Exercise 4.25. A Banach space is a complete normed linear space.

a) Show that any norm on $\mathbb{R}^N$ is complete and thus makes $\mathbb{R}^N$ into a Banach space.
b) Let $C([0,1])$ be the set of all continuous functions $f : [0,1] \to \mathbb{R}$. This is an $\mathbb{R}$-vector space under pointwise addition and scalar multiplication. For $f \in C([0,1])$, put $\|f\|_{\infty} = \max_{x \in [0,1]} |f(x)|$. Show that $(C([0,1]), \| \cdot \|_{\infty})$ is a Banach space.

c) Let $c_0$ be the set of all sequences $x$ of real numbers which converge to 0. This is an $\mathbb{R}$-vector space under componentwise addition and scalar multiplication. For $x \in c_0$, put $\|x\| = \max_n |x_n|$ (convince yourself that this maximum exists since $x \to 0$). Show that $(c_0, \| \cdot \|)$ is a Banach space.