MATH 4200 HW: PROBLEM SET FIVE: GENERAL TOPOLOGICAL SPACES

PETE L. CLARK

5. General Topological Spaces In Earnest

Please solve 5.1; two of 5.2, 5.3, 5.4; and three further problems.

In any set X a sequence is just a function from $\mathbb{Z}^+$ to X, $\mathbf{x} \mapsto \mathbf{x}_n$. Let X be a sequence in a topological space X. We say that **x converges** to a point $p \in X$ if for every neighborhood U of p, there is $N = N(U)$ such that $\mathbf{x}_n \in U$ for all $n \geq N$.

Let (X, τ) be a topological space. A **base** for the topology τ is a subset $\mathcal{B} \subset \tau$ such that every element of τ is a union of elements of $\mathcal{B}$.
(Silly example: τ is a base for itself. Important example: in any metric space, the open balls form a base for the metric topology.)
A topological space is **second countable** if it admits a countable base.

Let p be a point of a topological space X. A **neighborhood base at p** is a family $\mathcal{B}_p$ of neighborhoods of p such that for every neighborhood N of p, we have $B \subset N$ for some $B \in \mathcal{B}_p$.
(Silly examples: the collection of all neighborhoods of p is a neighborhood base, as is the family of all open neighborhoods. Important examples: if X is a metric space and $p \in X$, then all of the following are neighborhood bases at p: the balls centered at p; the open balls centered at p; the closed balls of positive radius centered at p; the open balls of radius $\frac{1}{n}$ for $n \in \mathbb{Z}^+$ centered at p.)
A topological space is **first countable at p** if there is a countable neighborhood base at p. A topological space is **first countable** if it is first countable at each of its points.

A topological space is **Lindelöf** if every open cover admits a countable subcover.

A topological space is **Kolmogorov** if for all points $p \neq q$ in X, either there is an open set U containing p and not q or there is an open set V contianing q and not p (or both: recall that "or" in mathematics is always inclusive unless otherwise mentioned). A topological space is **separated** if for all $p \neq q$ in X, there is an open set U containing p and not q *and* and open set V containing q and not p. A topological space is **Hausdorff** if two distinct points admit disjoint open neighborhoods. A topological space is **quasi-regular** if for every closed subset A and every point $p \not{}nA$ there are disjoint open sets $U \supset \{p\}$ and $V \supset A$. A topological space is **regular** if it is quasi-regular and Hausdorff. A topological space is **quasi-normal** if for all disjoint closed subsets A and B there are disjoint open subsets $U \supset A$ and $V \supset B$. A topological space is **normal** if it is quasi-normal and Hausdorff.

5.1. **Foot Wetting.**

Exercise 5.1. *Let X be a set. Determine whether each of the following topological spaces has each of the following properties: quasi-compactness, connectedness, separability, existence of a countable base, Lindelöfness, quasi-normality, quasi-regularity, Hausdorff(ness), separated(ness), Kolmogorov(ness). In some cases the answer will depend on the cardinality of the set X.*
a) The discrete topology on X: $\tau = 2^X$.
b) The indiscrete topology on X: $\tau = \{\varnothing, X\}$.
c) The cofinite topology on X: the closed *subsets are X and the finite subsets.*
d) The cocountable topology on X: the closed *subsets are X and the countable subsets.*

5.2. **Sequences and First Countability.**

Exercise 5.2. *A sequence $\mathbf{x}$ in a set X is* ***eventually constant*** *if there is $p \in X$ and $N \in \mathbb{Z}^+$ such that $\mathbf{x}_n = p$ for all $n \geq N$. The point p is called the* ***eventual value*** *of a sequence.*
a) Show that the eventual value of an eventually constant sequence is unique. (This is pretty trivial, but worth recording.)
b) Show that in any topological space, every eventually constant sequence converges to its eventual value. Show by example that an eventually constant sequence may converge to other points as well!
c) Call a topological space X ***sequentially discrete*** *if for a sequence $\mathbf{x}$ in X and $p \in X$, we have that $\mathbf{x} \to p$ iff $\mathbf{x}$ is eventually constant with eventual value p. Show that a discrete space is sequentially discrete.*
d) Show that the cocountable topology on an uncountable set is sequentially discrete but not discrete.

Exercise 5.3.
a) Show that in an indiscrete space, every sequence converges to every point.
b) Let X be a Hausdorff topological space, let $\mathbf{x}_n$ be a sequence in X, let $p, q \in X$ and suppose that $\mathbf{x}_n \to p$ and $\mathbf{x}_n \to q$. Show that $p = q$.
c) Let X be a topological space in which each point converges to at most one point. Show that X is separated. (Hint: show the contrapositive. More specifically, show that if $p \in X$ is such that $\{p\}$ is not closed, then the constant sequence $p, p, p \ldots$ converges to more than one point!)
d) Exhibit a space which is separated, not Hausdorff and in which every sequence converges to at most one point.
e)$*$ Suppose X is first countable and that every sequence in X converges to at most one point. Show that X is Hausdorff.

Exercise 5.4. *a) Show that any metric space is first countable.*
b) Let X be a topological space. For a subspace Y, we define the ***sequential closure*** $\operatorname{sc}(Y)$ *to be the set of $x \in X$ such that there is a sequence $\mathbf{y}$ with $\mathbf{y}_n \in Y$ for all Y and $y \to X$. Show that $Y \subset \operatorname{sc}(Y) \subset \overline{Y}$.*
b) Let X be a first countable topological space. Show that for all subsets Y of X we have $\operatorname{sc}(Y) = \overline{Y}$. Deduce that in a first countable space the topology is determined by the convergent sequences.
c)$*$ Give an example of a topological space Y and a proper subset $Y \subsetneq X$ such that $Y = \operatorname{sc}(Y) \subset \overline{Y} = X$.

5.3. **Continuous Functions.**

Exercise 5.5. *Let X be a topological space.*
a) Show that the following are equivalent:
(i) For every topological space Y, every map $f : X \to Y$ is continuous.
(ii) X has the discrete topology.
b) Show that the following are equivalent:
(i) For every topological space Y, every map $f : Y \to X$ is continuous.
(ii) X has the indiscrete topology.

Exercise 5.6. *a) Let $f : X \to Y$ be a map of topological spaces. Show that the following are equivalent:*
(i) f is continuous.
(ii) For all subsets $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.
b) There is a branch of mathematics called ***nonstandard analysis*** *(NSA) in which one studies an altered version of the real numbers in which there exist nonzero infinitesimal elements and uses it to do calculus in a manner which is much more in the spirit of Newton and Leibniz than the ϵ-δ arguments we were taught (and which were introduced by Cauchy and Weierstrass, roughly 200 years after Newton and Leibniz). Unfortunately to get NSA off the ground requires some nontrivial preliminaries in set theory and/or mathematical logic. Explain however why part a) implies – morally – that if a point is infinitesimally close to a set before applying a continuous function, it remains so after applying the function.*

5.4. **Axioms of Countability.**

Exercise 5.7. *Let $f : X \to Y$ be a continuous surjective map of topological spaces. (We say "Y is a continuous image of X.")*
a) Show: if X is quasi-compact, so is Y.
b) Show: if X is Lindelöf, so is Y.
c) Show: if X is connected, so is Y.
d) Show: if X is separable, so is Y.

Exercise 5.8.
a) Show that a topological space with a countable base is separable and Lindelöf.
b) Show that a subspace of a space with a countable base has a countable base.

5.5. **Separation Axioms.**

Exercise 5.9. *Let X be a topological space.*
a) Show that the following are equivalent:
(i) For all $x \in X$, $\{x\}$ is closed.
(ii) For all $x \neq y$ in X, there are open subsets U and V in X with $x \in U$, $y \notin U$, $y \in V$, $x \notin V$.
b) Show: if X is finite and separated, then it is discrete.

Exercise 5.10.
Subsets A and B of a topological space X are ***separated*** *if $\overline{A} \cap B = A \cap \overline{B} = \varnothing$.*
a) Show that a topological space is separated iff for all $x \neq y \in X$, $\{x\}$ and $\{y\}$ are separated. (I hope this explains the terminology!)
b) Find an open subset A and a closed subset B of $\mathbb{R}$ which are disjoint but not separated.
c) Show that if (X, d) is a metric space then A and B are separated iff $d(A, B) > 0$.

Exercise 5.11. *a) Show that for a topological space X, the following are equivalent:*
(i) X is separated.
(ii) For all $p \in X$, $\{p\}$ is closed.
(iii) For all $p \in X$, the intersection over all neighborhoods of p is $\{p\}$.
b) Show that a topological space X is Hausdorff iff for all $p \in X$, the intersection of all closed *neighborhoods of pis $\{p\}$.*

5.6. **Bases and Subbases.**

Exercise 5.12. *Let X be a set and let $\mathcal{B} \subset 2^X$ be a collection of subsets of X. Let τ be the collection of all unions of elements of $\mathcal{B}$.*
a) Suppose that $\mathcal{B}$ satisfies the following properties:
(B1) For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.
(B2) $\bigcup_{B \in \mathcal{B}} = X$.
Show that τ is a topology on X.
b) Suppose that τ is a topology on X and $\mathcal{B} \subset \tau$ is a base for τ. Show that $\mathcal{B}$ satisfies (B1) and (B2) above.
c) Revisit the verification that the open sets in a metric space form a topology in light of the above considerations.

Exercise 5.13. *a) Show that continuity can be checked on any base: if $f : X \to Y$ is a map of topological spaces and $f^{-1}(V)$ is open for all V in a base $\mathcal{B}$ of Y, then f is continuous.*
b) Show that quasi-compactness can be checked on any base: if $\mathcal{B}$ is a topological space such that every open covering of X by elements of $\mathcal{B}$ has a finite subcovering, then X is quasi-compact.

Exercise 5.14. *Let (X, τ) be a topological space. A family of open sets $\mathcal{S}$ is a **subbase** for τ if every $U \in \tau$ is a union of sets of the form $\bigcap_{j=i}^{n} S_{i_j}$ where each $S_{i_j} \in \mathcal{S}$. In other words, $\mathcal{S}$ is a subbase for τ if by taking finite intersections and then arbitrary unions, we get every element of τ.*
(Convention: The intersection over the empty family of subsets of X is X. The union over the empty family of subsets of X is $\varnothing$.)
a) Show that for any set X and any family $\mathcal{S}$ of subsets of X, there is a unique topology τ on X for which $\mathcal{S}$ is a subbase.
(In other words, any family of subsets of a set whatsoever generates a topology by taking finite intersections and then arbitrary unions.)
b) Let X be a (totally) ordered set. For $a, b \in X$ we define

$$(-\infty, b) = \{x \in X \mid x < b\},\ (a, \infty) = \{x \in X \mid a < x\}.$$

Show that the family $\{(-\infty, b), (a, \infty)\}_{a,b \in X}$ is a subbase for the order topology.
c) Show that continuity can be checked on any subbase: if $f : X \to Y$ is a map of topological spaces and $f^{-1}(V)$ is open for all V in a subbase $\mathcal{S}$ of Y, then f is continuous.
d)$*$ Show that quasi-compactness can be checked on any subbase: if $\mathcal{S}$ is a topological space such that every open covering of X by elements of $\mathcal{S}$ has a finite subcovering, then X is quasi-compact.
*(*WARNING*: the name of this result is the **Alexander Subbase Theorem**. If you solved the previous exercise on checking quasi-compactness on a base, you might think that this should be relatively straightforward. If so, you'd be dead wrong.)*

5.7. Specialization.

Exercise 5.15. *Let X be a topological space. Consider the following relation $\prec$ on X: we write $x \prec y$ if $y \in \overline{\{x\}}$. (We say that x* ***specializes to*** *y and call the relation "specialization".)*
a) Explicitly determine the specialization relation on the Sierpinski space $\{\circ, \bullet\}$.
b) Show that the specialization relation $\prec$ is a quasi-ordering on X: that is, it is symmetric and transitive.
c) Show that the specialization relation $\prec$ is anti-symmetric – and thus, by part a), a ***partial ordering*** *– iff X is Kolmogorov.*
d) Show that the specialization relation is the identity – i.e., $x \prec y \iff x = y$ – iff X is separated.

Exercise 5.16. *Let $(X, \prec)$ be a quasi-ordered set: i.e., $\prec$ is a reflexive, transitive binary relation on X. A subset Y of X is* ***downward closed*** *if for all $y \in Y$ and $x \in X$, if $x \prec y$, then $x \in Y$.*
a) Let τ_X be the set of all downward closed subsets of X. Show: τ_X is a topology.
b) A topological space is called ***Alexandroff*** *if arbitrary intersections of open subsets are closed. (Of course this is much stronger than the property required in the definition of a topology.) Show that any finite topological space is Alexandroff. Show that the topology τ_X associated to a quasi-ordered set in part a) above is Alexandroff.*
c) Show that the specialization relation on the topology τ_X associated to a quasi-ordered set is precisely the original quasi-ordering $\prec$.
d) (Equivalence Between Quasi-Ordered Sets and Alexandroff Spaces) Let $(X, \prec)$ and $(Y, \prec)$ be quasi-ordered sets. A ***quasi-order homomorphism*** *is a map $f : X \to Y$ such that for all $x_1 \leq x_2 \in X$, if $x_1 \prec x_2$ then $f(x_1) \prec f(x_2)$. A* ***quasi-order isomorphism*** *$f : X \to Y$ is a bijective quasi-order homomorphism whose inverse function $f^{-1} : Y \to X$ is also a quasi-order homomorphism. Two quasi-ordered sets are* ***isomorphic*** *if there is a quasi-order isomorphism between them. Show: if $(X, \prec)$ and $(Y, \prec)$ are quasi-ordered sets, then $(X, \prec)$ and $(Y, \prec)$ are isomorphic iff the Alexandroff spaces (X, τ_X) and (Y, τ_Y) are homeomorphic.*
e) Deduce from part d) an equivalence between Kolmogorov Alexandroff spaces and partially ordered sets.

5.8. Finite Topologies.

Exercise 5.17. *a) Show: there are, up to homeomorphism, three topologies on a 2-point set: the discrete, Sierpinski and indiscrete.*
b$*$) Show: there are, up to homeomorphism, nine topologies on a 3-point set, five of which are Kolmogorov. (Suggestion: by the preceding exercise it is enough to classify quasi-orders on a 3-point set. This seems easier.)
c) For $n \in \mathbb{Z}^+$, let $T(n)$ be the number of topologies on an n-point set, up to homeomophism, and let $T_0(n)$ be the number of Kolmogorov topologies on an n-point set, up to homeomorphism.[1] Show: $T(n)$ and $T_0(n)$ are both increasing and unbounded.

[1] For some information on $T(n)$ including its values up to $n = 16$, see `http://oeis.org/A001930`. For some information on $T_0(n)$ including its values up to $n = 16$, see `http://oeis.org/A000112`.

d∗) Show: for all $n \in \mathbb{Z}^+$ we have

$$T(n) = \sum_{k=0}^{n} S(n,k) T_0(k),$$

*where $S(n,k)$ the **Stirling number of the second kind**.*

5.9. **The Zariski Topology.**
This exercise is for those with some background in commutative ring theory.

Exercise 5.18. *a) Let R be a commutative ring, and let $\operatorname{Spec} R$ be the set of prime ideals. For any ideal I of R, let $V(I)$ be the set of prime ideals containing I, and let $U(I) = \operatorname{Spec} R \setminus V(I)$. Show: $\{U(I) \mid I \text{ is an ideal of } R\}$ is a topology on $\operatorname{Spec} R$, called the **Zariski topology**.*
b) Show: the Zariski topology on $\operatorname{Spec} R$ is always Kolmogorov. Show: it is separated iff it is Hausdorff iff every prime ideal of R is maximal.
c) Show: the Zariski topology on $\mathbb{C}[t]$ is homeomorphic to the cofinite topology on $\mathbb{C}$.
d) Let K be a field and let $K[[t]] = \{\sum_{n=0}^{\infty} a_n t^n \mid a_n \in K\}$ be the ring of formal power series with coefficients in K. Show: $\operatorname{Spec} K[[t]]$ is homeomorphic to the Sierpinski space $\{\circ, \bullet\}$.
*e) Show that the conclusion of part d) holds for any **discrete valuation ring** R.*

5.10. **Examples of Nonmetrizable Spaces.**

Exercise 5.19. *a) Show that $\{[a,b) \mid a < b\}$ is the base for a topology on $\mathbb{R}$, called the **Sorgenfrey line**.*
b) Show that the Sorgenfrey line is strictly finer than the usual (Euclidean = order) topology on $\mathbb{R}$.
c) Show that the Sorgenfrey line is totally disconnected.
d) Show that the Sorgenfrey line is first-countable and separable but does not admit a countable base.
e) Show that the Sorgenfrey line is not metrizable.

Exercise 5.20. *Let $X = [0,1] \times [0,1]$ be given the **lexicographic order topology** (* not *the metric topology from $\mathbb{R}^2$!!).*
a) Show that X is compact. (I will allow you to use the theorem that an order topology is compact iff it is Dedekind complete and has top and bottom elements, which can be proven by ordered induction.)
b) Show that there is an uncountable family of pairwise disjoint nonempty open subsets of X. Deduce that X is not separable.
c) Deduce that X is not metrizable. (Comment: this is by far the most subtle example of a nonmetrizable space that we have seen yet. There is nothing "obviously wrong" with the space: it is compact, normal and first countable. This shows that metrizability is a subtle issue. In fact, in our course we will give a necessary and sufficient condition for a topological space to be separable *and metrizable but not to be metrizable with no extra conditions.)*