Sequences and Series: A Sourcebook

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Introduction

This is a text concerned with the theory of infinite sequences and series, largely at an intermediate undergraduate level. The proximate cause of the text is the Math 3100 course that I taught at UGA in Spring 2011, which (as usual) motivated me to type up my own notes.

The current text is admittedly a bit strange, and it is **not necessarily intended for use** in an undergraduate course. Rather the purpose upon conception was to be *supplementary* to what appears in more standard texts. In the 19th century, the theory of infinite series was the crown jewel of mathematics, and most of the leading mathematicians contributed to it in some way. In our day this is no longer the case. On the contrary, for a while now the "theory of infinite series" has been synonymous with certain chapters in undergraduate real analysis. One can even put a finer point on it: after the introduction of Rudin's breakthrough text [**R**], what an American student learns about infinite series is some (or better, all) of Chapters 3 and 7 of [**R**]. Let me be clear: I find these chapters (especially the latter) to be the high points of what must be the most influential and successful of all American mathematical textbooks. And yet there is – must be – more to infinite series than what fits in these 75 pages, and there must be other valid – even novel – takes on the material. In fact the most recent generation of American texts has followed up on the latter point, if not the former. It is now recognized that relatively few undergraduate math majors are prepared for a presentation like Rudin's, and the most successful contemporary texts are more student friendly. I do not in any way object to this trend, but the current text does not participate in it: though I know a positive number of undergraduate students who have read and learned from (various drafts of) this text, it is intended more for the instructors of such courses than the students than the students themselves. Most of all it is intended for me, and in places it is openly experimenal: I have tried out certain things in order to determine which aspects of them (if any) may be suitable to include in actual lectures to actual undergraduates.

The desire to be different at any price is most clearly evident in Chapter 0. This is an approach to the foundations of the subject as might be done by Little Nicky Bourbaki. The conceit here is that the theory of infinite series arises from the confluence of multiple structures: (i) sequences in a set $\mathbb{R}$; (ii) a total ordering on $\mathbb{R}$, which provides a notion of convergence, and which satisfies any of several completeness properties (iii) the compatible structure of a field on $\mathbb{R}$. Saying things this way makes one want to *analyze* the setup: how much can be said when some but not all of these structures are present? One may well say that this kind of general nonsense is antithetical to the contentful concreteness of "real real analysis," and

I have several colleagues who would say at least this much. I would say that I am participating in the recent trend of "real analysis in reverse" which – though motivated by any number of things which have little to do with the substance of real analysis, nevertheless has a way of cutting to the core of "what is really going on" in undergraduate analysis, with some nice payoffs both at the research level and (more importantly, to me) pedagogically.

Most "real real analysts" would agree that just about the worst thing to spend time on in any undergraduate analysis course is a formal construction of the real numbers. What is less acknowledged, but I think must be true, is that the reason for this is the full success of the structural approach to the real numbers: they are characterized as being the unique **complete** ordered field (here "complete" can be construed in several different ways and the statement remains true). Thus there is at least some promise beginning the theory of sequences and series with a structural approach.

Here is a final remark about Chapter 0: it is almost dispensable. Most of the concepts which are given in obnoxious generality in Chapter 0 are repeated in the case of $\mathbb{R}$ in Chapter 1. I would advise anyone using this text in an American undergraduate course to start with Chapter 1 and dip backwards into Chapter 0 if needed.

CHAPTER 0

Foundations

1. Prerequisites

These notes are for a student who has had some prior exposure to *proofs*, *basic mathematical structures* and *mathematical abstraction*. At UGA these concepts are taught in Math 3200. We will draw more strongly on having a certain minimum comfort level with such things than on any specific prior knowledge, but for instance the student should be familiar with the following terms:

set, subset, proper subset, power set, cardinality of a finite set, relation, equivalence relation, partition, function, injection, surjection, bijection.

2. Number Systems

We begin by reminding the reader about basic number systems. One should view this as being little more than fixing notation: sophisticated readers will know that there is some mathematical content involved in giving complete definitions of these structures. It is by no means our goal to do so here.

We define

$$\mathbb{Z}^+ = \{1, 2, 3, \ldots, n, \ldots\},$$

the set of **positive integers**. We define

$$\mathbb{N} = \{0, 1, 2, 3, \ldots, n, \ldots\},$$

the set of **natural numbers**. Note that the two sets $\mathbb{N}$ and $\mathbb{Z}^+$ differ precisely in that 0 is a member of the natural numbers and is not a member of the positive integers. There are some (very minor) mathematical reasons behind this distinction, but in practice it functions mostly as a convention. Moreover it is not a universally held one, and when you read or hear others using the terms "positive integers" or "natural numbers," you will need to figure out from the context – or ask – whether 0 is meant to be included. (And it is annoying, because the answer is not *very* important, but a misunderstanding could still cause some trouble.)

We define

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\},$$

the set of **integers**.

We define

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\},$$

the set of **rational numbers**.

We define $\mathbb{R}$ to be the set of **real numbers** . This is a point where a formal definition is challenging enough to be unhelpful for a beginning student, but roughly the real numbers are the mathematical structure which make precise the idea of "points on the number line". A non-negative real number can be thought of as a distance between two points (in the Euclidean plane or Euclidean space), which need not be a rational quantity. One can *represent* any real number by its decimal expansion:

$$a_0.a_1a_2\ldots a_n\ldots$$

Here a_0 is an integer and

$$a_1, a_2, \ldots \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

are **decimal digits**. To make real sense out of this definition requires the theory of sequences and series! Later on in this chapter we will give not a definition of $\mathbb{R}$ but a perfectly effective and useful operational description of them: they form a complete ordered field. (You are not supposed to know what these terms mean yet.)

We define

$$\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R},\ i^2 = -1\}$$

to be the set of **complex numbers**.

Each of these number systems is a proper subset of the next: we have

$$\mathbb{Z}^+ \subsetneq \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}.$$

3. Sequences

Let X be a set. A **sequence in X** is a function $x : \mathbb{Z}^+ \to X$. Thus, as for any function, for each element $n \in \mathbb{Z}^+$ we must assign exactly one element $x(n) \in X$. This has the effect of giving us an infinite ordered list of elements of X:

$$x(1), x(2), x(3), \ldots, x(n), \ldots.$$

We say that e.g. $x(3)$ is the **3rd term** of the sequence and in general that $x(n)$ is the **nth term** of the sequence.

EXAMPLE 0.1. *For a set X, we denote the set of sequences in X by $X^{\mathbb{Z}^+}$. (This is a case of the notation Y^X for the set of functions from a set X to a set Y.)*
a) Suppose $X = \varnothing$. Then a sequence in X is a function from the nonempty set $\mathbb{Z}^+$ to the empty set $\varnothing$. There are no such functions, so there are no sequences in X.
b) Suppose $\#X = 1$, i.e., $X = \{\bullet\}$ has a single element. A sequence in X is a function from the set $\mathbb{Z}^+$ to the one-point set $\{\bullet\}$. There is exactly one function: we must have $x(n) = \bullet$ for all $n \in \mathbb{Z}^+$.
*c) Things become considerably more interesting when X has at least two elements. Suppose that $\#X = 2$, and to fix notation, take $X = \{0, 1\}$. A sequence $\mathbb{Z}^+ \to \{0, 1\}$ is called an **(infinite) binary sequence**. This is an extremely basic, important and useful mathematical structure. We can represent every real number between 0 and 1 by an infinite binary sequence, for instance. (Unfortunately the representation is not always unique, an annoyance that we do not need to deal with at the moment.) More to the point, the collection of all infinite binary sequences $\{f : \mathbb{Z}^+ \to \{0, 1\}\}$ is naturally in bijection with the power set $2^{\mathbb{Z}^+}$, i.e., the set of all subsets of $\mathbb{Z}^+$.*

Namely, let $S \subset \mathbb{Z}^+$ be a subset. To S we associate the binary sequence $s(S)$: for $n \in \mathbb{Z}^+$, we put $s(S) = 1$ if $n \in S$ and $s(S) = 0$ if $n \notin S$. This gives us a map

$$s : 2^{\mathbb{Z}^+} \to \{0,1\}^{\mathbb{Z}^+}.$$

To see that s is a bijection we write down the inverse function

$$t : \{0,1\}^{\mathbb{Z}^+} \to 2^{\mathbb{Z}^+}.$$

Namely, to any binary sequence $x : \mathbb{Z}^+ \to \{0,1\}$ we associate the subset $t(x) = \{n \in \mathbb{Z}^+ \mid x(n) = 1\}$. I leave it to you to check that for any subset $S \subset \mathbb{Z}^+$ we have $t(s(S)) = S$ and that for any binary sequence $x : \mathbb{Z}^+ \to \{0,1\}$ we have $s(t(x)) = x$.
d) We will be most interested in the case of sequences in $\mathbb{R}$. This is a very *large set. (Those who are familiar with cardinalities of infinite sets may wish to show that the cardinality of the set $\mathbb{R}^{\mathbb{Z}^+}$ of all rael sequences has the same cardinality as that of the power set $2^{\mathbb{R}}$ of all real numbers. Those who are not: no worries.)*

The "theory of infinite sequences" really means the theory of sequences in $\mathbb{R}$, $\mathbb{C}$ or closely related things (e.g. $\mathbb{R}^N$). There is very little "theory of sequences in an arbitrary set X." Very little but not none: we now give some.

To a sequence $x : \mathbb{Z}^+ \to X$ we can associate the set

$$x(\mathbb{Z}^+) = \{x(n) \in X \mid n \in \mathbb{Z}^+\}$$

of all terms of the sequence. This is just the image of the corresponding function. Notice that knowing the sequence itself is more information than knowing its image: consider in particular the case of binary sequences: $X = \{0,1\}$. There is exactly one binary sequence with image 0, namely

$$0, 0, 0, \ldots, 0, \ldots$$

There is exactly one binary sequence with image 1, namely

$$1, 1, 1, \ldots, 1, \ldots$$

All others have image $\{0,1\}$: there are (uncountably) infinitely many of them.

Let $f : X \to Y$ be a function. Given a sequence $x : \mathbb{Z}^+ \to X$ in X, we can **push it forward** to a sequence $f_*(x)$ in X by

$$f_*(x) : n \mapsto f(x(n)).$$

In other words, $f_*(x)$ is simply the composite function $f \circ x : \mathbb{Z}^+ \to Y$.

Example 0.2. *Let $x : \mathbb{Z}^+ \to \mathbb{Z}^+$ by $x(n) = n^2$. Thus x is a sequence in $\mathbb{Z}$.*
a) Using the inclusion map $f : \mathbb{Z} \hookrightarrow \mathbb{R}$ we can view x as a sequence of real numbers: $f^*(x)(n) = n^2 \in \mathbb{R}$.
b) Let $f : \mathbb{Z}^+ \to \{0,1\}$ by $f(n) = 0$ if n is even and $f(n) = 1$ if n is odd. (That is: we map an integer to its class modulo 2*.) We push forward the integer sequence*

$$1, 4, 9, 16, 25, 36, 49, \ldots$$

to get the binary sequence

$$1, 0, 1, 0, 1, 0, 1, 0 \ldots.$$

(We have such a nice pattern here because n^2 is even iff n is even.)

Exercise 0.1. *Let $f : X \to Y$ be a map of nonempty sets. Consider the associated map*

$$f_* : X^{\mathbb{Z}^+} \to Y^{\mathbb{Z}^+},\ x \mapsto (n \mapsto f(x(n))).$$

a) Show: f_* is injective iff f is injective.
b) Show: f_* is surjecive iff f is surjective.
c) Deduce: f_* is bijective iff f is bijective.

However, if we have a sequence $y : \mathbb{Z}^+ \to Y$ in a set Y and a function $f : X \to Y$, it does not make sense to "pull back" y to a sequence in X: the functions don't compose correctly. However, if we have a sequence $x : \mathbb{Z}^+ \to X$ and a function $g : \mathbb{Z}^+ \to \mathbb{Z}^+$, we get a new sequence by precomposing with g:

$$g^*(x) = x \circ g : \mathbb{Z}^+ \to X,\ n \mapsto x(g(n)).$$

In full generality this is a rather strange operation. We single out two special cases which will become significant later.

Suppose $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ is strictly increasing: if $n_1 < n_2$ then $g(x_1) < g(x_2)$. Then the process of passing from x to g^*x is called **passing to a subsequence**.

Example 0.3. *The function $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ by $n \mapsto n^2$ is strictly increasing, and this process carries a sequence*

$$x(1), x(2), x(3), \ldots, x(n), \ldots$$

to

$$x(1), x(4), x(9), \ldots, x(n), \ldots.$$

Exercise 0.2. *Let $\mathcal{I}$ be the set of all strictly increasing functions $g : \mathbb{Z}^+ \to \mathbb{Z}^+$. We define a function*

$$\Phi : \mathcal{I} \to 2^{\mathbb{Z}^+}$$

from $\mathcal{I}$ to the power set of $\mathbb{Z}^+$ by mapping each function to its image:

$$\Phi : g \mapsto g(\mathbb{Z}^+) \subset \mathbb{Z}^+.$$

Show: Φ is an injection with image the set of all infinite subsets of $\mathbb{Z}^+$.

The previous exercise reconciles our definition of a subsequence with another (perhaps more natural and useful) way of thinking about them: namely, passage to a subsequence means choosing an infinite set S of positive integers and only keeping the terms x_n of the sequence with $n \in S$.

The other case in which precomposing a sequence with a function $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a reasonable thing to do in real life is when the function $g; \mathbb{Z}^+ \to \mathbb{Z}^+$ is a bijection. Then we may view g^*x as a **rearrangement** of x.

Example 0.4. *Let $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ be the map which for all $n \in \mathbb{Z}^+$ interchanges the $2n-1$ and $2n$: thus*

$$g(1) = 2,\ g(2) = 1,\ g(3) = 4,\ g(4) = 3,\ g(5) = 6,\ g(6) = 5, \ldots.$$

Then precomposing with g carries the sequence

$$x(1), x(2), x(3), x(4), x(5), x(6) \ldots$$

to the sequence

$$x(2), x(1), x(4), x(3), x(6), x(5) \ldots.$$

Remark: We wished to spend a little time with the notation $x(n)$ to drive the point home that a sequence *is* a function. However no one actually uses this notation for sequences: rather, everyone writes $n \mapsto x_n$....and so will we.

4. Binary Operations

Let S be a set. A **binary operation** on S is simply a function from $S \times S$ to S. Rather than using multivariate function notation – i.e., $(x,y) \mapsto f(x,y)$ – it is common to fix a symbol, say $*$, and denote the binary operation by $(x,y) \mapsto x*y$.

A binary operation is **commutative** if for all $x, y \in S$, $x * y = y * x$.

A binary operation is **associative** if for all $x, y, z \in S$, $(x * y) * z = x * (y * z)$.

EXAMPLE 0.5. *Let $S = \mathbb{Q}$ be the set of rational numbers. Then addition $(+)$, subtraction $(-)$ and multiplication $(\cdot)$ are all binary operations on S. Note that division is* not *a binary operation on $\mathbb{Q}$ because it is not everywhere defined: the symbol $a \div b$ has no meaning if $b = 0$.*

As is well known, $+$ and $\cdot$ are both commutative and associative whereas it is easy to see that $-$ is neither commutative nor associative: e.g.

$$3 - 2 = 1 \neq 2 - 3,$$

$$(1 - 0) - 1 = 0 \neq 2 = 1 - (0 - 1).$$

Perhaps the most familiar binary operation which is associative but not commutative is matrix multiplication. Precisely, fix an integer $n \geq 2$ and let $S = M_n(\mathbb{Q})$ be the set of $n \times n$ square matrices with rational entries.

EXERCISE 0.3. *Give an example of a binary operation which is commutative but not associative. In fact, give at least two examples, in which the first one has the set S finite and as small as possible (e.g. is it possible to take S to have two elements?). Give a second example in which the operation is as "natural" as you can think of: i.e., if you told it to someone else, they would recognize that operation and not accuse you of having made it up on the spot.*

Let $*$ be a binary operation on a set S. We say that an element $e \in S$ is an **identity element** (for $*$) if for all $x \in S$,

$$e * x = x * e = x.$$

Remark: If $*$ is commutative (as will be the case in the examples of interest to us here) then of course we have $e * x = x * e$ for all $x \in S$, so in the definition of an identity it would be enough to require $x * e = x$ for all $x \in S$.

PROPOSITION 0.6. *A binary operation $*$ on a set S has at most one identity element.*

PROOF. Suppose e and e' are identity elements for $*$. The dirty trick is to play them off each other: consider $e * e'$. Since e is an identity, we must have $e * e' = e'$, and similarly, since e' is an identity, we must have $e * e' = e$, and thus $e = e'$. $\square$

EXAMPLE 0.7. *For the operation $+$ on $\mathbb{Q}$, there is an identity: 0. For the operation $\cdot$ on $\mathbb{Q}$, there is an identity: 1. For matrix multiplication on $M_n(\mathbb{Q})$, there is an identity: the $n \times n$ identity matrix I_n.*

EXAMPLE 0.8. *The addition operation $+$ on $\mathbb{Q}$ restricts to a binary operation on $\mathbb{Z}^+$ (that is, the sum of any two positive integers is again a positive integer). Since by our convention $0 \notin \mathbb{Z}^+$, it is not clear that $(\mathbb{Z}^+, +)$ has an identity element, and indeed a moment's thought shows that it does not.*

Suppose $(S, *)$ is a binary operation on a set which possesses an identity element e. For an element $x \in S$, an element $x' \in S$ is said to be **inverse to x** (with respect to $*$) if $x * x' = x' * x = e$.

PROPOSITION 0.9. *Let $(S, *)$ be an associative binary oepration with an identity element e. Let $x \in S$ be any element. Then x has at most one inverse.*

PROOF. It's the same trick: suppose x' and x'' are both inverses of x with respect to $*$. We have $x * x' = e$, and now we apply x'' "on the left" to both sides:

$$x'' = x'' * e = x'' * (x * x') = (x'' * x) * x' = e * x' = x'. \qquad \square$$

EXAMPLE 0.10. *For the operation $+$ on $\mathbb{Q}$, every element x has an inverse, namely $-x = (-1) \cdot x$. For the operation $\cdot$ on $\mathbb{Q}$, every nonzero element has an inverse, but 0 does not have a multiplicative inverse: for any $x \in \mathbb{Q}$, $0 \cdot x = 0 \neq 1$. (Later we will prove that this sort of thing holds in any field.)*

5. Binary Relations

Let S be a set. Recall that a **binary relation** on S is given by a subset R of the Cartesian product $S^2 = S \times S$. Restated in equivalent but less formal terms, for each ordered pair $(x, y) \in S\times$, either (x, y) lies in the relation R – in which case we sometimes write xRy – or (x, y) does not lie in the relation R.

For any binary relation R on S and any $x \in S$, we define

$$[x]_R = \{y \in S \mid (x, y) \in R\},$$

$$_R[x] = \{y \in S \mid (y, x) \in R\}.$$

In words, $[x]_R$ is the set of elements of S which are related to R "on the right", and $_R[x]$ is the set of elements of S which are related to R "on the left".

EXAMPLE 0.11. *Let S be the set of people in the world, and let R be the parenthood relation, i.e., $(x, y) \in R$ iff x is the parent of y. Then $[x]_R$ is the set of x's children and $_R[x]$ is the set of x's parents.*

Here are some important properties that a relation $R \subset S^2$ may satisfy.

(R) **Reflexivity**: for all $x \in S$, $(x, x) \in R$.
(S) **Symmetry**: for all $x, y \in S$, $(x, y) \in R \implies (y, x) \in R$.
(AS) **Antisymmetry**: for all $x, y \in S$, if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$.
(Tr) **Transitivity**: for all $x, y, z \in S$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.
(To) **Totality**: for all $x, y \in S$, either $(x, y) \in R$ or $(y, x) \in R$ (or both).

EXERCISE 0.4. *Let R be a binary relation on a set S.*
a) Show: R is reflexive iff for all $x \in S$, $x \in [x]_R$ iff for all $x \in S$, $x \in {}_R[x]$.
b) Show: R is symmetric iff for all $x \in S$, $[x]_R = {}_R[x]$. In this case we put

$$[x] = [x]_R =_R[x].$$

c) Show: R is antisymmetric iff for all $x, y \in S$, $x \in {}_R[y]$, $y \in [x]_R$ implies $x = y$.
d) Show R is transitive iff for all $x, y, z \in S$, $x \in {}_R[y]$ and $y \in {}_R[z]$ implies $x \in {}_R[z]$.
e) Show that R is total iff for all $x \in R$, either $x \in {}_R[y]$ or $x \in [y]_R$.

An **equivalence relation** is a relation which is reflexive, symmetric and transitive.

If R is an equivalence relation on x, we call

$$[x] =_R [x] = [x]_R$$

the **equivalence class** of R.

Let R be an equivalence relation on S, suppose that for $x, y, z \in S$ we have $z \in [x] \cap [y]$. Then transitivity gives $(x, y) \in R$. Moreover if $x' \in [x]$ then $(x, x') \in R$, $(x, y) \in R$ and hence $(x', y) \in R$, i.e., $x' \in [y]$. Thus $[x] \subset [y]$. Arguing similarly with the roles of x and y interchanged gives $[y] \subset [x]$ and thus $[x] = [y]$. That is, any two distinct equivalence classes are disjoint. Since by reflexivity, for all $x \in S$, $x \in [x]$, it follows that the distinct equivalence classes give a **partition** of S.

The converse is also true: if $X = \bigcup_{i \in I} X_i$ is a partition of X – i.e., for all $i \in I$, X_i is nonempty, for all $i \neq j$, $X_i \cap X_j = \varnothing$ and every element of X lies in some X_i, then there is an induced equivalence relation R by defining $(x, y) \in R$ iff x, y lie in the same element of the partition.

6. Ordered Sets

A **partial ordering** on S is a relation R which is reflexive, anti-symmetric and transitive. We then commonly write $x \leq y$ for $(x, y) \in R$. We also refer to the pair $(S, \leq)$ as a **partially ordered set**. A **totally ordered set** is a partially ordered set $(S, \leq)$ satisfying totality: for all $x, y \in S$, either $x \leq y$ or $y \leq x$.

EXAMPLE 0.12. ***(The Key Example)*** *The real numbers $\mathbb{R}$ under the standard $\leq$ is a totally ordered set.*

It will be useful to borrow some auxiliary notation from the above case. Namely, for a partially ordered set $(S, \leq)$ and $x, y \in S$, we write $y \geq x$ to mean $x \leq y$. We also write $x < y$ to mean $x \leq y$ and $x \neq y$, and finally $y > x$ to mean $x < y$.

EXERCISE 0.5.
For a partially ordered set $(S, \leq)$, we may view $<$ as a relation on S.
a) Show that the relation $<$ satisfies:
(i) anti-reflexivitiy: for no $x \in S$ do we have $x < x$.
(ii) strict anti-symmetry: for no $x, y \in S$ do we have $x < y$ and $y < x$.
(iii) transitivity.
*b) Conversely, we say a relation $<$ on a set S is a **strict partial ordering** if it is anti-reflexive, strictly anti-symmetric and transitive. Given a strict partial ordering $<$ on S, define a new relation $\leq$ by $x \leq y$ iff $x < y$ or $x = y$. Show: $\leq$ is a partial ordering on S.*
c) Check that the processes of passing from a partial ordering to a strict partial ordering and of passing from a strict partial ordering to a partial ordering are mutually inverse. (The upshot is that we can work with $<$ or $\leq$ as the basic relation; nothing is lost either way.)

EXERCISE 0.6. *Let X be a set, and let R be the equality relation on X: that is, $(x,y) \in R \iff x = y$.*
a) Show: R is an equivalence relation on X.
b) Show: R is a partial ordering on X.
c) Show: conversely, if R' is a relation on X which is both an equivalence relation and a partial ordering, then $R' = R$.

We call a partially ordered set $(X, \leq)$ **totally disordered** if $\leq$ is the equality relation, i.e., $x \leq y \iff x = y$. Thus every set can be made into a totally disordered partially ordered set in a unique way.

EXAMPLE 0.13. *For a set X, let $S = 2^X$ be the power set of X (the set of all subsets of X). The* ***containment relation*** *on subsets of X is a partial ordering on S: that is, for subsets A, B of X, we put $A \leq B$ iff $A \subset B$. Note that as soon as X has at least two distinct elements x_1, x_2, there are subsets A and B such that neither is contained in the other, namely $A = \{x_1\}, B = \{x_2\}$. Thus we have* neither *$A \leq B$ nor $B \leq A$.*

In general, two elements x, y in a partially ordered set are **comparable** if $x \leq y$ or $y \leq x$; otherwise we say they are **incomparable**. A total ordering is thus a partial ordering in which any two elements are comparable.

A **bottom element** b in a partially ordered set $(S, \leq)$ is an element of S such that $b \leq x$ for all $x \in S$. Similarly a **top element** t in $(S, \leq)$ is an element of S such that $x \leq t$ for all $x \in S$.

EXERCISE 0.7. *a) Show: a partially ordered set has at most one bottom element and at most one top element.*
b) Show: every power set $(2^X, \leq)$ admits a bottom element and a top element. What are they?
c) Show: $(\mathbb{Z}, \leq)$ has neither a bottom element nor a top element.
d) Suppose that in a partially ordered set $(S, \leq)$ we have a bottom element m, a top element M and moreover $m = M$. What can be said about S?
e) Show: a finite, nonempty totally ordered set has bottom and top elements.

We now have a pair of closely related concepts: an element m in a partially ordered set $(S, \leq)$ is **minimal** if there does not exist $x \in S$ with $x < m$. An element M in a partially ordered set $(S, \leq)$ is **maximal** if there does not exist $\in S$ with $M < y$.

PROPOSITION 0.14. *a) In any partially ordered set, a bottom element is minimal and a top element is maximal.*
b) In a totally ordered set, a minimal element is a bottom element and a maximal element is a top element.
c) A partially ordered set can have a minimal element which is not a bottom element and a maximal which is not a top element.

PROOF. a) If b is a bottom element, then for all $x \neq b$ we have $b < x$, so we cannot also have $x < b$: b is minimal. The argument for top elements is similar and left to the reader.
b) Let m be a minimal element in a totally ordered set, and let $x \neq m$. We cannot have $x < m$ and we are totally ordered, so we must have $m < x$: m is a bottom element. The argument maximal elements is similar and left to the reader.

c) Let X be a set with more than one element, and endow it with the equality relation. Then for no $x, y \in X$ do we have $x < y$, so every element of X is both minimal and maximal, and X has neither bottom nor top elements. $\square$

Comment on terminology: I am sorry to tell you that some people say "maximum" instead of "top element" and "minimum" instead of "bottom element" of a partially ordered set. In a totally ordered set, Proposition 0.14b) ensures this is fine, and we will usually refer to the top and bottom elements of a totally ordered set in this way. However, in a partially ordered set that is not totally ordered, this terminology is most unfortunate: e.g. Proposition 0.14c) would read: "A partially ordered set can have a minimal element which is not a minimum and a maximal element which is not a maximum." Thus changing an adjective to a noun changes the mathematical meaning!

Now let $\{x_n\}_{n=1}^{\infty}$ be a sequence in the partially ordered set $(S, \leq)$. We say that the sequence is **increasing** (respectively, **strictly increasing**) if for all $m \leq n$, $x_m \leq x_n$ (resp. $x_m < x_n$). Similarly, we say that the sequence is **decreasing** (respectively, **strictly decreasing** if for all $m \leq n$, $x_m \geq x_n$ (resp. $x_m > x_n$).

A partially ordered set $(S, \leq)$ is **well-founded** if there is no strictly decreasing sequence $\{x_n\}_{n=1}^{\infty}$ in S. A **well-ordered set** is a well-founded totally ordered set.

Let R be a relation on S, and let $T \subset S$. We define the **restriction** of R to T as the set of all elements $(x, y) \in R \cap (T \times T)$. We denote this by $R|_T$. F

EXERCISE 0.8. *Let R be a binary relation on S, and let T be a subset of S.*
a) Show: if R is an equivalence relation, then $R|_T$ is an equivalence relation on T.
b) Show that if R is a partial ordering, then $R|_T$ is a partial ordering on T.
c) Show that if R is a total ordering, then $R|_T$ is a total ordering on T.
d) Show that if R is a well-founded partial ordering, then $R|_T$ is a well-founded partial ordering on T.

PROPOSITION 0.15. *For a totally ordered $(S, \leq)$, the following are equivalent:*
(i) S is well-founded: it admits no infinite strictly decreasing sequence.
(ii) For every nonempty subset T of S, the restricted totally ordered set $(T, \leq)$ has a bottom element.

PROOF. (i) $\implies$ (ii): We go by contrapositive: suppose that there is a nonempty subset $T \subset S$ without a minimum element. From this it follows immediately that there is a strictly decreasing sequence in T: indeed, by nonemptiness, choose $x_1 \in T$. Since x_1 is not the minimum in the totally ordered set T, there must exist $x_2 \in T$ with $x_2 < x_1$. Since x_2 is not the minimum in the totally ordered set T, there must exist x_3 in T with $x_3 < x_2$. Continuing in this way we build a strictly decreasing sequence in T and thus T is not well-founded. It follows that S itself is not well-founded, e.g. by Exercise 0.4d), but let's spell it out: an infinite strictly decreasing sequence in the subset T of S is, in particular, an infinite strictly decreasing sequence in S!
(ii) $\implies$ (i): Again we go by contrapositive: suppose that $\{x_n\}$ is a strictly decreasing sequence in S. Then let T be the underlying set of the sequence: it is a nonempty set without a minimum element! $\square$

EXAMPLE 0.16. ***(Key Example)**: For any integer N, the set $\mathbb{Z}^{\geq N}$ of integers greater than or equal to N is well-ordered: there are no strictly decreasing infinite sequences of integers all of whose terms are at least N.*[1]

EXERCISE 0.9. *Let $(X, \leq)$ be a totally ordered set. Define a subset Y of X to be **inductive** if for each $x \in X$, if $I_x = \{y \in X \mid y < x\} \subseteq Y$, then $x \in Y$.*
a) Show that X is well-ordered iff the only inductive subset of X is X itself.
b) Explain why the result of part a) is a generalization of the principle of mathematical induction.

Let x, y be elements of a totally ordered set X. We say that y **covers** x if $x < y$ and there *does not exist* $z \in X$ with $x < z < y$.

EXAMPLE 0.17. *In $X = \mathbb{Z}$, y covers X iff $y = x + 1$. In $X = \mathbb{Q}$, no element covers any other element.*

A totally ordered set X is **right discrete** if for all $x \in X$, either x is the top element of X or there exists y in X such that y covers x. A totally ordered set X is **left discrete** if for all $x \in X$, either x is the bottom element of X or there exists z in X such that x covers z. A totally ordered set is **discrete** if it is both left discrete and right discrete.

EXERCISE 0.10. *a) Show that $(\mathbb{Z}, \leq)$ is discrete.*
b) Show that any subset of a discrete totally ordered set is also discrete.

7. Upper and Lower Bounds

Let $(S, \leq)$ be a partially ordered set, and let $Y \subset S$. We say $M \in S$ is an **upper bound for Y** if $y \leq M$ for all $y \in Y$. We say $m \in S$ is a **lower bound for Y** if $m \leq y$ for all $y \in Y$.

Notice that if M is an upper bound for Y and $M' > M$ then also M' is an upper bound for Y. So in general a subset will have many upper bounds. If the set of all upper bounds of Y has a bottom element, we call that the **supremum** or **least upper bound** of Y and denote it by $\sup(Y)$. Similarly, if m is a lower bound for Y and $m' < m$ then also m' is a lower bound for Y. If the set of all lower bounds of Y has a top element, we call tha the **infimum** or **greatest lower bound** of Y and denote it by $\inf(Y)$. Since a partially ordered set can have at most one bottom element and at most one top element, $\sup(Y)$ is unique if it exists and so is $\inf(Y)$.

EXERCISE 0.11. *Let $(S, \leq)$ be a partially ordered set.*
a) Show: $\sup(\varnothing)$ *exists iff X has a bottom element b, in which case* $\sup(\varnothing) = b$.
b) Show: $\inf(\varnothing)$ *exists iff X has a top element t, in which case* $\inf(\varnothing) = t$.

A **lattice** is a partially ordered set $(S, \leq)$ in which every two element subset $\{a, b\}$ has an infimum and a supremum. In this case it is traditional to write

$$a \vee b = \sup(\{a, b\}), \;\; a \wedge b = \inf(\{a, b\}).$$

[1]As usual in these notes, we treat structures like $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ as being known to the reader. This is not to say that we expect the reader to have witnessed a formal construction of them. Such formalities would be necessary to *prove* that the natural numbers are well-ordered, and one day the reader may like to see it, but such considerations take us *backwards* into the foundations of mathematics in terms of formal set theory, rather than forwards as we want to go.

A totally ordered set is certainly a lattice, since we have

$$a \vee b = \max(a, b), \ a \wedge b = \min(a, b).$$

A **complete lattice** is a partially ordered set $(S, \leq)$ in which *every subset* Y (including $Y = \varnothing$) has a supremum and an infimum. A **Dedekind complete lattice** is a partially ordered set $(S, \leq)$ in which every nonempty subset which is bounded above admits a supremum.

EXAMPLE 0.18. *A finite nonempty totally ordered set is a complete lattice.*

EXAMPLE 0.19. *Let X be a set. Then the partially ordered set $(2^X, \subset)$ is Dedekind complete. If $Y \subset 2^X$, then Y is a family of subsets of X, and we have*

$$\sup(Y) = \bigcup_{A \in Y} A, \ \inf(Y) = \bigcap_{A \in Y} A.$$

EXERCISE 0.12. *a) Let $(X, \leq)$ be a partially ordered set in which every subset has a supremum. Show: X is a complete lattice.*
b) Let $(X, \leq)$ be a partially ordered set in which every nonempty bounded above subset has a supremum. Show: X is a Dedekind complete lattice.
c) Let $(X, \leq)$ be a partially ordered set in which every subset has an infimum. Show: X is a complete lattice.
d) Let $(X, \leq)$ be a partially ordered set in which every nonempty bounded below subset has an infimum. Show: X is a Dedekind complete lattice.
e) Let $(X, \leq)$ be a Dedekind complete lattice. Show: X is a complete lattice iff it has a top element and a bottom element.

8. Intervals and Convergence

Let $(S, \leq)$ be a totally ordered set, and let $x < y$ be elements of S.

We define **intervals**

$$(x, y) = \{z \in S \mid x < z < y\},$$
$$[x, y) = \{z \in S \mid x \leq z < y\},$$
$$(x, y] = \{z \in S \mid x < z \leq y\},$$
$$[x, y] = \{z \in S \mid x \leq z \leq y.$$

These are straightforward generalizations of the notion of intervals on the real line. The intervals (x, y) are called **open**. We also define

$$(-\infty, y) = \{z \in S \mid z < y\},$$
$$(-\infty, y] = \{z \in S \mid z \leq y\},$$
$$(x, \infty) = \{z \in S \mid x < z\},$$
$$[x, \infty) = \{z \in S \mid z \leq z\}.$$

The intervals $(-\infty, y)$ and $[x, \infty)$ are also called **open**. Finally,

$$(-\infty, \infty) = S$$

is also called **open**.

There are some subtleties in these definitions. Depending upon the ordered set S, some intervals which "do not look open" may in fact be open. For instance, suppose that $S = \mathbb{Z}$ with its usual ordering. Then for any $m \leq n \in \mathbb{Z}$, the interval

$[m, n]$ does not look open, but it is also equal to $(m-1, n+1)$, and so it is. For another example, let $S = [0, 1]$ the unit interval in $\mathbb{R}$ with the restricted ordering. Then by definition $(-\infty, \frac{1}{2})$ is an open interval, which by definition is the same set as $[0, \frac{1}{2})$. The following exercise nails down this phenomenon in the general case.

EXERCISE 0.13. *Let $(S, \leq)$ be a totally ordered set.*
a) If S has a minimum element m, then for any $x > m$, the interval $[m, x)$ is equal to $(-\infty, x)$ and is thus open.
b) If S has a maximum element M, then for any $x < M$, the interval $(x, M]$ is equal to (x, ∞) and is thus open.

Let $x_n : \mathbb{Z}^+ \to X$ be a sequence with values in a totally ordered set $(X, \leq)$. We say that the sequence **converges** to $x \in X$ if for every open interval I containing x, there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $x_n \in I$.

9. Fields

9.1. The Field Axioms.

A **field** is a set F endowed with two binary operations $+$ and $\cdot$ satisfying the following **field axioms**:
(F1) $+$ is commutative, associative, has an identity element – called 0 – and every element of F has an inverse with respect to $+$.
(F2) $\cdot$ is commutative, associative, has an identity element – called 1 – and every $x \neq 0$ has an inverse with respect to $\cdot$.
(F3) (Distributive Law) For all $x, y, z \in F$, $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$.
(F4) $0 \neq 1$.

Thus any field has at least two elements, 0 and 1.

EXAMPLE 0.20. *The rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are all fields under the familiar binary operations of $+$ and $\cdot$. We treat these as known facts: that is, it is not our ambition here to* construct *any of these fields from simpler mathematical structures (although this can be done and the reader may want to see it someday).*

It is common to abbreviate $x \cdot y$ to just xy, and we shall feel free to do so here.

Remark: The definition of a field consists of three structures: the set F, the binary operation $+ : F \times F \to F$ and the binary operation $\cdot : F \times F \to F$. Thus we should speak of "the field $(F, +, \cdot)$". But this is tedious and is in fact not done in practice: one speaks e.g. of "the field $\mathbb{Q}$", and the specific binary operations $+$ and $\cdot$ are supposed to be understood. This sort of practice in mathematics is so common that it has its own name: **abuse of notation**.

EXAMPLE 0.21. *There is a field F with precisely two elements 0 and 1. Indeed, the axioms completely determine the "addition and multiplication tables" of such an F: we must have $0 + 0 = 0$, $0 + 1 = 1$, $0 \cdot 0 = 0$, $0 \cdot 1 = 0$, $1 \cdot 1 = 1$. Keeping in mind the commutativity of $+$ and $\cdot$, the only missing entry from these "tables" is $1 + 1$: what is that? Well, there are only two possibilities: $1 + 1 = 0$ or $1 + 1 = 1$. But the second possibility is not tenable: letting z be the additive inverse of 1 and adding it both sides of $1 + 1 = 1$ cancels the 1 and gives $1 = 0$, in violation of (F4).*

*Thus $1 + 1 = 0$ is the only possibility. Converesely, it is easy to check that these addition and multiplication tables do indeed make $F = \{0, 1\}$ into a field. This is in fact quite a famous and important example of a field, denoted either $\mathbb{F}_2$ or $\mathbb{Z}/2\mathbb{Z}$ and called the **binary field** or the **field of integers modulo 2**. It is important in set theory, logic and computer science, due for instance to the interpretation of $+$ as "OR" and $\cdot$ as "AND".*

EXAMPLE 0.22. *Let $N \geq 2$ be an integer. Let $\mathbb{Z}/N\mathbb{Z}$ be the set $\{0, 1, \ldots, N-1\}$, and consider the following binary operations on $\mathbb{Z}/N\mathbb{Z}$: for $x, y \in \mathbb{Z}/N\mathbb{Z}$, $x + y$ (resp. xy) is defined by first taking the usual sum $x + y \in \mathbb{Z}$ (resp. product $xy \in \mathbb{Z}$) and then taking the* remainder *upon division by N. This generalizes the previous example, which took $N = 2$.*

It is not difficult to show that the set $\mathbb{Z}/N\mathbb{Z}$ endowed with these two binary operations satisfies axioms (F1), (F3) and (F4). Moreover it satisfies most of *(F2): the multiplication is commutative, associative, has an identity element – the integer 1 – however it need not be the case that every nonzero element has a multiplicative inverse. For example, if $N = 4$, then the element 2 has no multiplicative inverse: $2 \cdot 1 = 2 \cdot 3 = 2$ and $2 \cdot 0 = 2 \cdot 2 = 0$. More generally, if N is not a prime number, then there are integers $1 < a, b < N$ such that $ab = N$, and then then the element a has no multiplicative inverse: if $xa = 1$, then*

$$0 = x(0) = x(ab) = (xa)b = 1(b) = b.$$

*It turns out that when $N = p$ is a prime number, then every nonzero element has a multiplicative inverse, so $\mathbb{Z}/p\mathbb{Z}$ is a field, often denoted $\mathbb{F}_p$. This uses a nontrivial result from number theory, **Euclid's Lemma**, and as we do not need finite fields for anything in these notes, we will leave things there.*

9.2. Some simple consequences of the field axioms.

There are many, many different fields. Nevertheless the field axioms are rich enough to imply certain consequences for all fields.

PROPOSITION 0.23. *Let F be any field. For all $x \in F$, we have $0 \cdot x = 0$.*

PROOF. For all $x \in F$, $0 \cdot x = (0 + 0) \cdot x = (0 \cdot x) + (0 \cdot x)$. Let y be the additive inverse of $0 \cdot x$. Adding y to both sides has the effect of "cancelling" the $0 \cdot x$ terms:

$$\begin{aligned} 0 &= y + (0 \cdot x) = y + ((0 \cdot x) + (0 \cdot x)) \\ &= (y + (0 \cdot x)) + (0 \cdot x) = 0 + (0 \cdot x) = 0 \cdot x. \end{aligned}$$

□

PROPOSITION 0.24. *Let $(F, +, \cdot)$ be a set with two binary operations satisfying axioms (F1), (F2) and (F3) but not (F4): that is, we assume the first three axioms and also that $1 = 0$. Then $F = \{0\} = \{1\}$ consists of a single element.*

PROOF. Let x be any element of F. Then

$$x = 1 \cdot x = 0 \cdot x = 0.$$

□

Proposition 0.23 is a very simple, and familiar, instance of a relation between the additive structure and the multiplicative structure of a field. We want to further explore such relations and establish that certain familiar arithmetic properties hold in an arbitrary field.

To help out in this regard, we define the element -1 of a field F to be the additive inverse, i.e., the unique element of F such that $1 + (-1) = 0$. (Note that although -1 is denoted differently from 1, it need not actually be a distinct element: indeed in the binary field $\mathbb{F}_2$ we have $1 + 1 = 0$ and thus $-1 = 1$.)

PROPOSITION 0.25. *Let F be a field and let -1 be the additive inverse of 1. Then for any element $x \in F$, $(-1) \cdot x$ is the additive inverse of x.*

PROOF.

$$x + ((-1) \cdot x) = 1 \cdot x + (-1) \cdot x = (1 + (-1)) \cdot x = 0 \cdot x = 0. \qquad \square$$

In light of Proposition 0.25 we create an abbreviated notation: for any $x \in F$, we abbreviate $(-1) \cdot x$ to $-x$. Thus $-x$ is the additive inverse of x.

PROPOSITION 0.26. *For any $x \in F$, we have $-(-x) = x$.*

PROOF. We prove this in two ways. First, from the definition of additive inverses, suppose y is the additive inverse of x and z is the additive inverse of y. Then, what we are trying to show is that $z = x$. But indeed $x + y = 0$ and since additive inverses are unique, this means x is the additive inverse of y.

A second proof comes from identifying $-x$ with $(-1) \cdot x$: for this it will suffice to show that $(-1) \cdot (-1) = 1$. But

$$(-1) \cdot (-1) - 1 = (-1) \cdot (-1) + (1) \cdot (-1) = (-1 + 1) \cdot (-1) = 0 \cdot (-1) = 0.$$

Adding one to both sides gives the desired result. $\square$

EXERCISE 0.14. *Apply Proposition 0.26 to show that for all x, y in a field F:*
a) $(-x)(y) = x(-y) = -(xy)$.
b) $(-x)(-y) = xy$.

For a nonzero $x \in F$ our axioms give us a (necessarily unique) multiplicative inverse, which we denote – as usual – by x^{-1}.

PROPOSITION 0.27. *For all $x \in F \setminus \{0\}$, we have $(x^{-1})^{-1} = x$.*

EXERCISE 0.15. *Prove Proposition 0.27.*

PROPOSITION 0.28. *For $x, y \in F$, we have $xy = 0 \iff (x = 0)$ or $(y = 0)$.*

PROOF. If $x = 0$, then as we have seen $xy = 0 \cdot y = 0$. Similarly if $y = 0$, of course. Conversely, suppose $xy = 0$ and $x \neq 0$. Multiplying by x^{-1} gives $y = x^{-1}0 = 0$. $\square$

Let F be a field, $x \in F$, and let n be any positive integer. Then – despite the fact that we are *not* assuming that our field F contains a copy of the ordinary integers $\mathbb{Z}$ – we can make sense out of $n \cdot x$: this is the element $x + \ldots + x$ (n times).

EXERCISE 0.16. *Let x be an element of a field F.*
a) Give a reasonable definition of $0 \cdot x$, where 0 is the integer 0.
b) Give a reasonable definition of $n \cdot x$, where n is a negative integer.

Thus we can, in a reasonable sense, multiply any element of a field by any positive integer (in fact, by Exercise 0.16, by any integer). We should however beware not to carry over any false expectations about this operation. For instance, if F is the binary field $\mathbb{F}_2$, then we have $2 \cdot 0 = 0 + 0 = 0$ and $2 \cdot 1 = 1 + 1 = 0$: that is, for all $x \in \mathbb{F}_2$, $2x = 0$. It follows from this for all $x \in \mathbb{F}_2$, $4x = 6x = \ldots = 0$. This example should serve to motivate the following definition.

A field F has **finite characteristic** if there exists a positive integer n such that for all $x \in F$, $nx = 0$. If this is the case, the **characteristic** of F is the least positive integer n such that $nx = 0$ for all $x \in F$. A field which does not have finite characteristic is said to have **characteristic zero**.[2]

EXERCISE 0.17. *Show that the characteristic of a field is either* 0 *or a prime number. (Hint: suppose for instance that* $6x = 0$ *for all* $x \in F$*. In particular* $(2 \cdot 1)(3 \cdot 1) = 6 \cdot 1 = 0$*. It follows that either* $2 \cdot 1 = 0$ *or* $3 \cdot 1 = 0$.

EXAMPLE 0.29. *a) The binary field* $\mathbb{F}_2$ *has characteristic* 2*. More generally, for any prime number* p*, the integers modulo* p *form a field which has characteristic* p*. (These examples are perhaps misleading: for every prime number* p*, there are fields of characteristic* p *which are finite but contain more than* p *elements as well as fields of characteristic* p *which are infinite. However, as we shall soon see, as long as we are studying calculus / sequences and series / real analysis, we will not meet fields of positive characteristic.)*
b) The fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ *have characteristic zero.*

PROPOSITION 0.30. *Let* F *be a field of characteristic* 0*. Then* F *contains the rational numbers* $\mathbb{Q}$ *as a subfield.*

PROOF. Since F has characteristic zero, for all $n \in \mathbb{Z}^+$, $n \cdot 1 \neq 0$ in F. We claim that this implies that in fact for all pairs $m \neq n$ of distinct integers, $m \cdot 1 \neq n \cdot 1$ as elements of F. Indeed, we may assume that $m > n$ (otherwise interchange m and n) and thus $m \cdot 1 - n \cdot 1 = (m - n) \cdot 1 \neq 0$. Thus the assignment $n \mapsto n \cdot 1$ gives us a copy of the ordinary integers $\mathbb{Z}$ inside our field F. We may thus simply write n for $n \cdot 1$. Having done so, notice that for all $n \in \mathbb{Z}^+$, $0 \neq n \in F$, so $\frac{1}{n} \in F$, and then for any $m \in \mathbb{Z}$, $m\frac{1}{n} = \frac{m}{n} \in F$. Thus F contains a copy of $\mathbb{Q}$ *in a canonical way*, i.e., via a construction which works the same way for any field of characteristic zero. $\square$

10. Ordered Fields

An **ordered field** $(F, +, \cdot, <)$ is a set F equipped with the structure $(+, \cdot)$ of a field and also with a total ordering $<$, satisfying the following compatbility conditions:

(OF1) For all $x_1, x_2, y_1, y_2 \in F$, $x_1 < x_2$ and $y_1 < y_2$ implies $x_1 + y_1 < x_2 + y_2$.
(OF2) For all $x, y \in F$, $x > 0$ and $y > 0$ implies $xy > 0$.

The rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ are familiar examples of ordered fields. In fact the real numbers are a special kind of ordered field, the unique ordered field satisfying any one of a number of fundamental properties which are critical in the study of calculus and analysis. One of the main goals of this section is to understand these special properties of $\mathbb{R}$ and their interrelationships.

PROPOSITION 0.31. *For elements* x, y *in an ordered field* F*, if* $x < y$*, then* $-y < -x$.

[2]Perhaps you were expecting "infinite characteristic". This would indeed be reasonable. But the terminology is traditional and well-established, and we would not be doing you a favor by changing it here.

Proof. We rule out the other two possibilities: if $-y = -x$, then multiplying by -1 gives $x = y$, a contradiction. If $-x < -y$, then adding this to $x < y$ using (OF1) gives $0 < 0$, a contradiction. □

Proposition 0.32. *Let x, y be elements of an ordered field F.*
a) If $x > 0$ and $y < 0$ then $xy < 0$.
b) If $x < 0$ and $y < 0$ then $xy > 0$.

Proof. a) By Proposition 0.31, $y < 0$ implies $-y > 0$ and thus $x(-y) = (-xy) > 0$. Applying 0.31 again, we get $xy < 0$.
b) If $x < 0$ and $y < 0$, then $-x > 0$ and $-y > 0$ so $xy = (-x)(-y) > 0$. □

Proposition 0.33. *Let x_1, x_2, y_1, y_2 be elements of an ordered field F.*
a) (OF1′) If $x_1 < x_2$ and $y_1 \leq y_2$, then $x_1 + y_1 < x_2 + y_2$.
b) (OF1″) If $x_1 \leq x_2$ and $y_1 \leq y_2$, then $x_1 + y_1 \leq x_2 + y_2$.

Proof. a) If $y_1 < y_2$, then this is precisely (OF1), so we may assume that $y_1 = y_2$. We cannot have $x_1 + y_1 = x_2 + y_2$ for then adding $-y_1 = -y_2$ to both sides gives $x_1 = x_2$, contradiction. Similarly, if $x_2 + y_2 < x_1 + y_1$, then by Proposition 0.31) we have $-y_2 < -y_1$ and adding these two inequalities gives $x_2 < x_1$, a contradiction.
b) The case where both inequalities are strict is (OF1). The case where exactly one inequality is strict follows from part a). The case where we have equality both times is trivial:if $x_1 = x_2$ and $y_1 = y_2$, then $x_1 + y_1 = x_2 + y_2$, so $x_1 + y_1 \leq x_2 + y_2$. □

Remark: What we have shown is that given any field $(F, +, \cdot)$ with a total ordering $<$, the axiom (OF1) implies (OF1′) and also (OF1″). In fact (OF1′) implies (OF1) and (OF1″) and that (OF1″) implies (OF1)and (OF1′), so all three are equivalent.

Proposition 0.34. *Let x, y be elements of an ordered field F.*
(OF2′) If $x \geq 0$ and $y \geq 0$ then $xy \geq 0$.

Exercise 0.18. *Prove Proposition 0.34.*

Proposition 0.35. *Let F be an ordered field and x a nonzero element of F. Then exactly one of the following holds:*
(i) $x > 0$.
(ii) $-x > 0$.

Proof. First we show that (i) and (ii) cannot both hold. Indeed, if so, then using (OF1) to add the inequalities gives $0 > 0$, a contradiction. (Or apply Proposition 0.31: $x > 0$ implies $-x < 0$.) Next suppose neither holds: since $x \neq 0$ and thus $-x \neq 0$, we get $0 < x$ and $0 < -x$. Adding these gives $0 < 0$, a contradiction. □

Proposition 0.36. *Let $0 < x < y$ in an ordered field F. Then $0 < \frac{1}{y} < \frac{1}{x}$.*

Proof. First observe that for any $z > 0$ in F, $\frac{1}{z} > 0$. Indeed, it is certainly not zero, and if it were negative then $1 = z \cdot \frac{1}{z}$ would be negative, by Proposition 0.32. Moreover, for $x \neq y$, we have $\frac{1}{x} - \frac{1}{y} = (y - x) \cdot (xy)^{-1}$. Since $x < y$, $y - x > 0$; since $x, y > 0$, $xy > 0$ and thus $(xy)^{-1}$, so $\frac{1}{x} - \frac{1}{y} > 0$: equivalently, $\frac{1}{y} < \frac{1}{x}$. □

Proposition 0.37. *Let F be an ordered field.*
a) We have $1 > 0$ and hence $-1 < 0$. b) For any $x \in F$, $x^2 \geq 0$.
c) For any $x_1, \ldots, x_n \in F \setminus \{0\}$, $x_1^2 + \ldots + x_n^2 > 0$.

PROOF. a) By definition of a field, $1 \neq 0$, hence by Proposition 0.35 we have either $1 > 0$ or $-1 > 0$. In the former case we're done, and in the latter case (OF2) gives $1 = (-1)(-1) > 0$. that $-1 < 0$ follows from 0.31.
b) If $x \geq 0$, then $x^2 \geq 0$. Otherwise $x < 0$, so by Proposition 0.31, $-x > 0$ and thus $x^2 = (-x)(-x) > 0$.
c) This follows easily from part b) and is left as an exercise for the reader. □

Proposition 0.37 can be used to show that certains field *cannot be ordered*, i.e., cannot be endowed with a total ordering satisfying (OF1) and (OF2). First of all the field $\mathbb{C}$ of complex numbers cannot be ordered, because $\mathbb{C}$ contains an element i such that $i^2 = -1$, whereas by Proposition 0.37b) the square of any element in an ordered field is non-negative and by Proposition 0.37a) -1 is negative.

The binary field $\mathbb{F}_2$ of two elements cannot be ordered: because $0 = 1 + 1 = 1^2 + 1^2$ contradicts Proposition 0.37c). Similarly, the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of integers modulo p cannot be ordered, since $0 = 1 + 1 + \ldots + 1 = 1^2 + \ldots + 1^2$ (each sum having p terms). This generalizes as follows.

COROLLARY 0.38. *An ordered field F must have characteristic* 0.

PROOF. Seeking a contradiction, suppose there is $n \in \mathbb{Z}^+$ such that $n \cdot x = 0$ for all $x \in F$. Then $1^2 + \ldots + 1^2 = 0$, contradicting Proposition 0.37c). □

Combining Proposition 0.30 and Corollary 0.38, we see that every ordered field contains a copy of the rational numbers $\mathbb{Q}$.

11. Archimedean Ordered Fields

Recall that every ordered field $(F, +, \cdot, <)$ has contained inside it a copy of the rational numbers $\mathbb{Q}$ with the usual ordering. An ordered field F is **Archimedean** if the subset $\mathbb{Q}$ is *not* bounded above and **non-Archimedean** otherwise.

EXAMPLE 0.39. *The ordered field $\mathbb{Q}$ is Archimedean. From our intuitive understanding of $\mathbb{R}$ – e.g., by considerations involving the decimal expansion – it is clear that $\mathbb{R}$ is an Archimedean field. (Later we will assert a stronger axiom satisfied by $\mathbb{R}$ from which the Archimedean property follows.)*

Non-archimedean ordered fields do exist. Indeed, in a sense we will not try to make precise here, "most" ordered fields are non-Archimedean.

Suppose F is a non-Archimedean ordered field, and let M be an upper bound for $\mathbb{Q}$. Then one says that M is **infinitely large**. Dually, if m is a positive element of F such that $m < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$, one says that m is **infinitely small**.

EXERCISE 0.19. *Let F be an ordered field, and M an element of F.*
a) Show that M is infinitely large iff $\frac{1}{M}$ is infinitely small.
b) Conclude that F is Archimedean iff it does not have infinitely large elements iff it does not have infinitely small elements.

CHAPTER 1

Real Sequences

1. Least Upper Bounds

PROPOSITION 1.1. *Let F be an Archimedean ordered field, $\{a_n\}$ a sequence in F, and $L \in F$. Then $a_n \to L$ iff for all $m \in \mathbb{Z}^+$ there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $|a_n - L| < \frac{1}{m}$.*

PROOF. Let $\epsilon > 0$. By the Archimedean property, there exists $m > \frac{1}{\epsilon}$; by Proposition 0.32 this implies $0 < \frac{1}{m} < \epsilon$. Therefore if N is as in the statement of the proposition, for all $n \geq N$,

$$|a_n - L| < \frac{1}{m} < \epsilon.$$

□

EXERCISE 1.1. *Show that for an ordered field F the following are equivalent:*
(i) $\lim_{n\to\infty} n = \infty$.
(ii) F is Archimedean.

PROPOSITION 1.2. *For an ordered field F, the following are equivalent:*
*(i) $\mathbb{Q}$ is **dense** in F: for every $a < b \in F$, there exists $c \in \mathbb{Q}$ with $a < c < b$.*
(ii) The ordering on F is Archimedean.

PROOF. (i) $\implies$ (ii): we show the contrapositive. Suppose the ordering on F is non-Archimedean, and let M be an infinitely large element. Take $a = M$ and $b = M + 1$: there is no rational number in between them.
(ii) $\implies$ (i): We will show that for $a, b \in F$ with $0 < a < b$, there exists $x \in \mathbb{Q}$ with $a < x < b$: the other two cases are $a < 0 < b$ – which reduces immediately to this case – and $a < b < 0$, which is handled very similarly.

Because of the nonexistence of infinitesimal elements, there exist $x_1, x_2 \in \mathbb{Q}$ with $0 < x_1 < a$ and $0 < x_2 < b - a$. Thus $0 < x_1 + x_2 < b$. Therefore the set $S = \{n \in \mathbb{Z}^+ \mid x_1 + nx_2 < b\}$ is nonempty. By the Archimedean property S is finite, so let N be the largest element of S. Thus $x_1 + Nx_2 < b$. Moreover we must have $a < x_1 + Nx_2$, for if $x_1 + Nx_2 \leq a$, then $x_1 + (N+1)x_2 = (x_1 + Nx_2) + x_2 < a + (b - a) = b$, contradicting the definition of N. □

Let $(X, \leq)$ be an ordered set. Earlier we said that X satisfies the **least upper bound axiom** (LUB) if every nonempty subset $S \subset F$ which is bounded above has a least upper bound, or supremum. Similarly, X satisfies the **greatest lower bound axiom** (GLB) if every nonempty subset $S \subset F$ which is bounded below has a greatest lower bound, or infimum. We also showed the following basic result.

PROPOSITION 1.3. *An ordered set $(X, \leq)$ satisfies the least upper bound axiom if and only if it satisfies the greatest lower bound axiom.*

The reader may find all of this business about (LUB)/(GLB) in ordered fields quite abstract. And so it is. But our study of the foundations of real analysis takes a great leap forward when we innocently inquire which ordered fields $(F, +, \cdot, <)$ satisfy the least upper bound axiom. Indeed, we have the following assertion.

FACT 1. *The real numbers $\mathbb{R}$ satsify (LUB): every nonempty set of real numbers which is bounded above has a least upper bound.*

Note that we said "fact" rather than "theorem". The point is that stating this result as a theorem cannot, strictly speaking, be meaningful until we give a precise *definition* of the real numbers $\mathbb{R}$ as an ordered field. This will come later, butf what is remarkable is that we can actually go about our business *without needing to know* a precise definition (or construction) of $\mathbb{R}$.

(It is not my intention to be unduly mysterious, so let me say now – for whatever it is worth – that the reason for this is that there is, in the strongest reasonable sense, exactly one Dedekind complete ordered field. Thus, whatever we can say about any such field is really being said about $\mathbb{R}$.)

2. Monotone Sequences

A sequence $\{x_n\}$ with values in an ordered set $(X, \leq)$ is **increasing** if for all $n \in \mathbb{Z}^+$, $x_n \leq x_{n+1}$. It is **strictly increasing** if for all $n \in \mathbb{Z}^+$, $x_n < x_{n+1}$. It is **decreasing** if for all $n \in \mathbb{Z}^+$, $x_n \geq x_{n+1}$. It is **strictly decreasing** if for all $n \in \mathbb{Z}^+$, $x_n < x_{n+1}$. A sequence is **monotone** if it is either increasing or decreasing.

EXAMPLE 1.4. *A constant sequence $x_n = C$ is both increasing and decreasing. Conversely, a sequence which is both increasing and decreasing must be constant.*

EXAMPLE 1.5. *The sequence $x_n = n^2$ is strictly increasing. Indeed, for all $n \in \mathbb{Z}^+$, $x_{n+1} - x_n = (n+1)^2 - n^2 = 2n + 1 > 0$. The sequence F $x_n = -n^2$ is strictly decreasing: for all $n \in \mathbb{Z}^+$, $x_{n+1} - x_n = -(n+1)^2 + n^2 = -(2n+1) < 0$.*

EXERCISE 1.2. *Let $\{x_n\}$ be an increasing (resp. strictly increasing) sequence in an ordered field. Show: $\{-x_n\}$ is decreasing (resp. strictly decreasing).*

We claim that an increasing sequence is the *discrete analogue* of an increasing function in calculus. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is increasing if for all $x_1 < x_2$, $f(x_1) \leq f(x_2)$, is strictly increasing if for all $x_1 < x_2$, $f(x_1) < f(x_2)$, and similarly for decreasing and strictly decreasing functions. Looking carefully, we notice that this does not line up preicsely with the definition of increasing. But we can resolve the situation happily.

PROPOSITION 1.6. *Let $\{x_n\}$ be a sequence with values in an ordered field. Then $\{x_n\}$ is increasing if and only if: for all $m, n \in \mathbb{Z}^+$, if $m \leq n$ then $x_m \leq x_n$.*

PROOF. Suppose $\{x_n\}$ is increasing. Let $m \in \mathbb{Z}^+$. Then for all $k \in \mathbb{Z}^+$ we have

$$x_{m+k} - x_m = (x_{m+k} - x_{m+k-1}) + (x_{m+k-1} - x_{m+k-2}) + \ldots + (x_{m+1} - x_m).$$

Since $\{x_n\}$ is increasing, each term on the right is non-negative, so $x_{m+k} - x_m$ is non-negative. Taking $n = m + k$ this gives $x_m \leq x_n$ for all $n > m$ (and the case $n = m$ is trivial).

The converse is immediate: if we suppose that for all $m \leq n \in \mathbb{Z}^+$ we have $x_m \leq x_n$, then taking $n = m + 1$ shows that $\{x_n\}$ is increasing. $\square$

EXERCISE 1.3. *Show that Proposition 1.6 holds verbatim for sequences in any ordered set. (The point here is not to make any use of the field operations.) Suggestion: let $\mathcal{S}$ be the set of positive integers k such that for all $m \in \mathbb{Z}^+$, we have $x_m \leq x_{m+k}$. Use induction to show $\mathcal{S} = \mathbb{Z}^+$.*

EXERCISE 1.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.*
a) Recall from calculus (or, if it is more to your taste, prove using the Mean Value Theorem) that the following are equivalent:
(i) For all $x, y \in \mathbb{R}$, if $x \leq y$ then $f(x) \leq f(y)$.
(ii) For all $x \in \mathbb{R}$ we have $f'(x) \geq 0$.
b) I claim that Proposition 1.6 is the discrete analogue of part a). Do you see why?

EXAMPLE 1.7. *Let us show that the sequence $\{\frac{\log n}{n}\}_{n=3}^{\infty}$ is strictly decreasing. Consider the function $f : [1, \infty) \to \mathbb{R}$ given by $f(x) = \frac{\log x}{x}$. Then*

$$f'(x) = \frac{x \cdot \frac{1}{x} - \log x \cdot 1}{x^2} = \frac{1 - \log x}{x^2}.$$

Thus $f'(x) < 0 \iff 1 - \log x < 0 \iff \log x > 1 \iff x > e$. Thus f is strictly decreasing on $[e, \infty)$ so $\{x_n\}_{n=3}^{\infty} = \{f(n)\}_{n=3}^{\infty}$ is strictly decreasing.

This example illustrates something else: many naturally occurring sequences are not increasing or decreasing for all values of n, but are *eventually* increaasing/decreasing: that is, they become increasing or decreasing upon omitting some finite number of initial terms of the sequence. Since we are mostly interested in the limiting behavior of sequences here, being eventually monotone is just as good as being monotone.

Speaking of limits of monotone sequences...let's. It turns out that such sequences are especially well-behaved, as catalogued in the following simple result.

THEOREM 1.8. *Let $\{x_n\}_{n=1}^{\infty}$ be an increasing sequence in an ordered field F.*
a) If $x_n \to L$, then L is the least upper bound of the term set $X = \{x_n : n \in \mathbb{Z}^+\}$.
b) Conversely, if the term set $X = \{x_n : n \in \mathbb{Z}^+\}$ has a least upper bound $L \in F$, then $x_n \to L$.
c) Suppose that some subsequence x_{n_k} converges to L. Then also the original sequence x_n converges to L.

PROOF. a) First we claim that $L = \lim_{n\to\infty} x_n$ is an upper bound for the term set X. Indeed, assume not: then there exists $N \in \mathbb{Z}^+$ such that $L < x_N$. But since the sequence is increasing, this implies that for all $n \geq N$, $L < x_N \leq x_n$. Thus if we take $\epsilon = x_N - L$, then for no $n \geq N$ do we have $|x_n - L| < \epsilon$, contradicting our assumption that $x_n \to L$. Second we claim that L is the least upper bound, and the argument for this is quite similar: suppose not, i.e., that there is L' such that for all $n \in \mathbb{Z}^+$, $x_n \leq L' < L$. Let $\epsilon = L - L'$. Then for no n do we have $|x_n - L| < \epsilon$, contradicting our asumption that x_n converges to L.
b) Let $\epsilon > 0$. We need to show that for all but finitely many $n \in \mathbb{Z}^+$ we have $-\epsilon < L - x_n < \epsilon$. Since L is the least upper bound of X, in particular $L \geq x_n$ for all $n \in \mathbb{Z}^+$, so $L - x_n \geq 0 > -\epsilon$. Next suppose that there are infinitely many terms x_n with $L - x_n \geq \epsilon$, or $L \geq x_n + \epsilon$. But if this inequality holds for ifninitely many terms of the sequence, then because x_n is increasing, it holds for all terms of the sequence, and this implies that $L - \epsilon \geq x_n$ for all n, so that $L - \epsilon$ is a smaller upper bound for X than L, contradiction.

c) By part a) applied to the subsequence $\{x_{n_k}\}$, we have that L is the least upper bound for $\{x_{n_k} : k \in \mathbb{Z}^+\}$. But as above, since the sequence is increasing, the least upper bound for this subsequence is also a least upper bound for the original sequence $\{x_n\}$. Therefore by part b) we have $x_n \to L$. $\square$

EXERCISE 1.5. *State and prove an analogue of Theorem 1.8 for decreasing sequences.*

Theorem 1.8 makes a clear connection between existence of least upper bounds in an ordered field F and limits of monotone sequences in F. In fact, we deduce the following important fact almost immediately.

THEOREM 1.9. *(Monotone Sequence Lemma) Let F be an ordered field satisfying (LUB): every nonempty, bounded above subset has a least upper bound. Then:*
a) Every bounded increasing sequence converges to its least upper bound.
b) Every bounded decreasing sequence converges to its greatest upper bound.

PROOF. a) If $\{x_n\}$ is increasing and bounded above, then the term set $X = \{x_n : n \in \mathbb{Z}^+\}$ is nonempty and bounded above, so by (LUB) it has a least upper bound L. By Theorem 1.8b), $x_n \to L$.
b) We can either reduce this to the analogue of Theorem 1.8 for decreasing sequences (c.f. Exercise X.X) or argue as follows: if $\{x_n\}$ is bounded and increasing, then $\{-x_n\}$ is bounded and increasing, so by part a) $-x_n \to L$, where L is the least upper bound of the term set $-X = \{-x_n : n \in \mathbb{Z}^+$. Since L is the least upper bound of $-X$, $-L$ is the greatest lower bound of X and $x_n \to -L$. $\square$

We can go further by turning the conclusion of Theorem 1.9 into axioms for an ordered field F. Namely, consider the following conditions on F:

(MSA) An increasing sequence in F which is bounded above is convergent.
(MSB) A decreasing sequence in F which is bounded below is convergent.

Comment: The title piece of the distinguished Georgian writer F. O'Connor's[1] last collection of short stories is called *Everything That Rises Must Converge*. Despite the fact that she forgot to mention boundedness, (MSA) inevitably makes me think of her, and thus I like to think of it as the **Flannery O'Connor axiom**. But this says more about me than it does about mathematics.

PROPOSITION 1.10. *An ordered field F satisfies (MSA) iff it satisfies (MSB).*

EXERCISE 1.6. *Prove Proposition 1.10.*

We may refer to either or both of the axioms (MSA), (MSB) as the "Monotone Sequence Lemma", although strictly speaking these are properties of a field, and the Lemma is that these properties are implied by property (LUB).

Here is what we know so far about these axioms for ordered fields: the least upper bound axiom (LUB) is equivalent to the greatest lower bound axiom (GLB); either of these axioms implies (MSA), which is equivalent to (MSB). Symbolically:

(LUB) $\iff$ (GLB) $\implies$ (MSA) $\iff$ (MSB).

[1]Mary Flannery O'Connor, 1925-1964

Our next order of business is to show that in fact (MSA)/(MSB) implies (LUB)/(GLB). For this we need a preliminary result.

PROPOSITION 1.11. *An ordered field satisfying the Monotone Sequence Lemma is Archimedean.*

PROOF. We prove the contrapositive: let F be a non-Archimedean ordered field. Then the sequence $x_n = n$ is increasing and bounded above. Suppose that it were convergent, say to $L \in F$. By Theorem 1.8, L must be the least upper bound of $\mathbb{Z}^+$. But this is absurd: if $n \leq L$ for all $n \in \mathbb{Z}^+$ then $n+1 \leq L$ for all $n \in \mathbb{Z}^+$ and thus $n \leq L-1$ for all $n \in \mathbb{Z}^+$, so $L-1$ is a smaller upper bound for $\mathbb{Z}^+$. □

THEOREM 1.12. *The following properties of an ordered field are equivalent:*
(ii) The greatest lower bound axiom (GLB).
(ii) The least upper bound axiom (LUB).
(iii) Every increasing sequence which is bounded above is convergent (MSA).
(iv) Everu decreasing sequence which is bounded below is convergent (MSB).

PROOF. Since we have already shown (i) $\iff$ (ii) $\implies$ (iii) $\iff$ (iv), it will suffice to show (iv) $\implies$ (i). So assume (MSB) and let $S \subset \mathbb{R}$ be nonempty and bounded above by M_0.
CLAIM For all $n \in \mathbb{Z}^+$, there exists $y_n \in S$ such that for any $x \in S$, $x \leq y_n + \frac{1}{n}$.
Proof of claim: Indeed, first choose any element z_1 of S. If for all $x \in S$, $x \leq z_1 + \frac{1}{n}$, then we may put $y_n = z_1$. Otherwise there exists $z_2 \in S$ with $z_2 > z_1 + \frac{1}{n}$. If for all $x \in S$, $x \leq z_2 + \frac{1}{n}$, then we may put $y_n = z_2$. Otherwise, there exists $z_3 \in S$ with $z_3 > z_2 + \frac{1}{n}$. If this process continues infinitely, we get a sequence with $z_k \geq z_1 + \frac{k-1}{n}$. But by Proposition 1.11, F is Archimedean, so that for sufficiently large k, $z_k > M$, contradiction. Therefore the process musts terminate and we may take $y_n = z_k$ for sufficiently large k.
Now we define a sequence of upper bounds $\{M_n\}_{n=1}^{\infty}$ of S as follows: for all $n \in \mathbb{Z}^+$, $M_n = \min(M_{n-1}, y_n + \frac{1}{n})$. This is a decreasing sequence bounded below by any element of S, so by (MSB) it converges, say to M, and by Theorem 1.8a) M is the greatest lower bound of the set $\{M_n\}$. Moreover M must be the least upper bound of S, since again by the Archimedean nature of the order, for any $m < M$, for sufficiently large n we have $m + \frac{1}{n} < M \leq M_n \leq y_n + \frac{1}{n}$ and thus $m < y_n$. □

We now have four important and equivalent properties of an ordered field: (LUB), (GLB), (MSA) and (MSB). It is time to make a new piece of terminology for an ordered field that satisfies any one, and hence all, of these equivalent properties. Let us say – somewhat mysteriously, but following tradition – that such an ordered field is **Dedekind complete**. As we go on we will find a similar pattern with respect to many of the key theorems in this chapter: (i) in order to prove them we need to use Dedekind completeness, and (ii) in fact assuming that the conclusion of the theorem holds in an ordered field implies that the field is Dedekind complete. In particular, all of these facts hold in $\mathbb{R}$, and what we are trying to say is that many of them hold *only* in $\mathbb{R}$ and in no other ordered field.

3. The Bolzano-Weierstrass Theorem

LEMMA 1.13. *(Rising Sun) Each infinite sequence has a monotone subsequence.*

PROOF. Let us say that $m \in \mathbb{Z}^+$ is a **peak** of the sequence $\{a_n\}$ if for all $n > m$, we have $a_n < a_m$. Suppose first that there are infinitely many peaks. Then any sequence of peaks forms a strictly decreasing subsequence, hence we have found a monotone subsequence. So suppose on the contrary that there are only finitely many peaks, and let $N \in \mathbb{N}$ be such that there are no peaks $n \geq N$. Since $n_1 = N$ is not a peak, there exists $n_2 > n_1$ with $a_{n_2} \geq a_{n_1}$. Similarly, since n_2 is not a peak, there exists $n_3 > n_2$ with $a_{n_3} \geq a_{n_2}$. Continuing in this way we construct an infinite (not necessarily strictly) increasing subsequence $a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots$. Done! $\square$

I learned Lemma 1.13 (with its suggestive name) from Evangelos Kobotis in my first quarter of college at the University of Chicago (1994). It seems that this argument first appeared in a short note of Newman and Parsons [**NP88**].

THEOREM 1.14. *(Bolzano[2]-Weierstrass[3]) Every bounded sequence of real numbers admits a convergent subsequence.*

PROOF. By the Rising Sun Lemma, every real sequence admits a monotone subsequence which, as a subsequence of a bounded sequence, is bounded. By the Monotone Sequence Lemma, every bounded monotone sequence converges. QED! $\square$

Remark: Thinking in terms of ordered fields, we may say that an ordered field F has the **Bolzano-Weierstrass Property** if every bounded sequence in F admits a convergent subsequence. In this more abstract setting, Theorem 1.14 may be rephrased as: a Dedekind complete ordered field has the Bolzano-Weierstrass property. The converse is also true:

THEOREM 1.15. *An ordered field satisfying the Bolzano-Weierstrass property – every bounded sequence admits a convergent subsequence – is Dedekind complete.*

PROOF. One of the equivalent formulations of Dedekind completeness is the Monotone Sequence Lemma. By contraposition it suffices to show that the negation of the Monotone Sequence Lemma implies the existence of a bounded sequence with no convergent subsequence. But this is easy: if the Monotone Sequence Lemma fails, then there exists a bounded increasing sequence $\{x_n\}$ which does not converge. We claim that $\{x_n\}$ admits no convergent subsequence. Indeed, suppose that there exists a subsequence $x_{n_k} \to L$. Then by Theorem 1.8, L is the least upper bound of the subsequence, which must also be the least upper bound of the original sequence. But an increasing sequence converges iff it admits a least upper bound, so this contradicts the divergence of $\{x_n\}$. $\square$

THEOREM 1.16. *(Supplements to Bolzano-Weierstrass)*
a) A real sequence which is unbounded above admits a subsequence diverging to ∞.
b) A real sequence which is unbounded below admits a subsequence diverging to $-\infty$.

PROOF. We will prove part a) and leave the task of adapting the given argument to prove part b) to the reader.

Let $\{x_n\}$ be a real sequence which is unbounded above. Then for every $M \in \mathbb{R}$, there exists at least one n such that $x_n \geq M$. Let n_1 be the least positive integer such that $x_{n_1} > 1$. Let n_2 be the least positive integer such that $x_{n_2} > \max(x_{n_1}, 2)$.

[2]Bernhard Placidus Johann Nepomuk Bolzano, 1781-1848
[3]Karl Theodor Wilhelm Weierstrass, 1815-1897

And so forth: having defined n_k, let n_{k+1} be the least positive integer such that $x_{n_{k+1}} > \max(x_{n_k}, k+1)$. Then $\lim_{k\to\infty} x_{n_k} = \infty$.[4] $\square$

4. The Extended Real Numbers

We define the set of **extended real numbers** to be the usual real number $\mathbb{R}$ together with two elements, ∞ and $-\infty$. We denote the extended real numbers by $[-\infty, \infty]$. They naturally have the structure of an ordered set just by decreeing

$$\forall x \in (-\infty, \infty], -\infty < x,$$

$$\forall x \in [-\infty, \infty), x < \infty.$$

We can also partially extend the addition and multiplication operations to $[-\infty, \infty]$ in a way which is compatible with how we compute limits. Namely, we define

$$\forall x \in (-\infty, \infty],\ x + \infty = \infty,$$

$$\forall x \in [-\infty, \infty),\ x - \infty = -\infty,$$

$$\forall x \in (0, \infty),\ x \cdot \infty = \infty,\ x \cdot (-\infty) = -\infty,$$

$$\forall x \in (-\infty, 0),\ x \cdot \infty = -\infty,\ x \cdot (-\infty) = \infty.$$

However some operations need to remain undefined since the corresponding limits are indeterminate: there is no constant way to define $0 \cdot (\pm\infty)$ or $\infty - \infty$. So certainly $[-\infty, \infty]$ is not a field.

Proposition 1.17. *Let $\{a_n\}$ be a monotone sequence of extended real numbers. Then there exists $L \in [-\infty, \infty]$ such that $a_n \to L$.*

5. Partial Limits

For a real sequence $\{a_n\}$, we say that an extended real number $L \in [-\infty, \infty]$ is a **partial limit** of $\{a_n\}$ if there exists a subsequence a_{n_k} such that $a_{n_k} \to L$.

Lemma 1.18. *Let $\{a_n\}$ be a real sequence. Suppose that L is a partial limit of some subsequence of $\{a_n\}$. Then L is also a partial limit of $\{a_n\}$.*

Exercise 1.7. *Prove Lemma 1.18. (Hint: this comes down to the fact that a subsequence of a subsequence is itself a subsequence.)*

Theorem 1.19. *Let $\{a_n\}$ be a real sequence.*
a) $\{a_n\}$ has at least one partial limit $L \in [-\infty, \infty]$.
b) The sequence $\{a_n\}$ is convergent iff it has exactly one partial limit L and L is finite, *i.e., $L \neq \pm\infty$.*
c) $a_n \to \infty$ iff ∞ is the only partial limit.
d) $a_n \to -\infty$ iff $-\infty$ is the only partial limit.

[4] This is one of these unforunate situations in which the need to spell things out explicitly and avoid a very minor technical pitfall – namely, making that we are not choosing the same term of the sequence more than once – makes the proof twice as long and look significantly more elaborate than it really is. I apologize for that but feel honorbound to present complete, correct proofs in this course.

Proof. a) If $\{a_n\}$ is bounded, then by Bolzano-Weierstrass there is a finite partial limit L. If $\{a_n\}$ is unbounded above, then by Theorem 1.16a) $+\infty$ is a partial limit. It $\{a_n\}$ is unbounded below, then by Theorem 1.16b) $-\infty$ is a partial limit. Every sequence is either bounded, unbounded above or unbounded below (and the last two are not mutually exclusive), so there is always at least one partial limit.
b) Suppose that $L \in \mathbb{R}$ is the unique partial limit of $\{a_n\}$. We wish to show that $a_n \to L$. First observe that $\{a_n\}$ must be bounded above and below, for otherwise it would have an infinite partial limit. So choose $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all n.

Now suppose that a_n *does not converge* to L: then there exists $\epsilon > 0$ such that *it is not the case* that there exists $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. What this means is that there are infinitely many values of n such that $|a_n - L| \geq \epsilon$. Moreover, since $|a_n - L| \geq \epsilon$ means *either* $-M \leq a_n \leq L - \epsilon$ or $L + \epsilon \leq a_n \leq M$, there must in fact be in infinite subset $S \subset \mathbb{N}$ such that either for all $n \in S$ we have $a_n \in [-M, L - \epsilon]$ or for all $n \in S$ we have $a_n \in [L + \epsilon, M]$.

Let us treat the former case. The reader who understands the argument will have no trouble adapting to the latter case. Writing the elements of S in increasing order as $n_1, n_2, \ldots, n_k$, we have shown that there exists a subsequence $\{a_{n_k}\}$ all of whose terms lie in the closed interval $[-M, L - \epsilon]$. Applying Bolzano-Weierstrass to this subsequence, we get a subsubsequence (!) $a_{n_{k_\ell}}$ which converges to some L'. We note right away that a subsubsequence of a_n is also a subsequence of a_n: we still have an infinite subset of $\mathbb{N}$ whose elements are being taken in increasing order. Moreover, since every term of $a_{n_{k_\ell}}$ is bounded above by $L - \epsilon$, its limit L' must satisfy $L' \leq L - \epsilon$. But then $L' \neq L$ so the sequence has a second partial limit L': contradiction.
c) Suppose $a_n \to \infty$. Then also every subsequence diverges to $+\infty$, so $+\infty$ is a partial limit and there are no other partial limits. We will prove the converse via its contrapositive (the inverse): suppose that a_n does not diverge to ∞. Then there exists $M \in \mathbb{R}$ and infinitely many $n \in \mathbb{Z}^+$ such that $a_n \leq M$, and from this it follows that there is a subsequence $\{a_{n_k}\}$ which is bounded above by M. This subsequence does not have $+\infty$ as a partial limit, hence by part a) it has some partial limit $L < \infty$. By Lemma 1.18, L is also a partial limit of the original sequence, so it is not the case that $+\infty$ is the only partial limit of $\{a_n\}$.
d) Left to the reader to prove, by adapting the argument of part c) or otherwise. ☐

Exercise 1.8. *Let $\{x_n\}$ be a real sequence. Suppose that:*
(i) Any two convergent subsequences converge to the same limit.
(ii) $\{x_n\}$ is bounded.
Show that $\{x_n\}$ is convergent. (Suggestion: Combine Theorem 1.19b) with the Bolzano-Weierstrass Theorem.)

Exercise 1.9. *Let $\{x_n\}$ be a real sequence, and let $a \leq b$ be extended real numbers. Suppose that there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $a \leq x_n \leq b$. Show that $\{a_n\}$ has a partial limit L with $a \leq L \leq b$.*

6. The Limit Supremum and Limit Infimum

For a real sequence $\{a_n\}$, let $\mathcal{L}$ be the set of all partial limits of $\{a_n\}$.

We define the **limit supremum** $\overline{L}$ of a real sequence to be $\sup \mathcal{L}$, i.e., the supremum of the set of all partial limits.

THEOREM 1.20. *For any real sequence* $\{a_n\}$, $\overline{L}$ *is a partial limit of the sequence and is thus the largest partial limit.*

PROOF. Case 1: The sequence is unbounded above. Then $+\infty$ is a partial limit, so $\overline{L} = +\infty$ is a partial limit.

Case 2: The sequence diverges to $-\infty$. Then $-\infty$ is the only partial limit and thus $\overline{L} = -\infty$ is the largest partial limit.

Case 3: The sequence is bounded above and does not diverge to $-\infty$. Then it has a finite partial L (it may or may not also have $-\infty$ as a partial limit), so $\overline{L} \in (-\infty, \infty)$. We need to find a subsequence converging to $\overline{L}$.

For each $k \in \mathbb{Z}^+$, $\overline{L} - \frac{1}{k} < \overline{L}$, so there exists a subsequence converging to some $L' > \overline{L} - \frac{1}{k}$. In particular, there exists n_k such that $a_{n_k} > \overline{L} - \frac{1}{k}$. It follows from these inequalities that the subsequence a_{n_k} cannot have any partial limit which is less than $\overline{L}$; moreover, by the definition of $\overline{L} = \sup \mathcal{L}$ the subsequence cannot have any partial limit which is strictly greater than $\overline{L}$: therefore by the process of elimination we must have $a_{n_k} \to \overline{L}$. □

Similarly we define the **limit infimum** $\underline{L}$ of a real sequence to be $\inf \mathcal{L}$, i.e., the infimum of the set of all partial limits. As above, $\underline{L}$ is a partial limit of the sequence, i.e., there exists a subsequence a_{n_k} such that $a_{n_k} \to \underline{L}$.

Here is a very useful characterization of the limit supremum of a sequence $\{a_n\}$ it is the unique extended real number L such that for any $M > L$, $\{n \in \mathbb{Z}^+ \mid a_n \geq M\}$ is finite, and such that for any $m < L$, $\{n \in \mathbb{Z}^+ \mid \ a_n \geq m\}$ is infinite.

EXERCISE 1.10. *a) Prove the above characterization of the limit supremum.*
b) State and prove an analogous characterization of the limit infimum.

PROPOSITION 1.21. *For any real sequence* a_n*, we have*

$$\overline{L} = \lim_{n\to\infty} \sup_{k \geq n} a_k \tag{1}$$

and

$$\underline{L} = \lim_{n\to\infty} \inf_{k \geq n} a_k. \tag{2}$$

Because of these identities it is traditional to write $\limsup a_n$ *in place of* $\overline{L}$ *and* $\liminf a_n$ *in place of* $\underline{L}$*.*

PROOF. As usual, we will prove the statements involving the limit supremum and leave the analogous case of the limit infimum to the reader.

Our first order of business is to show that $\lim_{n\to\infty} \sup_{k\geq n} a_k$ exists as an extended real number. To see this, define $b_n = \sup_{k\geq n} a_k$. The key observation is that $\{b_n\}$ is decreasing. Indeed, when we pass from a set of extended real numbers to a subset, its supremum either stays the same or decreased. Now it follows from Proposition 1.17 that $b_n \to L' \in [-\infty, \infty]$.

Now we will show that $\overline{L} = L'$ using the characterization of the limit supremum stated above. First suppose $M > L'$. Then there exists $n \in \mathbb{Z}^+$ such that

$\sup_{k\geq n} a_k < M$. Thus there are only finitely many terms of the sequence which are at least M, so $M \geq \overline{L}$. It follows that $L' \geq \overline{L}$.

On the other hand, suppose $m < L'$. Then there are infinitely many $n \in \mathbb{Z}^+$ such that $m < a_n$ and hence $m \leq \overline{L}$. It follows that $\overline{L} \leq L'$, and thus $\overline{L} = L'$. $\square$

The merit of these considerations is the following: if a sequence converges, we have a number to describe its limiting behavior, namely its limit L. If a sequence diverges to $\pm\infty$, again we have an "extended real number" that we can use to describe its limiting behavior. But a sequence can be more complicated than this: it may be highly oscillatory and therefore its limiting behavior may be hard to describe. However, to every sequence we have now associated two numbers: the limit infimum $\underline{L}$ and the limit supremum $\overline{L}$, such that

$$-\infty \leq \underline{L} \leq \overline{L} \leq +\infty.$$

For many purposes – e.g. for making upper estimates – we can use the limit supremum $\overline{L}$ in the same way that we would use the limit L if the sequence were convergent (or divergent to $\pm\infty$). Since $\overline{L}$ exists *for any sequence*, this is very powerful and useful. Similarly for $\underline{L}$.

Corollary 1.22. *A real sequence $\{a_n\}$ is convergent iff $\underline{L} = \overline{L} \in (-\infty,\infty)$.*

Exercise 1.11. *Prove Corollary 1.22.*

7. Cauchy Sequences

Let $(F,<)$ be an ordered field. A sequence $\{a_n\}_{n=1}^\infty$ in F is **Cauchy** if for all $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that for all $m,n \geq N$, $|a_m - a_n| < \epsilon$.

Exercise 1.12. *Show that each subsequence of a Cauchy sequence is Cauchy.*

Proposition 1.23. *In any ordered field, a convergent sequence is Cauchy.*

Proof. Suppose $a_n \to L$. Then there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $|a_n - L| < \frac{\epsilon}{2}$. Thus for all $m,n \geq N$ we have

$$|a_n - a_m| = |(a_n - L) - (a_m - L)| \leq |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad \square$$

Proposition 1.24. *In any ordered field, a Cauchy sequence is bounded.*

Proof. Let $\{a_n\}$ be a Cauchy sequence in the ordered field F. There exists $N \in \mathbb{Z}^+$ such that for all $m,n \geq N$, $|a_m - a_n| < 1$. Therefore, taking $m = N$ we get that for all $n \geq N$, $|a_n - a_N| < 1$, so $|a_n| \leq |a_N| + 1 = M_1$, say. Moreover put $M_2 = \max_{1\leq n\leq N} |a_n|$ and $M = \max(M_1, M_2)$. Then for all $n \in \mathbb{Z}^+$, $|a_n| \leq M$. $\square$

Proposition 1.25. *In any ordered field, a Cauchy sequence which admits a convergent subsequence is itself convergent.*

Proof. Let $\{a_n\}$ be a Cauchy sequence in the ordered field F and suppose that there exists a subsequence a_{n_k} converging to $L \in F$. We claim that a_n converges to L. Fix any $\epsilon > 0$. Choose $N_1 \in \mathbb{Z}^+$ such that for all $m,n \geq N_1$ we have $|a_n - a_m| = |a_m - a_n| < \frac{\epsilon}{2}$. Further, choose $N_2 \in \mathbb{Z}^+$ such that for all $k \geq N_2$ we have $|a_{n_k} - L| < \frac{\epsilon}{2}$, and put $N = \max(N_1,N_2)$. Then $n_N \geq N$ and $N \geq N_2$, so

$$|a_n - L| = |(a_n - a_{n_N}) - (a_{n_N} - L)| \leq |a_n - a_{n_N}| + |a_{n_N} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square$$

THEOREM 1.26. *Any real Cauchy sequence is convergent.*

PROOF. Let $\{a_n\}$ be a real Cauchy sequence. By Proposition 1.24, $\{a_n\}$ is bounded. By Bolzano-Weierstrass there exists a convergent subeqeuence. Finally, by Proposition 1.25, this implies that $\{a_n\}$ is convergent. □

THEOREM 1.27. *For an Archimedean ordered field F, the following are equivalent:*
(i) F is ***Dedekind complete****.*
(ii) F is ***sequentially complete****: every Cauchy sequence converges.*

PROOF. The implication (i) $\implies$ (ii) is the content of Theorem 1.26, since the Bolzano-Weierstrass Theorem holds in any ordered field satisfying (LUB).
(ii) $\implies$ (i): Let $S \subset F$ be nonempty and bounded above, and write $\mathcal{U}(S)$ for the set of least upper bounds of S. Our strategy will be to construct a decreasing Cauchy sequence in $\mathcal{U}(S)$ and show that its limit is $\sup S$.

Let $a \in S$ and $b \in \mathcal{U}(S)$. Using the Archimedean property, we choose a negative integer $m < a$ and a positive integer $M > b$, so

$$m < a \leq b \leq M.$$

For each $n \in \mathbb{Z}^+$, we define

$$S_n = \{k \in \mathbb{Z} \mid \frac{k}{2^n} \in \mathcal{U}(A) \text{ and } k \leq 2^n M\}.$$

Every element of S_n lies in the interval $[2^n m, 2^n M]$ and $2^n M \in S_n$, so each S_n is finite and nonempty. Put $k_n = \min S_n$ and $a_n = \frac{k_n}{2^n}$, so $\frac{2k_n}{2^{n+1}} = \frac{k_n}{2^n} \in \mathcal{U}(S)$ wihle $\frac{2k_n-2}{2^{n+1}} = \frac{k_n-1}{2^n} \notin \mathcal{U}(S)$. It follows that we have either $k_{n+1} = 2k_n$ or $k_{n+1} = 2k_n - 1$ and thus either $a_{n+1} = a_n$ or $a_{n+1} = a_n - \frac{1}{2^{n+1}}$. In particular $\{a_n\}$ is decreasing. For all $1 \leq m < n$ we have

$$\begin{aligned} 0 \leq a_m - a_n &= (a_m - a_{m+1}) + (a_{m+1} - a_{m+2}) + \ldots + (a_{n-1} - a_n) \\ &\leq 2^{-(m+1)} + \ldots + 2^{-n} = 2^{-m}. \end{aligned}$$

This shows that $\{a_n\}$ is a Cauchy sequence, hence by our assumption on F $a_n \to L \in F$.

We CLAIM $L = \sup(S)$. Seeking a contradiction we suppose that $L \notin \mathcal{U}(S)$. Then there exists $x \in S$ such that $L < x$, and thus there exists $n \in \mathbb{Z}^+$ such that

$$a_n - L = |a_n - L| < x - L.$$

It follows that $a_n < x$, contradicting $a_n \in \mathcal{U}(S)$. So $L \in \mathcal{U}(S)$. Finally, if there exists $L' \in \mathcal{U}(S)$ with $L' < L$, then (using the Archimedean property) choose $n \in \mathbb{Z}^+$ with $\frac{1}{2^n} < L - L'$, and then

$$a_n - \frac{1}{2^n} \geq L - \frac{1}{2^n} > L',$$

so $a_n - \frac{1}{2^n} = \frac{k_n-1}{2^n} \in \mathcal{U}(S)$, contradicting the minimality of k_n. □

Remark: The proof of (ii) $\implies$ (i) in Theorem 1.27 above is taken from [**HS**] by way of [**Ha11**]. It is rather unexpectedly complicated, but I do not know a simpler proof at this level. However, if one is willing to introduce the notion of convergent and Cauchy **nets**, then one can show first that in an Archimedean ordered field, the convergence of all Cauchy sequences implies the convergence of all Cauchy nets, and second use the hypothesis that all Cauchy nets converge to give a proof which is (in my opinion of course) more conceptually transparent. This is the approach taken in my (more advanced) notes on Field Theory [**FT**].

8. Sequentially Complete Non-Archimedean Ordered Fields

We have just seen that in an *Archimedean* ordered field, the convergence of all Cauchy sequences holds iff the field satisfies (LUB). This makes one wonder what happens if the hypothesis of "Archimedean" is dropped. In fact there are many non-Archimedean (hence not Dedekind complete) fields in which all Cauchy sequences converge. We will attempt to give two very different examples of such fields here. We hasten to add that an actual sophomore/junior undergraduate learning about sequences for the first time should skip past this section and come back to it in several years' time (if ever!).

Example: Let $F = \mathbb{R}((t))$ be the field of **formal Laurent series** with $\mathbb{R}$-coefficients: an element of F is a formal sum $\sum_{n \in \mathbb{Z}} a_n t^n$ where there exists $N \in \mathbb{Z}$ such that $a_n = 0$ for all $n < N$. We add such series term by term and multiply them in the same way that we multiply polynomials. It is not so hard to show that K is actually a field: we skip this.

We need to equip K with an ordering; equivalently, we need to specify a set of positive elements. For every nonzero element $x \in F$, we put $v(x)$ to be the smallest $n \in \mathbb{Z}$ such that $a_n \neq 0$. Then we say that x is positive if the coefficient $a_{v(x)}$ of the smallest nonzero term is a positive real number. It is straightforward to see that the sum and product of positive elements is positive and that for each nonzero $x \in F$, exactly one of x and $-x$ is positive, so this gives an ordering on F in the usual way: we decree that $x < y$ iff $y - x$ is positive.

We observe that this ordering is non-Archimedean. Indeed, the element $\frac{1}{t}$ is positive – its one nonzero coefficient is 1, which is a positive real number – and infinitely large: for any $n \in \mathbb{Z}$, $\frac{1}{t} - n$ is still positive – recall that we look to the smallest degree coefficient to check positivity – so $\frac{1}{t} > n$ for all n.

Next we observe that the set $\{\frac{1}{t^n}\}$ is unbounded in F. Taking reciprocals, it follows that the sequence $\{t^n\}$ converges to 0 in K: explicitly, given any $\epsilon > 0$ – here ϵ is not necessarily a real number but any positive element of F! – for all sufficiently large n we have that $\frac{1}{t^n} > \frac{1}{\epsilon}$, so $|t^n| = t^n < \epsilon$. We will use this fact to give a simple explicit description of all convergent sequences in F. First, realize that a sequence in F consists of, for each $m \in \mathbb{Z}^+$ a formal Laurent series $x_m = \sum_{n \in \mathbb{Z}} a_{m,n} t^n$, so in fact for each $n \in \mathbb{Z}$ we have a real sequence $\{a_{m,n}\}_{m=1}^{\infty}$. Now consider the following conditions on a sequence $\{x_m\}$ in K:

(i) There is an integer N such that for all $m \in \mathbb{Z}^+$ and $n < N$, $a_{m,n} = 0$, and
(ii) For each $n \in \mathbb{Z}$ the sequence $a_{m,n}$ is eventually constant: i.e., for all sufficiently large m, $a_{m,n} = C_n \in \mathbb{R}$. (Because of (i) we must have $C_n = 0$ for all $n < N$.)

Then condition (i) is equivalent to boundedness of the sequence.

I claim that if the sequence converges – say $x_m \to x = \sum_{n=N}^{\infty} a_n t^n \in F$ – then (i) and (ii) both hold. Indeed convergent sequences are bounded, so (i) holds. Then for all $n \geq N$, $a_{m,n}$ is eventually constant in m iff $a_{m,n} - a_n$ is eventually constant in m, so we may consider $x_m - x$ instead of x_m and thus we may assume that $x_m \to 0$ and try to show that for each fixed n, $a_{m,n}$ is eventually equal to 0. As above, this holds iff for all $k \geq 0$, there exists M_k such that for all $m \geq M_k$, $|x_m| \leq t^k$. This latter condition holds iff the coefficient $a_{m,n}$ of t^n in x_n is zero for all $N < k$. Thus, for all $m \geq M_k$, $a_{m,-N} = a_{m,-N+1} = \ldots = a_{m,k-1} = 0$, which is what we wanted to show.

Conversely, suppose (i) and (ii) hold. Then since for all $n \geq N$ the sequence $a_{m,n}$ is eventually constant, we may define a_n to be this eventual value, and an argument very similar to the above shows that $x_m \to x = \sum_{n \geq N} a_n t^n$.

Next I claim that if a sequence $\{x_n\}$ is Cauchy, then it satisfies (i) and (ii) above, hence is convergent. Again (i) is immediate because every Cauchy sequence is bounded. The Cauchy condition here says: for all $k \geq 0$, there exists M_k such that for all $m, m' \geq M_k$ we have $|x_m - x'_m| \leq t^k$, or equivalently, for all $n < k$, $a_{m,n} - a_{m',n} = 0$. In other words this shows that for each fixed $n < k$ and all $m \geq M_k$, the sequence $a_{m,n}$ is constant, so in particular for all $n \geq N$ the sequence $a_{m,n}$ is eventually constant in m, so the sequence x_m converges.

In the above example of a non-Archimedean sequentially complete ordered field, there were plenty of convergent sequences, but they all took a rather simple form that enabled us to show that the condition for convergence was the same as the Cauchy criterion. It is possible for a non-Archimedean field to be sequentially complete in a completely different way: there are non-Archimedean fields for which every Cauchy sequence is eventually constant. Certainly every eventually constant sequence is convergent, so such a field F must be sequentially complete!

In fact a certain property will imply that every Cauchy sequence is eventually constant. It is the following: **every sequence in F is bounded**. This is a "weird" property for an ordered field to have: certainly no Archimedean ordered field has this property, because by definition in an Archimedean field the sequence $x_n = n$ is unbounded. And there are plenty of non-Archimedean fields which *do not* have this property, for instance the field $\mathbb{R}((t))$ discussed above, in which $\{t^{-n}\}$ is an unbounded sequence. Nevertheless there are such fields.

Suppose F is an ordered field in which every sequence is bounded, and let $\{x_n\}$ be a Cauchy sequence in F. Seeking a contradiction, we suppose that $\{x_n\}$ is *not* eventually constant. Then there is a subsequence which takes distinct values, i.e., for all $k \neq k'$, $x_{n_k} \neq x_{n_{k'}}$, and a subsequence of a Cauchy sequence is still a Cauchy sequence. Thus if there is a non-eventually constant Cauchy sequence, there is a noneventually constant Cauchy sequence with distinct terms, so we may assume from the start that $\{x_n\}$ has distinct terms. Now for $n \in \mathbb{Z}^+$, put $z_n = |x_{n+1} - x_n|$, so $z_n > 0$ for all n, hence so is $Z_n = \frac{1}{z_n}$. By our assumption on K, the sequence Z_n is bounded: there exists $\alpha > 0$ such that $Z_n < \alpha$ for all n. Now put $\epsilon = \frac{1}{\alpha}$. Taking reciprocals we find that for all $n \in \mathbb{Z}^+$,

$$|x_{n+1} - x_n| = z_n = \frac{1}{Z_n} > \frac{1}{\alpha} = \epsilon.$$

This shows that the sequence $\{x_n\}$ is not Cauchy and completes the argument.

It remains to construct an ordered field having the property that every sequence is bounded. At the moment the only constructions I know are wildly inappropriate for an undergraduate audience. For instance, one can start with the real numbers $\mathbb{R}$ and let K be an **ultrapower** of $\mathbb{R}$ corresponding to a nonprincipal ultrafilter on the positive integers. Unfortunately even if you happen to know what this means, the proof that such a field K has the desired property that all sequences are bounded uses the saturation properties of ultraproducts...which I myself barely understand. There must be a better way to go: please let me know if you have one.

9. The Stolz-Cesaro Theorem

THEOREM 1.28. *(Stolz-Cesaro) a) Let $\{a_n\}$ and $\{b_n\}$ be real sequences. Suppose that $b_n > 0$ for all $n \in \mathbb{Z}^+$ and that $\lim_{n\to\infty}(b_1 + \ldots + b_n) = \infty$. Then*

$$\liminf \frac{a_n}{b_n} \overset{(A)}{\leq} \liminf \frac{a_1 + \ldots + a_n}{b_1 + \ldots + b_n} \overset{(B)}{\leq} \limsup \frac{a_1 + \ldots + a_n}{b_1 + \ldots + b_n} \overset{(C)}{\leq} \limsup \frac{a_n}{b_n}$$

In particular, if for some $L \in [-\infty, \infty]$ we have $\frac{a_n}{b_n} \to L$, then also $\frac{a_1+\ldots+a_n}{b_1+\ldots+b_n} \to L$.
b) Let $\{a_n\}$ and $\{b_n\}$ be real sequences. Suppose that $\{b_n\}$ is strictly increasing and $\lim_{n\to\infty} b_n = \infty$. Then

$$\liminf \frac{a_n - a_{n-1}}{b_n - b_{n-1}} \leq \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} \leq \limsup \frac{a_n - a_{n-1}}{b_n - b_{n-1}}.$$

PROOF. (G. Nagy) For $n \in \mathbb{Z}^+$ we put $A_n = a_1 + \ldots + a_n$ and $B_n = b_1 + \ldots + b_n$.
a) There are three inequalities here: (A), (B) and (C). Of these, (B) holds for any real sequence. Moreover, (C) holds for the sequence $\{a_n\}$ iff (A) holds for the sequence $\{-a_n\}$, so it suffices to prove (C). Let $L = \limsup \frac{a_n}{b_n}$. If $L = \infty$ there is nothing to show, so suppose $L \in [-\infty, \infty)$. Let $\ell > L$; then there is $N \in \mathbb{N}$ such that for all $n > N$ we have

$$\frac{a_n}{b_n} \leq \ell. \tag{3}$$

From (3) it follows that for all $n > N$ we have

$$A_n \leq A_N + \ell(B_n - B_N)$$

and thus

$$\frac{A_n}{B_n} \leq \ell + \frac{A_N - \ell B_N}{B_n}. \tag{4}$$

In (4), take $n \to \infty$; since $B_n \to \infty$, we get

$$\limsup \frac{a_1 + \ldots + a_n}{b_1 + \ldots + b_n} \leq \ell.$$

Since this holds for all $\ell > L = \limsup \frac{a_n}{b_n}$, we conclude

$$\limsup \frac{a_1 + \ldots + a_n}{b_1 + \ldots + b_n} \leq L = \limsup \frac{a_n}{b_n}.$$

b) For a sequence $\{c_n\}$, put $c_1' = c_1$ and put $c_n' = c_n - c_{n-1}$ for all $n \geq 2$. Then $\{c_n\}$ is strictly increasing iff $c_n' > 0$ for all $n \in \mathbb{Z}^+$; moreover, we have

$$\lim_{n\to\infty}(c_1' + \ldots + c_n') = \lim_{n\to\infty} c_n.$$

So under the hypotheses of part b), we may apply the result of part a) with $\{a_n\}$ replaced by $\{a_n'\}$ and $\{b_n\}$ replaced by $\{b_n'\}$, getting the conclusion of part b). $\square$

CHAPTER 2

Real Series

1. Introduction

1.1. Zeno Comes Alive: a historico-philosophical introduction.

Humankind has had a fascination with, but also a suspicion of, infinite processes for well over two thousand years. Historically, the first kind of infinite process that received detailed infomation was the idea of adding together infinitely many quantitties; or, to put a slightly different emphasis on the same idea, to divide a whole into infinitely many parts.

The idea that any sort of infinite process can lead to a finite answer has been deeply unsettling to philosophers going back at least to Zeno,[1] who believed that a convergent infinite process was absurd. Since he had a sufficiently penetrating eye to see convergent infinite processes all around him, he ended up at the lively conclusion that many everyday phenomena are in fact absurd (so, in his view, illusory).

We will get the flavor of his ideas by considering just one paradox, Zeno's **arrow paradox**. Suppose that an arrow is fired at a target one stadium away. Can the arrow possibly hit the target? Call this event E. Before E can take place, the arrow must arrive at the halfway point to its target: call this event E_1. But before it does *that* it must arrive halfway to the halfway point: call this event E_2. We may continue in this way, getting infinitely many events $E_1, E_2, \ldots$ all of which must happen *before* the event E. That infinitely many things can happen before some predetermined thing Zeno regarded as absurd, and he concluded that the arow never hits its target. Similarly he deduced that all motion is impossible.

Nowadays we have the mathematical tools to retain Zeno's basic insight (that a single interval of finite length can be divided into infinitely many subintervals) without regarding it as distressing or paradoxical. Indeed, assuming for simplicity that the arrow takes one second to hit its target and (rather unrealistically) travels at uniform velocity, we know exactly when these events E_i take place: E_1 takes place after $\frac{1}{2}$ seconds, E_2 takes place after $\frac{1}{4}$ seconds, and so forth: E_n takes place after $\frac{1}{2^n}$ seconds. Nevertheless there is something interesting going on here: we have divided the total time of the trip into infinitely many parts, and the conclusion seems to be that

$$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} + \ldots = 1. \tag{5}$$

[1] Zeno of Elea, ca. 490 BC - ca. 430 BC.

So now we have not a problem not in the philosophical sense but in the mathematical one: what meaning can be given to the left hand side of (5)? Certainly we ought to proceed with some caution in our desire to add infinitely many things together and get a finite number: the expression

$$1 + 1 + \ldots + 1 + \ldots$$

represents an infinite sequence of events, each lasting one second. Surely the aggregate of these events takes forever.

We see then that we dearly need a mathematical definition of an infinite series of numbers and also of its sum. Precisely, if $a_1, a_2, \ldots$ is a sequence of real numbers and S is a real number, we need to give a precise meaning to the equation

$$a_1 + \ldots + a_n + \ldots = S.$$

So here it is. We *do not* try to add everything together all at once. Instead, we form from our sequence $\{a_n\}$ an auxiliary sequence $\{S_n\}$ whose terms represent adding up the fist n numbers. Precisely, for $n \in \mathbb{Z}^+$, we define

$$S_n = a_1 + \ldots + a_n.$$

The associated sequence $\{S_n\}$ is said to be the **sequence of partial sums** of the sequence $\{a_n\}$; when necessary we call $\{a_n\}$ the sequence of **terms**. Finally, we say that the **infinite series** $a_1 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n$ **converges to S** – or **has sum S** – if $\lim_{n \to \infty} S_n = S$ in the familiar sense of limits of seqeunces. If the sequence of partial sums $\{S_n\}$ converges to some number S we say the infinite series is **convergent** (or sometimes **summable**, although this term will not be used here); if the sequence $\{S_n\}$ diverges then the infinite series $\sum_{n=1}^{\infty} a_n$ is **divergent**.

Thus the trick of defining the infinite sum $\sum_{n=1}^{\infty} a_n$ is to do everything in terms of the associated sequence of partial sums $S_n = a_1 + \ldots + a_n$.

In particular by $\sum_{n=1}^{\infty} a_n = \infty$ we mean the sequence of partial sums diverges to ∞, and by $\sum_{n=1}^{\infty} a_n = -\infty$ we mean the sequence of partial sums diverges to $-\infty$. So to spell out the first definition completely, $\sum_{n=1}^{\infty} a_n = \infty$ means: for every $M \in \mathbb{R}$ there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $a_1 + \ldots + a_n \geq M$.

Let us revisit the examples above using the formal definition of convergence.

Example 1: Consider the infinite series $1 + 1 + \ldots + 1 + \ldots$, in which $a_n = 1$ for all n. Then $S_n = a_1 + \ldots + a_n = 1 + \ldots + 1 = n$, and we conclude

$$\sum_{n=1}^{\infty} 1 = \lim_{n \to \infty} n = \infty.$$

Thus this infinite series indeed diverges to infinity.

Example 2: Consider $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} + \ldots$, in which $a_n = \frac{1}{2^n}$ for all n, so

$$S_n = \frac{1}{2} + \ldots + \frac{1}{2^n}. \tag{6}$$

There is a standard trick for evaluating such finite sums. Namely, multiplying (6) by $\frac{1}{2}$ and subtracting it from (6) all but the first and last terms cancel, and we get

$$\frac{1}{2}S_n = S_n - \frac{1}{2}S_n = \frac{1}{2} - \frac{1}{2^{n+1}},$$

and thus

$$S_n = 1 - \frac{1}{2^n}.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty}(1 - \frac{1}{2^n}) = 1.$$

So Zeno was right!

Remark: It is not necessary for the sequence of terms $\{a_n\}$ of an infinite series to start with a_1. In our applications it will be almost as common to consider series starting with a_0. More generally, if N is any integer, then by $\sum_{n=N}^{\infty} a_n$ we mean the sequence of partial sums a_N $, a_N + a_{N+1},$ $a_N + a_{N+1} + a_{N+2}, \ldots$.

1.2. Geometric Series.

Recall that a **geometric sequence** is a sequence $\{a_n\}_{n=0}^{\infty}$ of nonzero real numbers such that the ratio between successive terms $\frac{a_{n+1}}{a_n}$ is equal to some fixed number r, the **geometric ratio**. In other words, if we write $a_0 = A$, then for all $n \in \mathbb{N}$ we have $a_n = Ar^n$. A geometric sequence with geometric ratio r converges to zero if $|r| < 1$, converges to A if $r = 1$ and otherwise diverges.

We now define a **geometric series** to be an infinite series whose terms form a geometric sequence, thus a series of the form $\sum_{n=0}^{\infty} Ar^n$. Geometric series will play a singularly important role in the development of the theory of all infinite series, so we want to study them carefully here. In fact this is quite easily done.

Indeed, for $n \in \mathbb{N}$, put $S_n = a_0 + \ldots + a_n = A + Ar + \ldots + Ar^n$, the nth partial sum. It happens that we can give a closed form expression for S_n, for instance using a technique the reader has probably seen before. Namely, consider what happens when we multiply S_n by the geometric ratio r: it changes, but in a very clean way:

$$S_n = A + Ar + \ldots + Ar^n,$$

$$rS_n \quad = \qquad Ar + \ldots + Ar^n + Ar^{n+1}.$$

Subtracting the two equations, we get

$$(r-1)S_n = A(r^{n+1} - 1)$$

and thus

$$S_n = \sum_{k=0}^{n} Ar^k = A\left(\frac{1 - r^{n+1}}{1-r}\right). \tag{7}$$

Note that the division by $r - 1$ is invalid when $r = 1$, but this is an especially easy case: then we have $S_n = A + A(1) + \ldots + A(1)^n = (n+1)A$, so that $|\lim_{n\to\infty} S_n| = \infty$ and the series diverges. For $r \neq 1$, we see immediately from (7) that $\lim_{n\to\infty} S_n$ exists iff $\lim_{n\to\infty} r^n$ exists iff $|r| < 1$, in which case the latter limit is 0 and thus $S_n \to \frac{A}{1-r}$. We record this simple computation as a theorem.

THEOREM 2.1. *Let A and r be nonzero real numbers, and consider the geometric series $\sum_{n=0}^{\infty} Ar^n$. Then the series converges iff $|r| < 1<$ in which case the sum is $\frac{A}{1-r}$.*

EXERCISE 2.1. *Show: for all $N \in \mathbb{Z}$, $\sum_{n=N}^{\infty} Ar^n = \frac{Ar^N}{1-r}$.*

EXERCISE 2.2. *Many ordinary citizens are uncomfortable with the identity*

$$0.999999999999\ldots = 1.$$

Interpret it as a statement about geometric series, and show that it is correct.

1.3. Telescoping Series.

Example: Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$. We have

$$S_1 = \frac{1}{2},$$
$$S_2 = S_1 + a_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$
$$S_3 = S_2 + a_3 = \frac{2}{3} + \frac{1}{12} = \frac{3}{4},$$
$$S_4 = S_3 + a_4 = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}.$$

It certainly seems as though we have $S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$ for all $n \in \mathbb{Z}^+$. If this is the case, then we have

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

How to prove it?

First Proof: As ever, induction is a powerful tool to prove that an identity holds for all positive integers, *even if we don't really understand why the identity should hold!*. Indeed, we don't even have to fully wake up to give an induction proof: we wish to show that for all $n \in \mathbb{Z}^+$,

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2+k} = \frac{n}{n+1}. \tag{8}$$

Indeed this is true when $n = 1$: both sides equal $\frac{1}{2}$. Now suppose that (8) holds for some $n \in \mathbb{Z}^+$; we wish to show that it also holds for $n+1$. But indeed we may just calculate:

$$S_{n+1} = S_n + \frac{1}{(n+1)^2 + (n+1)} \stackrel{\text{IH}}{=} \frac{n}{n+1} + \frac{1}{n^2+3n+2} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$
$$= \frac{(n+2)n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}.$$

This proves the result.

As above, this is certainly a way to go, and the general technique will work whenever we have some reason to look for and successfully guess a simple closed form identity for S_n. But in fact, as we will see in the coming sections, in practice it is *exceedingly rare* that we are able to express the partial sums S_n in a simple closed form. Trying to do this for each given series would turn out to be a discouraging waste of time. We need some *insight* into why the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ happens to work out so nicely.

Well, if we stare at the induction proof long enough we will eventually notice how convenient it was that the denominator of $\frac{1}{(n+1)^2+(n+1)}$ factors into $(n+1)(n+2)$. Equivalently, we may look at the factorization $\frac{1}{n^2+n} = \frac{1}{(n+1)(n)}$. Does this remind us of anything? I hope so, yes – recall from calculus that every rational function admits a **partial fraction decomposition**. In this case, we know there are constants A and B such that

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}.$$

I leave it to you to confirm – in whatever manner seems best to you – that we have

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

This makes the behavior of the partial sums much more clear! Indeed we have

$$S_1 = 1 - \frac{1}{2}.$$

$$S_2 = S_1 + a_2 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) = 1 - \frac{1}{3}.$$

$$S_3 = S_2 + a_3 = (1 - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) = 1 - \frac{1}{4},$$

and so on. This much simplifies the inductive proof that $S_n = 1 - \frac{1}{n+1}$. In fact induction is not needed: we have that

$$S_n = a_1 + \ldots + a_n = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1},$$

the point being that every term except the first and last is cancelled out by some other term. Thus once again $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$.

Finite sums which cancel in this way are often called **telescoping sums**, I believe after those old-timey collapsible nautical telescopes. In general an infinite sum $\sum_{n=1}^{\infty} a_n$ is telescoping when we can find an auxiliary sequence $\{b_n\}_{n=1}^{\infty}$ such that $a_1 = b_1$ and for all $n \geq 2$, $a_n = b_n - b_{n-1}$, for then for all $n \geq 1$ we have

$$S_n = a_1 + a_2 + \ldots + a_n = b_1 + (b_2 - b_1) + \ldots + (b_n - b_{n-1}) = b_n.$$

But looking at these formulas shows something curious: every infinite series is telescoping: we need only take $b_n = S_n$ for all n! Another, less confusing, way to say this is that if we start with any infinite sequence $\{S_n\}_{n=1}^{\infty}$, then there is a unique sequence $\{a_n\}_{n=1}^{\infty}$ such that S_n is the sequence of partial sums $S_n = a_1 + \ldots + a_n$. Indeed, the key equations here are simply

$$S_1 = a_1,$$

$$\forall n \geq 2,\ S_n - S_{n-1} = a_n,$$

which tells us how to define the a_n's in terms of the S_n's.

In practice all this seems to amount to the following: if you can find a simple closed form expression for the nth partial sum S_n (in which case you are very lucky), then in order to prove it you do not need to do anything so fancy as mathematical induction (or fancier!). Rather, it will suffice to just compute that $S_1 = a_1$

and for all $n \geq 2$, $S_n - S_{n-1} = a_n$. This is the discrete analogue of the fact that if you want to show that $\int f dx = F$ – i.e., you already have a function F which you believe is an antiderivative of f – then you need not use any integration techniques whatsoever but may simply check that $F' = f$.

EXERCISE 2.3. *Let $n \in \mathbb{Z}^+$. We define the* ***nth harmonic number*** $H_n = \sum_{k=1}^n \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}$. *Show that for all $n \geq 2$, $H_n \in \mathbb{Q} \setminus \mathbb{Z}$. (Suggestion: more specifically, show that for all $n \geq 2$, when written as a fraction $\frac{a}{b}$ in lowest terms, then the denominator b is divisible by 2.)*[2]

EXERCISE 2.4. *Let $k \in \mathbb{Z}^+$. Use the method of telescoping sums to give an exact formula for $\sum_{n=1}^\infty \frac{1}{n(n+k)}$ in terms of the harmonic number H_k of the previous exercise.*

2. Basic Operations on Series

Given an infinite series $\sum_{n=1}^\infty a_n$ there are two basic questions to ask:

QUESTION 2.2. *For an infinite series $\sum_{n=1}^\infty a_n$:*
a) Is the series convergent or divergent?
b) If the series is convergent, what is its sum?

It may seem that this is only "one and a half questions" because if the series diverges we cannot ask about its sum (other than to ask whether it diverges to $\pm\infty$ or "due to oscillation"). However, later on we will revisit this missing "half a question": if a series diveges we may ask *how rapidly* it diverges, or in more sophisticated language we may ask for an **asymptotic estimate** for the sequence of partial sums $\sum_{n=1}^N a_n$ as a function of N as $N \to \infty$.

Note that we have just seen an instance in which we asked and answered both of these questions: for a geometric series $\sum_{n=N}^\infty cr^n$, we know that the series converges iff $|r| < 1$ and in that case its sum is $\frac{cr^N}{1-r}$. We should keep this success story in mind, both because geometric series are ubiquitous and turn out to play a distinguished role in the theory in many ways, *but* also because other examples of series in which we can answer Question 2.2b) – i.e., determine the sum of a convergent series – are much harder to come by. Frankly, in a standard course on infinite series one all but forgets about Question 2.2b) and the game becomes simply to decide whether a given series is convergent or not. In these notes we try to give a little more attention to the second question in some of the optional sections.

In any case, there is a certain philosophy at work when one is, for the moment, interested in determining the convergence / divergence of a given series $\sum_{n=1}^\infty a_n$ rather than the sum. Namely, there are certain operations that one can perform on an infinite series that will preserve the convergence / divergence of the series – i.e., when applied to a convergent series yields another convergent series and when applied to a divergent series yields another divergent series – but will in general change the sum.

The simplest and most useful of these is simply that we may add or remove any

[2]This is a number theory exercise which has, so far as I know, nothing to do with infinite series. But I am a number theorist...

finite number of terms from an infinite series without affecting its convergence. In other words, suppose we start with a series $\sum_{n=1}^{\infty} a_n$. Then, for any integer $N > 1$, consider the series $\sum_{n=N+1}^{\infty} a_n = a_{N+1} + a_{N+2} + \ldots$. Then the first series converges iff the second series converges. Here is one (among many) ways to show this formally: write $S_n = a_1 + \ldots + a_n$ and $T_n = a_{N+1} + a_{N+2} + \ldots + a_{N+n}$. Then for all $n \in \mathbb{Z}^+$

$$\left(\sum_{k=1}^{N} a_k\right) + T_n = a_1 + \ldots + a_N + a_{N+1} + \ldots + a_{N+n} = S_{N+n}.$$

It follows that if $\lim_{n\to\infty} T_n = \sum_{n=N+1}^{\infty} a_n$ exists, then so does $\lim_{n\to\infty} S_{N+n} = \lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$ exists. Conversely if $\sum_{n=1}^{\infty} a_n$ exists, then so does $\lim_{n\to\infty} \sum_{k=1}^{N} a_k + T_n = \sum_{k=1}^{n} a_k + \lim_{n\to\infty} T_n$, hence $\lim_{n\to\infty} T_n = \sum_{n=N+1}^{\infty} a_n$ exists.

Similarly, if we are so inclined (and we will be, on occasion), we could add finitely many terms to the series, or for that matter *change* finitely many terms of the series, without affecting the convergence. We record this as follows.

PROPOSITION 2.3. *The addition, removal or altering of any finite number of terms in an infinite series does not affect the convergence or divergence of the series (though of course it may change the sum of a convergent series).*

As the reader has probably already seen for herself, reading someone else's formal proof of this result can be more tedious than enlightening, so we leave it to the reader to construct a proof that she finds satisfactory.

Because the convergence or divergence of a seies $\sum_{n=1}^{\infty} a_n$ is not affected by changing the lower limit 1 to any other integer, we often employ a simplified notation $\sum_n a_n$ when discussing series only up to convergence.

PROPOSITION 2.4. *Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be two infinite series, and let α be any real number.*
a) If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are both convergent, then the series $\sum_{n=1}^{\infty} a_n + b_n$ is also convergent, with sum $A + B$.
b) If $\sum_{n=1}^{\infty} a_n = S$ is convergent, then so also is $\sum_{n=1}^{\infty} \alpha a_n$, with sum αS.

PROOF. a) Let $A_n^a = a_1 + \ldots + a_n$, $B_n^b = b_1 + \ldots + b_n$ and $C_n = a_1 + b_1 + \ldots + a_n + b_n$. By definition of convergence of infinite series we have $A_n \to S_a$ and $B_n \to B$. Thus for any $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $|A_n - A| < \frac{\epsilon}{2}$ and $|B_n - B| < \frac{\epsilon}{2}$. It follows that for all $n \geq N$,

$$|C_n - (A + B)| = |A_n + B_n - A - B| \leq |A_n - A| + |B_n - B| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

b) We leave the case $\alpha = 0$ to the reader as an (easy) exercise. So suppose that $\alpha \neq 0$ and put $S_n = a_1 + \ldots + a_n$, and our assumption that $\sum_{n=1}^{\infty} a_n = S$ implies that for all $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $|S_n - S| < \frac{\epsilon}{|\alpha|}$. It follows that

$$|\alpha a_1 + \ldots + \alpha a_n - \alpha S| = |\alpha||a_1 + \ldots + a_n - S| = |\alpha||S_n - S| < |\alpha|\left(\frac{\epsilon}{|\alpha|}\right) = \epsilon.$$

$\square$

EXERCISE 2.5. *Let $\sum_n a_n$ be an infinite series and $\alpha \in \mathbb{R}$.*
a) If $\alpha = 0$, show that $\sum_n \alpha a_n = 0$.[3]
b) Suppose that $\alpha \neq 0$. Show that $\sum_n a_n$ converges iff $\sum_n \alpha a_n$ converges. Thus multiplying every term of a series by a nonzero real number does not affect its convergence.

EXERCISE 2.6. *Prove the* ***Three Series Principle****: let $\sum_n a_n$, $\sum_n b_n$, $\sum_n c_n$ be three infinite series with $c_n = a_n + b_n$ for all n. If any two of the three series $\sum_n a_n$, $\sum_n b_n$, $\sum_n c_n$ converge, then so does the third.*

2.1. The Nth Term Test.

The following result is the first of the "convergence tests" that one encounters in freshman calculus.

THEOREM 2.5. *(Nth Term Test) Let $\sum_n a_n$ be an infinite series. If $\sum_n a_n$ converges, then $a_n \to 0$.*

PROOF. Let $S = \sum_{n=1}^{\infty} a_n$. Then for all $n \geq 2$, $a_n = S_n - S_{n-1}$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - S_{n-1} = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0. \qquad \square$$

The result is often applied in its contrapositive form: if $\sum_n a_n$ is a series such that $a_n \not\to 0$ (i.e., either a_n converges to some nonzero number, or it does not converge), then the series $\sum_n a_n$ diverges.

Warning: The converse of Theorem 2.5 is not valid! It may well be the case that $a_n \to 0$ but $\sum_n a_n$ diverges. Later we will see many examples. Still, when put under duress (e.g. while taking an exam) many students can will themselves into believing that the converse might be true. Don't do it!

EXERCISE 2.7. *Let $\frac{P(x)}{Q(x)}$ be a rational function, i.e., a quotient of polynomials with real coefficients (and, to be completely precise, such that Q is not the identically zero polynomial!). The polynomial $Q(x)$ has only finitely many roots, so we may choose $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $Q(x) \neq 0$. Show that if the degree of $P(x)$ is at least at large as the degree of $Q(x)$, then $\sum_{n=N}^{\infty} \frac{P(n)}{Q(n)}$ is divergent.*

2.2. Associativity of infinite sums.

Since we write infinite series as $a_1 + a_2 + \ldots + a_n + \ldots$, it is natural to wonder whether the familiar algebraic properties of addition carry over to the context of infinite sums. For instance, one can ask whether the commutative law holds: if we were to give the terms of the series in a different order, could this change the convergence/divergence of the series, or change the sum. This has a surprisingly complicated answer: *yes* in general, but *no* provided we place an additional requirement on the convergence of the series. In fact this is more than we wish to discuss at the moment and we will come back to the commmutativity problem in the context of our later discussion of **absolute convergence**.

[3]This is a rare case in which we are interested in the sum of the series but the indexing does not matter!

What about the associative law: may we at least insert and delete parentheses as we like? For instance, is it true that the series

$$a_1 + a_2 + a_3 + a_4 + \ldots + a_{2n-1} + a_{2n} + \ldots$$

is convergent iff the series

$$(a_1 + a_2) + (a_3 + a_4) + \ldots + (a_{2n-1}) + a_{2n}) + \ldots$$

is convergent? Here it is easy to see that the answer is *no*: take the geometric series with $r = -1$, so

$$1 - 1 + 1 - 1 + \ldots. \tag{9}$$

The sequence of partial sums is $1, 0, 1, 0, \ldots$, which is divergent. However, if we group the terms together as above, we get

$$(1 - 1) + (1 - 1) + \ldots + (1 - 1) + \ldots = 0,$$

so this regrouped series is convergent. This makes one wonder whether perhaps our definition of convergence of a series is overly fastidious: should we perhaps try to widen our definition so that the given series converges to 1? In fact this does not seem fruitful, because a different regrouping of terms leads to a convergent series with a different sum:

$$1 + (-1 + 1) + (-1 + 1) + \ldots + (-1 + 1) + \ldots = 1.$$

(In fact there *are* more permissive notions of summability of a series than the one we have given, and in this case most of them agree that the right sum to associate to the series 9 is $\frac{1}{2}$. For instance this was the value associated to the series by the eminent 18th century analyst Euler.[4])

It is not difficult to see that adding parentheses to a series can only help it to converge. Indeed, suppose we add parentheses in blocks of lengths $n_1, n_2, \ldots$. For example, the case of no added parentheses at all is $n_1 = n_2 = \ldots = 1$ and the two cases considered above were $n_1 = n_2 = \ldots = 2$ and $n_1 = 1,\ n_2 = n_3 = \ldots = 2$. The point is that adding parentheses has the effect of passing from the sequence $S_1, S_2, \ldots, S_n$ of partial sums to the *subsequence*

$$S_{n_1}, S_{n_1+n_2}, S_{n_1+n_2+n_3}, \ldots.$$

Since we know that any subsequence of a convergent sequence remains convergent and has the same limiting value, it follows that adding parentheses to a convergent series is harmless: the series will still converge and have the same sum. On the other hand, by suitably adding parentheses we can make any series converge to any partial limit of the sequence of partial sums. Thus it follows from the Bolzano-Weierstrass theorem that for any series $\sum_n a_n$ with bounded partial sums it is possible to add parentheses so as to make the series converge.

The following result gives a condition under which parentheses can be *removed* without affecting convergence.

[4]Leonhard Euler, 1707-1783 (pronounced "oiler")

PROPOSITION 2.6. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series such that* $a_n \to 0$. *Let* $\{n_k\}_{k=1}^{\infty}$ *be a* bounded *sequence of positive integers. Suppose that when we insert parentheses in blocks of length* $n_1, n_2, \ldots$ *the series converges to* S. *Then the original series* $\sum_{n=1}^{\infty} a_n$ *converges to* S.

EXERCISE 2.8. *Prove Proposition 2.6.*

2.3. The Cauchy criterion for convergence.

Recall that we proved that a sequence $\{x_n\}$ of real numbers is convergent iff it is Cauchy: that is, for all $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that for all $m, n \geq N$ we have $|x_n - x_m| < \epsilon$.

Applying the Cauchy condition to the sequence of partial sums $\{S_n = a_1 + \ldots + a_n\}$ of an infinite series $\sum_{n=1}^{\infty} a_n$, we get the following result.

PROPOSITION 2.7. *(Cauchy criterion for convergence of series) An infinite series* $\sum_{n=1}^{\infty} a_n$ *converges iff: for every* $\epsilon > 0$, *there exists* $N_0 \in \mathbb{Z}^+$ *such that for all* $N \geq N_0$ *and all* $k \in \mathbb{N}$, $|\sum_{n=N}^{N+k} a_n| < \epsilon$.

Note that taking $k = 0$ in the Cauchy criterion, we recover the Nth Term Test for convergence (Theorem 2.5). It is important to compare these two results: the Nth Term Test gives a very weak *necessary* condition for the convergence of the series. In order to turn this condition into a necessary and sufficient condition we must require not only that $a_n \to 0$ but also $a_n + a_{n+1} \to 0$ and indeed that $a_n + \ldots + a_{n+k} \to 0$ for a k which is allowed to be (in a certain precise sense) arbitrarily large.

Let us call a sum of the form $\sum_{n=N}^{N+k} = a_N + a_{N+1} + \ldots + a_{N+k}$ a **finite tail** of the series $\sum_{n=1}^{\infty} a_n$. As a matter of notation, if for a fixed $N \in \mathbb{Z}^+$ and all $k \in \mathbb{N}$ we have $|\sum_{n=N}^{N+k} a_n| \leq \epsilon$, let us abbreviate this by

$$|\sum_{n=N}^{\infty} a_n| \leq \epsilon.$$

In other words the *supremum* of the absolute values of the finite tails $|\sum_{n=N}^{N+k} a_n|$ is at most ϵ. This gives a nice way of thinking about the Cauchy criterion.

PROPOSITION 2.8. *An infinite series* $\sum_{n=1}^{\infty} a_n$ *converges iff: for all* $\epsilon > 0$, *there exists* $N_0 \in \mathbb{Z}^+$ *such that for all* $N \geq N_0$, $|\sum_{n=N}^{\infty} a_n| < \epsilon$.

In other (less precise) words, an infinite series converges iff by removing sufficiently many of the initial terms, we can make what remains arbitrarily small.

3. Series With Non-Negative Terms I: Comparison

3.1. The sum is the supremum.

Starting in this section we get down to business by restricting our attention to series $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ for all $n \in \mathbb{Z}^+$. This simplifies matters considerably and places an array of powerful tests at our disposal.

Why? Well, assume $a_n \geq 0$ for all $n \in \mathbb{Z}^+$ and consider the sequence of partial sums. We have

$$S_1 = a_1 \leq a_1 + a_2 = S_2 \leq a_1 + a_2 + a_3 = S_3,$$

and so forth. In general, we have that $S_{n+1} - S_n = a_{n+1} \geq 0$, so that the sequence of partial sums $\{S_n\}$ is increasing. Applying the Monotone Sequence Lemma we immediately get the following result.

PROPOSITION 2.9. *Let $\sum_n a_n$ be an infinite series with $a_n \geq 0$ for all n. Then the series converges iff the partial sums are bounded above, i.e., iff there exists $M \in \mathbb{R}$ such that for all n, $a_1 + \ldots + a_n \leq M$. Moroever if the series converges, its sum is precisely the least upper bound of the sequence of partial sums. If the partial sums are unbounded, the series diverges to ∞.*

Because of this, when dealing with series with non-negative terms we may express convergence by writing $\sum_n a_n < \infty$ and divergence by writing $\sum_n a_n = \infty$.

3.2. The Comparison Test.

Example: Consider the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$. Its sequence of partial sums is

$$T_n = 1 \cdot \left(\frac{1}{2}\right) + \frac{1}{2} \cdot \left(\frac{1}{4}\right) + \ldots + \frac{1}{n} \cdot \left(\frac{1}{2^n}\right).$$

Unfortunately we do not (yet!) know a closed form expression for T_n, so it is not possible for us to compute $\lim_{n\to\infty} T_n$ directly. But if we just want to decide whether the series converges, we can compare it with the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$:

$$S_n = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}.$$

Since $\frac{1}{n} \leq 1$ for all $n \in \mathbb{Z}^+$, we have that for all $n \in \mathbb{Z}^+$, $\frac{1}{n2^n} \leq \frac{1}{2^n}$. Summing these inequalities from $k = 1$ to n gives $T_n \leq S_n$ for all n. By our work with geometric series we know that $S_n \leq 1$ for all n and thus also $T_n \leq 1$ for all n. Therefore our given series has partial sums bounded above by 1, so $\sum_{n=1}^{\infty} \frac{1}{n2^n} \leq 1$. In particular, the series converges.

Example: conside the series $\sum_{n=1}^{\infty} \sqrt{n}$. Again, a closed form expression for $T_n = \sqrt{1} + \ldots + \sqrt{n}$ is not easy to come by. But we don't need it: certainly $T_n \geq 1 + \ldots + 1 = n$. Thus the sequence of partial sums is unbounded, so $\sum_{n=1}^{\infty} \sqrt{n} = \infty$.

THEOREM 2.10. *(Comparison Test) Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be two series with non-negative terms, and suppose that $a_n \leq b_n$ for all $n \in \mathbb{Z}^+$. Then*

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n.$$

In particular: if $\sum_n b_n < \infty$ then $\sum_n a_n < \infty$, and if $\sum_n a_n = \infty$ then $\sum_n b_n = \infty$.

PROOF. There is really nothing new to say here, but just to be sure: write

$$S_n = a_1 + \ldots + a_n, \; T_n = b_1 + \ldots + b_n.$$

Since $a_k \leq b_k$ for all k we have $S_n \leq T_n$ for all n and thus

$$\sum_{n=1}^{\infty} a_n = \sup_n S_n \leq \sup_n T_n = \sum_{n=1}^{\infty} b_n.$$

The assertions about convergence and divergence follow immediately. □

3.3. The Delayed Comparison Test.

The Comparison Test is beautifully simple when it works. It has two weaknesses: first, given a series $\sum_n a_n$ we need to find some other series to compare it to. Thus the test will be more or less effective according to the size of our repertoire of known convergent/divergent series with non-negative terms. (At the moment, we don't know much, but that will soon change.) Second, the requirement that $a_n \leq b_n$ for all $n \in \mathbb{Z}^+$ is rather inconveniently strong. Happily, it can be weakened in several ways, resulting in minor variants of the Comparison Test with a much wider range of applicability. Here is one for starters.

Example: Consider the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \ldots + \frac{1}{2 \cdot 3 \cdot \ldots \cdot n} + \ldots.$$

We would like to show that the series converges by comparison, but what to compare it to? Well, there is always the geometric series! Observe that the sequence $n!$ grows faster than any geometric r^n in the sense that $\lim_{n \to \infty} \frac{n!}{r^n} = \infty$. Taking reciprocals, it follows that for any $0 < r < 1$ we will have $\frac{1}{n!} < \frac{1}{r^n}$ – not necessarily for all $n \in \mathbb{Z}^+$, but at least for all sufficiently large n. For instance, one easily establishes by induction that $\frac{1}{n!} < \frac{1}{2^n}$ iff $n \geq 4$. Putting $a_n = \frac{1}{n!}$ and $b_n = \frac{1}{2^n}$ we cannot apply the Comparison Test because we have $a_n \geq b_n$ for all $n \geq 4$ rather than for all $n \geq 0$. But this objection is more worthy of a bureaucrat than a mathematician: certainly *the idea* of the Comparison Test is applicable here:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = \sum_{n=0}^{3} \frac{1}{n!} + \sum_{n=4}^{\infty} \frac{1}{n!} \leq 8/3 + \sum_{n=4}^{\infty} \frac{1}{2^n} = \frac{8}{3} + \frac{1}{8} = \frac{67}{24} < \infty.$$

So the series converges. More than that, we still retain a quantitative estimate on the sum: it is at most (in fact strictly less than, as a moment's thought will show) $\frac{67}{24} = 2.79166666\ldots$. (Perhaps this reminds you of $e = 2.718281828459045235360287471 4\ldots$, which also happens to be a bit less than $\frac{67}{24}$. It should! More on this later...)

We record the technique of the preceding example as a theorem.

Theorem 2.11. *(Delayed Comparison Test) Let $\sum_{n=1}^{\infty}$, $\sum_{n=1}^{\infty} b_n$ be two series with non-negative terms. Suppose that there exists $N \in \mathbb{Z}^+$ such that for all $n > N$, $a_n \leq b_n$. Then*

$$\sum_{n=1}^{\infty} a_n \leq \left(\sum_{n=1}^{N} a_n - b_n \right) + \sum_{n=1}^{\infty} b_n.$$

In particular: if $\sum_n b_n < \infty$ then $\sum_n a_n < \infty$, and if $\sum_n a_n = \infty$ then $\sum_n b_n = \infty$.

Exercise 2.9. *Prove Theorem 2.11.*

Thus the Delayed Comparison Test assures us that we do not need $a_n \leq b_n$ for all n but only for all sufficiently large n. A different issue occurs when we wish to apply the Comparison Test and the inequalities do not go our way.

3.4. The Limit Comparison Test.

THEOREM 2.12. *(Limit Comparison Test) Let $\sum_n a_n$, $\sum_n b_n$ two series. Suppose that there exists $N \in \mathbb{Z}^+$ and $M \in \mathbb{R}^{\geq 0}$ such that for all $n \geq N$, $0 \leq a_n \leq Mb_n$. Then if $\sum_n b_n$ converges, $\sum_n a_n$ converges.*

EXERCISE 2.10. *Prove Theorem 2.12.*

COROLLARY 2.13. *(Calculus Student's Limit Comparison Test) Let $\sum_n a_n$ and $\sum_n b_n$ be two series. Suppose that for all sufficiently large n both a_n and b_n are positive and $\lim_{n\to\infty} \frac{a_n}{b_n} = L \in [0,\infty]$.*
a) If $0 < L < \infty$, the series $\sum_n a_n$ and $\sum_n b_n$ converge or diverge together (i.e., either both convege or both diverge).
b) If $L = \infty$ and $\sum_n a_n$ converges, then $\sum_n b_n$ converges.
c) If $L = 0$ and $\sum_n b_n$ converges, then $\sum_n a_n$ converges.

PROOF. In all three cases we deduce the result from the Limit Comparison Test (Theorem 2.12).
a) If $0 < L < \infty$, then there exists $N \in \mathbb{Z}^+$ such that $0 < \frac{L}{2}b_n \leq a_n \leq (2L)b_n$. Applying Theorem 2.12 to the second inequality, we get that if $\sum_n b_n$ converges, then $\sum_n a_n$ converges. The first inequality is equivalent to $0 < b_n \leq \frac{2}{L}a_n$ for all $n \geq N$, and applying Theorem 2.12 to this we get that if $\sum_n a_n$ converges, then $\sum_n b_n$ converges. So the two series $\sum_n a_n$, $\sum_n b_n$ converge or diverge together.
b) If $L = \infty$, then there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $a_n \geq b_n \geq 0$. Applying Theorem 2.12 to this we get that if $\sum_n$ converges, then $\sum_n b_n$ converges.
c) This case is left to the reader as an exercise. $\square$

EXERCISE 2.11. *Prove Theorem 2.13.*

Example: We will show that for all $p \geq 2$, the **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. In fact it is enough to show this for $p = 2$, since for $p > 2$ we have for all $n \in \mathbb{Z}^+$ that $n^2 < n^p$ and thus $\frac{1}{n^p} < \frac{1}{n^2}$ so $\sum_n \frac{1}{n^p} \leq \sum_n \frac{1}{n^2}$. For $p = 2$, we happen to know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1,$$

and in particular that $\sum_n \frac{1}{n^2+n}$ converges. For large n, $\frac{1}{n^2+n}$ is close to $\frac{1}{n^2}$. Indeed, the precies statement of this is that putting $a_n = \frac{1}{n^2+n}$ and $b_n = \frac{1}{n^2}$ we have $a_n \sim b_n$, i.e.,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$$

Applying Theorem 2.13, we find that $\sum_n \frac{1}{n^2+n}$ and $\sum_n \frac{1}{n^2}$ converge or diverge together. Since the former series converges, we deduce that $\sum_n \frac{1}{n^2}$ converges, even though the Direct Comparison Test does not apply.

EXERCISE 2.12. *Let $\frac{P(x)}{Q(x)}$ be a rational function such that the degree of the denominator minus the degree of the numerator is at least 2. Show that $\sum_{n=N}^{\infty} \frac{P(n)}{Q(n)}$. (Recall from Exercise X.X our convention that we choose N to be larger than all the roots of $Q(x)$, so that every term of the series is well-defined.)*

EXERCISE 2.13. *Determine whether each of the following series converges or diverges:*

a) $\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$.
b) $\sum_{n=1}^{\infty} \cos \frac{1}{n^2}$.

3.5. Cauchy products I: non-negative terms.

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two infinite series. Is there some notion of a **product** of these series?

In order to forestall possible confusion, let us point out that many students are tempted to consider the following "product" operation on series:

$$(\sum_{n=0}^{\infty} a_n) \cdot (\sum_{n=0}^{\infty} b_n) \stackrel{??}{=} \sum_{n=0}^{\infty} a_n b_n.$$

In other words, given two sequences of terms $\{a_n\}$, $\{b_n\}$, we form a new sequence of terms $\{a_n b_n\}$ and then we form the associated series. In fact this is not a very useful candidate for the product. What we surely want to happen is that if $\sum_n a_n = A$ and $\sum_n b_n = B$ then our "product series" should converge to AB. But for instance, take $\{a_n\} = \{b_n\} = \frac{1}{2^n}$. Then $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n = \frac{1}{1-\frac{1}{2}} = 2$, so $AB = 4$, whereas $\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$. Of course $\frac{4}{3} < 4$. What went wrong?

Plenty! We have ignored the laws of algebra for finite sums: e.g.

$$(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_0 b_1 + a_1 b_0 + a_0 b_2 + a_1 b_1 + a_2 b_0.$$

The product is different and more complicated – and indeed, if all the terms are positive, strictly lager – than just $a_0 b_0 + a_1 b_1 + a_2 b_2$. We have forgotten about the cross-terms which show up when we multiply one expression involving several terms by another expression involving several terms.[5]

Let us try again at formally multiplying out a product of infinite series:

$$(a_0 + a_1 + \ldots + a_n + \ldots)(b_0 + b_1 + \ldots + b_n + \ldots)$$

$$= a_0 b_0 + a_0 b_1 + a_1 b_0 + a_0 b_2 + a_1 b_1 + a_2 b_0 + \ldots + a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0 + \ldots.$$

So it is getting a bit notationally complicated. In order to shoehorn the right hand side into a single infinite series, we need to either (i) choose some particular ordering to take the terms $a_k b_\ell$ on the right hand side, or (ii) collect some terms together into an nth term.

For the moment we choose the latter: we define for any $n \in \mathbb{N}$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_n$$

and then we define the **Cauchy product** of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ to be the series

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right).$$

[5]To the readers who did not forget about the cross-terms: my apologies. But it is a common enough misconception that it had to be addressed.

THEOREM 2.14. *Let* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ *be two series with non-negative terms. Let* $\sum_{n=0}^{\infty} a_n = A$ *and* $\sum_{n=0}^{\infty} b_n = B$. *Putting* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ *we have that* $\sum_{n=0}^{\infty} c_n = AB$. *In particular, the Cauchy product series converges iff the two "factor series"* $\sum_n a_n$ *and* $\sum_n b_n$ *both converge.*

PROOF. It is instructive to define yet another sequence, the **box product**, as follows: for all $N \in \mathbb{N}$,

$$\square_N = \sum_{0 \le i,j \le N} a_i b_j = (a_0 + \ldots + a_N)(b_0 + \ldots + b_N) = A_N B_N.$$

Thus by the usual product rule for sequences, we have

$$\lim_{N \to \infty} \square_N = \lim_{N \to \infty} A_N B_N = AB.$$

So the box product clearly converges to the product of the sums of the two series. This suggests that we compare the Cauchy product to the box product. The entries of the box product can be arranged to form a square, viz:

$$\begin{aligned} \square_N = a_0 b_0 + a_0 b_1 + \ldots + a_0 b_N \\ + a_1 b_0 + a_1 b_1 + \ldots + a_1 b_N \\ \vdots \\ + a_N b_0 + aN b_1 + \ldots + a_N b_N. \end{aligned}$$

On the other hand, the terms of the Nth partial sum of the Cauchy product can naturally be arranged in a triangle:

$$\begin{aligned} C_N = \qquad\qquad\qquad\qquad\qquad\qquad a_0 b_0 \\ + a_0 b_1 + a_1 b_0 \\ + a_0 b_2 + a_1 b_1 + a_2 b_0 \\ + a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 \\ \vdots \\ + a_0 b_N + a_1 b_{N-1} + a_2 b_{N-2} + \ldots + a_N b_0. \end{aligned}$$

Thus while $\square_N$ is a sum of $(N+1)^2$ terms, C_N is a sum of $1 + 2 + \ldots + N + 1 = \frac{(N+1)(N+2)}{2}$ terms: those lying on our below the diagonal of the square. Thus in considerations involving the Cauchy product, the question is to what extent one can neglect the terms in the upper half of the square – i.e., those with $a_i b_j$ with $i + j > N$ – as N gets large.

Here, since all the a_i's and b_j's are non-negative and $\square_N$ contains all the terms of C_N and others as well, we certainly have

$$C_N \le \square_N = A_N B_N \le AB.$$

Thus $C = \lim_{N \to \infty} C_N \le AB$. For the converse, the key observation is that if we make the sides of the triangle twice as long, it will contain the box: that is, every term of $\square_N$ is of the form $a_i b_j$ with $0 \le i, j \le N$; thus $i + j \le 2N$ so $a_i b_j$ appears as a term in C_{2N}. It follows that $C_{2N} \ge \square_N$ and thus

$$C = \lim_{N \to \infty} C_N = \lim_{N \to \infty} C_{2N} \ge \lim_{N \to \infty} \square_N = \lim_{N \to \infty} A_N B_N = AB.$$

Having shown both that $C \leq AB$ and $C \geq AB$, we conclude

$$C = \sum_{n=0}^{\infty} a_n = AB = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

$\square$

4. Series With Non-Negative Terms II: Condensation and Integration

We have recently been studying criteria for convergence of an infinite series $\sum_n a_n$ which are valid under the assumption that $a_n \geq 0$ for all n. In this section we place ourselves under more restrictive hypotheses: that for all $n \in \mathbb{N}$, $a_{n+1} \geq a_n \geq 0$, i.e., that the sequence of terms is **non-negative** and **decreasing**.

Remark: It is in fact no loss of generality to assume that $a_n > 0$ for all n. Indeed, if not we have $a_N = 0$ for some N and then since the terms are assumed to be decreasing we have $0 = a_N = a_{N+1} = \ldots$ and our infinite series reduces to the finite series $\sum_{n=1}^{N-1} a_n$: this converges!

4.1. The Harmonic Series.

Consider $\sum_{n=1}^{\infty} \frac{1}{n}$, the **harmonic series**. Does it converge? None of the tests we have developed so far are up to the job: especially, $a_n \to 0$ so the Nth Term Test is inconclusive.

Let us take a computational approach by looking at various partial sums. S_{100} is approximately 5.187. Is this close to a familiar real number? Not really. Next we compute $S_{150} \approx 5.591$ and $S_{200} \approx 5.878$. Perhaps the partial sums never exceed 6? (If so, the series would converge.) Let's try a significantly larger partial sums: $S_{1000} \approx 7.485$, so the above guess is incorrect. Since $S_{1050} \approx 7.584$, we are getting the idea that whatever the series is doing, it's doing it rather slowly, so let's instead start stepping up the partial sums multiplicatively:

$$S_{100} \approx 5.878.$$

$$S_{10^3} \approx 7.4854.$$

$$S_{10^4} \approx 9.788.$$

$$S_{10^5} \approx 12.090.$$

Now there is a pattern for the perceptive eye to see: the difference $S_{10^{k+1}} - S_{10^k}$ appears to be approaching $2.30\ldots = \log 10$. This points to $S_n \approx \log n$. If this is so, then since $\log n \to \infty$ the series would diverge. I hope you notice that the relation between $\frac{1}{n}$ and $\log n$ is one of a function and its antiderivative. We ask the reader to hold this thought until we discuss the integral test a bit late on.

For now, we give the following brilliant and elementary argument due to Cauchy.

Consider the terms arranged as follows:

$$\left(\frac{1}{1}\right) + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \ldots,$$

i.e., we group the terms in blocks of length 2^k. Now observe that the power of $\frac{1}{2}$ which beings each block is larger than every term in the preceding block, so if we replaced every term in the current block the the first term in the next block, we would only decrease the sum of the series. But this latter sum is much easier to deal with:

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots = \infty.$$

Therefore the harmonic series $\sum_{n=1}^{\infty}$ diverges.

EXERCISE 2.14. *Determine the convergence of* $\sum_n \frac{1}{n^{1+\frac{1}{n}}}$.

EXERCISE 2.15. *Let* $\frac{P(x)}{Q(x)}$ *be a rational function such that the degree of the denominator is exactly one greater than the degree of the numerator. Show that* $\sum_{n=N}^{\infty} \frac{P(n)}{Q(n)}$ *diverges.*

Collecting Exercises 2.12, 2.13 and 2.14 on rational functions, we get the following result.

PROPOSITION 2.15. *For a rational function* $\frac{P(x)}{Q(x)}$, *the series* $\sum_{n=N} \frac{P(n)}{Q(n)}$ *converges iff the degree of the denominator minus the degree of the numerator is at least two.*

4.2. Criterion of Abel-Olivier-Pringsheim.

THEOREM 2.16. *(Abel[6]-Olivier-Prinsgheim[7]* **[Ol27]***) Let* $\sum_n a_n$ *be a convergent infinite series with* $a_n \geq a_{n+1} \geq 0$ *for all* n. *Then* $\lim_{n\to\infty} na_n = 0$.

PROOF. (Hardy) By the Cauchy Criterion, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|\sum_{k=n+1}^{2n} a_k| < \epsilon$. Since the terms are decreasing we get

$$|na_{2n}| = a_{2n} + \ldots + a_{2n} \leq a_{n+1} + \ldots + a_{2n} = |\sum_{k=n+1}^{2n} a_n| < \epsilon.$$

It follows that $\lim_{n\to\infty} na_{2n} = 0$, hence $\lim_{n\to\infty} 2na_{2n} = 2 \cdot 0 = 0$. Thus also

$$(2n+1)a_{2n+1} \leq \left(\frac{2n+1}{2n}\right)(2na_{2n}) \leq 4\,(na_{2n}) \to 0.$$

Thus both the even and odd subsequences of na_n converge to 0, so $na_n \to 0$. $\square$

In other words, for a series $\sum_n a_n$ with decreasing terms to converge, its nth term must be qualitatively smaller than the nth term of the harmonic series.

EXERCISE 2.16. *Let* a_n *be a decreasing non-negative sequence, and let* $n_1 < n_2 < \ldots < n_k$ *be a strictly increasing sequence of positive integers. Suppose that* $\lim_{k\to\infty} n_k a_{n_k} = 0$. *Deduce that* $\lim_{n\to\infty} na_n = 0$.

Warning: The condition given in Theorem 2.16 is necessary for a series with decreasing terms to converge, but it is not sufficient. Soon enough we will see that e.g. $\sum_n \frac{1}{n \log n}$ does not converge, even though $n(\frac{1}{n \log n}) = \frac{1}{\log n} \to 0$.

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[6]Niels Henrik Abel, 1802-1829

[7]Alfred Israel Pringsheim, 1850-1941

4.3. Condensation Tests.

The apparently *ad hoc* argument used to prove the divergence of the harmonic series can be adapted to give the following useful test.

THEOREM 2.17. *(Cauchy Condensation Test) Let $\sum_{n=1}^{\infty} a_n$ be an infinite series such that $a_n \geq a_{n+1} \geq 0$ for all $n \in \mathbb{N}$. Then:*
a) We have $\sum_{n=1}^{\infty} a_n \leq \sum_{n=0}^{\infty} 2^n a_{2^n} \leq 2 \sum_{n=1}^{\infty} a_n$.
b) Thus the series $\sum_n a_n$ converges iff the ***condensed series*** *$\sum_n 2^n a_{2^n}$ converges.*

PROOF. We have

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots$$

$$\leq a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + 8a_8 + \ldots = \sum_{n=0}^{\infty} 2^n a_{2^n}$$

$$= (a_1 + a_2) + (a_2 + a_4 + a_4 + a_4) + (a_4 + a_8 + a_8 + a_8 + a_8 + a_8 + a_8 + a_8) + (a_8 + \ldots)$$

$$\leq (a_1 + a_1) + (a_2 + a_2 + a_3 + a_3) + (a_4 + a_4 + a_5 + a_5 + a_6 + a_6 + a_7 + a_7) + (a_8 + \ldots)$$

$$= 2 \sum_{n=1}^{\infty} a_n.$$

This establishes part a), and part b) follows immediately. □

The Cauchy Condensation Test is, I think, an *a priori* interesting result: it says that, under the given hypotheses, in order to determine whether a series converges we need to know only a very sparse set of the terms of the series – whatever is happening in between a_{2^n} and $a_{2^{n+1}}$ is immaterial, *so long as the sequence remains decreasing.* This is a very curious phenomenon, and of couse without the hypothesis that the terms are decreasing, nothing like this could hold.

On the other hand, it may be less clear that the Condensation Test is of any practical use: after all, isn't the condensed series $\sum_n 2^n a_{2^n}$ more complicated than the original series $\sum_n a_n$? In fact the opposite is often the case: passing from the given series to the condensed series preserves the convergence or divergence but tends to exchange subtly convergent/divergent series for more obviously (or better: more rapidly) converging/diverging series.

Example: Fix a real number p and consider the **p-series**[8] $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Our task is to find all values of p for which the series converges.

Step 1: The sequence $a_n = \frac{1}{n^p}$ has positive terms. The terms are decreasing iff the sequence n^p is increasing iff $p > 0$. So we had better treat the cases $p \leq 0$ separately. First, if $p < 0$, then $\lim_{n \to \infty} \frac{1}{n^p} = \lim_{n \to \infty} n^{|p|} = \infty$, so the p-series diverges by the nth term test. Second, if $p = 0$ then our series is simply $\sum_n \frac{1}{n^0} = \sum_n 1 = \infty$. So the p-series "obviously diverges" when $p \leq 0$.

[8] Or sometimes: **hyperharmonic series**.

Step 2: Henceforth we assume $p > 0$, so that the hypotheses of Cauchy's Condensation Test apply. We get that $\sum_n n^{-p}$ converges iff $\sum_n 2^n(2^n)^{-p} = \sum_n 2^n 2^{-np} = \sum_n (2^{1-p})^n$ converges. But the latter series is a geometric series with geometric ratio $r = 2^{1-p}$, so it converges iff $|r| < 1$ iff $2^{p-1} > 1$ iff $p > 1$.

Thus we have proved the following important result.

THEOREM 2.18. *For $p \in \mathbb{R}$, the p-series $\sum_n \frac{1}{n^p}$ converges iff $p > 1$.*

Example (p-series continued): Let $p > 1$. By applying part b) of Cauchy's Condensation Test we showed that $\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$. What about part a)? It gives an explicit upper bound on the sum of the series, namely

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq \sum_{n=0}^{\infty} 2^n(2^n)^{-p} = \sum_{n=0}^{\infty} (2^{1-p})^n = \frac{1}{1-2^{1-p}}.$$

For instance, taking $p = 2$ we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{1}{1-2^{1-2}} = 2.$$

Using a computer algebra package I get

$$1 \leq \sum_{n=1}^{1024} \frac{1}{n^2} = 1.643957981030164240100762569\ldots.$$

So it seems like $\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.64$, whereas the Condensation Test tells us that it is at most 2. (Note that since the terms are positive, simply adding up any finite number of terms gives a lower bound.)

The following exercise gives a technique for using the Condensation Test to estimate $\sum_{n=1}^{\infty} \frac{1}{n^p}$ to arbitrary accuracy.

EXERCISE 2.17. *Let N be a non-negative integer.*
a) Show that under the hypotheses of the Condensation Test we have

$$\sum_{n=2^N+1}^{\infty} a_n \leq \sum_{n=0}^{\infty} 2^n a_{2^{n+N}}.$$

b) Apply part a) to show that for any $p > 1$,

$$\sum_{n=2^N+1} \frac{1}{n^p} \leq \frac{1}{2^{Np}\left(1-2^{1-p}\right)}.$$

Example: $\sum_{n=2}^{\infty} \frac{1}{n\log n}$. $a_n = \frac{1}{n\log n}$ is positive and decreasing (since its reciprocal is positive and increasing) so the Condensation Test applies. We get that the convergence of the series is equivalent to the convergence of

$$\sum_n \frac{2^n}{2^n \log 2^n} = \frac{1}{\log 2}\sum_n \frac{1}{n} = \infty,$$

so the series diverges. This is rather subtle: we know that for any $\epsilon > 0$, $\sum_n \frac{1}{nn^{\epsilon}}$ converges, since it is a p-series with $p = 1 + \epsilon$. But $\log n$ grows more slowly than n^{ϵ} for any $\epsilon > 0$, indeed slowly enough so that replacing n^{ϵ} with $\log n$ converts a convergent series to a divergent one.

Exercise 2.18. *Determine whether the series* $\sum_n \frac{1}{\log(n!)}$ *converges.*

Exercise 2.19. *Let* p, q, r *be positive real numbers.*
a) Show that $\sum_n \frac{1}{n(\log n)^q}$ *converges iff* $q > 1$.
b) Show that $\sum_n \frac{1}{n^p(\log n)^q}$ *converges iff* $p > 1$ *or* ($p = 1$ *and* $q > 1$).
c) Find all values of p, q, r *such that* $\sum_n \frac{1}{n^p(\log n)^q(\log\log n)^r}$ *converges.*

The pattern of Exercise 2.18 could be continued indefinitely, giving series which converge or diverge excruciatingly slowly, and showing that the difference between convergence and divergence can be arbitarily subtle.

Exercise 2.20. *(*[**H**, §182]*) Use the Cauchy Condensation Test and Exercise X.X to give another proof of the Abel-Olivier-Pringsheim Theorem.*

It is natural to wonder at the role played by 2^n in the Condensation Test. Could it not be replaced by some other subsequence of $\mathbb{N}$? For instance, could we replace 2^n by 3^n? It is not hard to see that the answer is *yes*, but let us pursue a more ambitious generalization.

Suppose we are given two positive decreasing sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ and a subsequence $g(1) < g(2) < \ldots < g(n)$ of the positive integers such that $a_{g(n)} = b_{g(n)}$ for all $n \in \mathbb{Z}^+$. In other words, we have two monotone sequences which agree on the subsequence given by g. Under what circumstances can we assert that $\sum_n a_n < \infty \iff \sum_n b_n < \infty$? Let us call this the **monotone interpolation problem**. Note that Cauchy's Condensation Test tells us, among other things, that the monotone interpolation problem has an affirmative answer when $g(n) = 2^n$: since $\sum_n a_n < \infty$ iff $\sum_n 2^n a_{2^n} < \infty$, evidently in order to tell whether a series $\sum_n a_n$ with decreasing terms converges, we only need to know $a_2, a_4, a_8, \ldots$: whatever it is doing in between those values is not enough to affect the convergence.

But quite generally, given two sequences $\{a_n\}$, $\{b_n\}$ as above such that $a_{g(n)} = b_{g(n)}$ for all n, we can give upper and lower bounds on the size of b_n's in terms of the $a_{g(n)}$'s, using exactly the same reasoning we used to prove the Cauchy Condensation Test. Indeed, putting $g(0) = 0$, we have

$$\sum_{n=1}^\infty b_n = b_1 + b_2 + b_3 + \ldots$$

$$\leq (g(1) - 1)b_1 + (g(2) - g(1))a_{g(1)} + (g(3) - g(2))a_{g(2)} + \ldots$$

$$= (g(1) - 1)b_1 + \sum_{n=1}^\infty (g(n+1) - g(n))a_{g(n)}$$

and also

$$\sum_{n=1}^\infty b_n = b_1 + b_2 + b_3 + \ldots$$

$$\geq (g(1) - g(0))a_{g(1)} + (g(2) - g(1))a_{g(2)} + \ldots = \sum_{n=1}^\infty (g(n) - g(n-1))a_{g(n)}.$$

Putting these inequalities together we get

$$(10)\quad \sum_{n=1}^{\infty}(g(n)-g(n-1))a_{g(n)} \leq \sum_{n=1}^{\infty} b_n \leq (g(1)-1)b_1 + \sum_{n=1}^{\infty}(g(n+1)-g(n))a_{g(n)}.$$

Let us stop and look at what we have. In the special case $g(n) = 2^n$, then $g(n) - g(n-1) = 2^{n-1}$ and $g(n+1) - g(n) = 2^n = 22^{n-1}$, and since these two sequences have the same order of magnitude, it follows that the third series is finite iff the first series is finite iff the second series is finite, and we get that $\sum_n b_n$ and $\sum_n g(n+1) - g(n)a_{g(n)} = \sum_n (g(n+1) - g(n))b_{g(n)}$ converge or diverge together. Therefore by *imposing* the condition that $g(n+1) - g(n)$ and $g(n) - g(n-1)$ agree up to a constant, we get a more general condensation test originally due to O. Schlömilch.[9]

THEOREM 2.19. *(Schlömilch Condensation Test* [**Sc73**]*) Let $\{a_n\}$ be a sequence with $a_n \geq a_{n+1} \geq 0$ for all $n \in \mathbb{N}$. Let $g : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function satisfying Hypothesis (S): there exists $M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,*

$$\frac{\Delta g(n)}{\Delta g(n-1)} := \frac{g(n+1) - g(n)}{g(n) - g(n-1)} \leq M.$$

Then the two series $\sum_n a_n$, $\sum_n (g(n) - g(n-1))\, a_{g(n)}$ converge or diverge together.

PROOF. Indeed, take $a_n = b_n$ for all n in the above discussion, and put $A = (g(1) - 1)a_1$. Then (10) gives

$$\sum_{n=1}^{\infty}(g(n) - g(n-1))a_{g(n)} \leq \sum_{n=1}^{\infty} a_n \leq A + \sum_{n=1}^{\infty}(g(n+1) - g(n))a_{g(n)}$$

$$\leq A + M\sum_{n=1}^{\infty}(g(n) - g(n-1))a_{g(n)},$$

which shows that $\sum_n a_n$ and $\sum_n (g(n)-g(n-1))a_{g(n)}$ converge or diverge together. $\square$

EXERCISE 2.21. *Use Theorem 2.19 to show that for any sequence $\sum_n a_n$ with decreasing terms and any integer $r \geq 2$, $\sum_n a_n$ converges iff $\sum_n r^n a_{r^n}$ converges.*

EXERCISE 2.22. *Use Theorem 2.19 with $g(n) = n^2$ to show that the series $\sum_n \frac{1}{2^{\sqrt{n}}}$ converges. (It is also possible to do this with a Limit Comparison argument, but using Theorem 2.19 is a pleasantly clean way to go.)*

Note that we did not solve the monotone interpolation problem: we got "distracted" by Theorem 2.19. It is not hard to use (10) to show that if the difference sequence $(\Delta g)(n)$ grows sufficiently rapidly, then the convergence of $\sum_n a_n$ does not imply the convergence of the interpolated series $\sum_n b_n$. For instance, take $a_n = \frac{1}{n^2}$; then the estimates of (10) correspond to taking the largest possible monotone interpolation b_n – but the key here is "possible" – in which case we get

$$\sum_n b_n = \sum_n \frac{g(n+1) - g(n)}{g(n)^2}.$$

[9] Oscar Xavier Schlömilch, 1823-1901

We can then define a sequence $g(n)$ so as to ensure that $\frac{g(n+1)-g(n)}{g(n)^2} \geq 1$ for all n, and then the series diverges.[10] Some straightforward (and rough) analysis shows that the function $g(n)$ is doubly exponential in n.

4.4. The Integral Test.

THEOREM 2.20. *(Integral Test) Let $f : [1,\infty) \to \mathbb{R}$ be a positive decreasing function, and for $n \in \mathbb{N}$ put $a_n = f(n)$. Then*

$$\sum_{n=2}^{\infty} a_n \leq \int_1^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} a_n.$$

Thus the series $\sum_n a_n$ converges iff the improper integral $\int_1^{\infty} f(x)dx$ converges.

PROOF. This is a rare opportunity in analysis in which a picture supplies a perfectly rigorous proof. Namely, we divide the interval $[1,\infty)$ into subintervals $[n, n+1]$ for all $n \in \mathbb{N}$ and for any $N \in N$ we compare the integral $\int_1^N f(x)dx$ with the upper and lower Riemann sums associated to the partition $\{1, 2, \ldots, N\}$. From the picture one sees immediately that – since f is decreasing – the lower sum is $\sum_{n=2}^{N+1} a_n$ and the upper sum is $\sum_{n=1}^{N} a_n$, so that

$$\sum_{n=2}^{N+1} a_n \leq \int_1^{N} f(x)dx \leq \sum_{n=1}^{N} a_n.$$

Taking limits as $N \to \infty$, the result follows. $\square$

Remark: The Integral Test is due to Maclaurin[11] [**Ma42**] and later in more modern form to A.L. Cauchy [**Ca89**]. I don't know why it is traditional to attach Cauchy's name to the Condensation Test but not the Integral Test, but I have preserved the tradition nevetheless.

It happens that, at least among the series which arise naturally in calculus and undergraduate analysis, it is usually the case that the Condensation Test can be successfully applied to determine convergence / divergence of a series iff the Integral Test can be successfully applied.

Example: Let us use the Integral Test to determine the set of $p > 0$ such that $\sum_n \frac{1}{n^p}$ converges. Indeed the series converges iff the improper integral $\int_1^{\infty} \frac{dx}{x^p}$ is finite. If $p \neq 1$, then we have

$$\int_1^{\infty} \frac{dx}{x^p} = \frac{x^{1-p}}{1-p}|_{x=1}^{x=\infty}.$$

The upper limit is 0 if $p - 1 < 0 \iff p > 1$ and is ∞ if $p < 1$. Finally,

$$\int_1^{\infty} \frac{dx}{x} = \log x|_{x=1}^{\infty} = \infty.$$

So, once again, the p-series diverges iff $p > 1$.

---

[10] In analysis one often encounters increasing sequences $g(n)$ in which each term is chosen to be *much* larger than the one before it – sufficiently large so as drown out some other quantity. Such sequences are called **lacunary** (this is a description rather than a definition) and tend to be very useful for producing counterexamples.

[11] Colin Maclaurin, 1698-1746

EXERCISE 2.23. *Verify that all of the above examples involving the Condensation Test can also be done using the Integral Test.*

Given the similar applicability of the Condensation and Integral Tests, it is perhaps not so surprising that many texts content themselves to give one or the other. In calculus texts, one almost always finds the Integral Test, which is logical since often integration and then improper integation are covered earlier in the same course in which one studies infinite series. In elementary analysis courses one often develops sequences and series before the study of functions of a real variable, which is logical because a formal treatment of the Riemann integral is necessarily somewhat involved and technical. Thus many of these texts give the Condensation Test.

From an aesthetic standpoint, the Condensation Test is more appealing (to me). On the other hand, under a mild additional hypothesis the Integral Test can be used to give **asymptotic expansions** for divergent series.[12]

LEMMA 2.21. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive real numbers with $a_n \sim b_n$ and $\sum_n a_n = \infty$. Then $\sum_n b_n = \infty$ and $\sum_{n=1}^N a_n \sim \sum_{n=1}^N b_n$.*

PROOF. That $\sum_n a_n = \infty$ follows from the Limit Comparison Test. Now fix $\epsilon > 0$ and choose $K \in \mathbb{N}$ such that for all $n \geq K$ we have $a_n \leq (1+\epsilon)b_n$. Then for $N \geq K$,

$$\sum_{n=1}^N a_n = \sum_{n=1}^{K-1} a_n + \sum_{n=K}^N a_n \leq \sum_{n=1}^{K-1} a_n + \sum_{n=K}^N (1+\epsilon)b_n$$

$$= \left(\sum_{n=1}^{K_1} a_n - \sum_{n=1}^{K-1}(1+\epsilon)b_n\right) + \sum_{n=1}^N (1+\epsilon)b_n = C_{\epsilon,K} + (1+\epsilon)\sum_{n=1}^N b_n,$$

say, where $C_{\epsilon,K}$ is a constant independent of N. Dividing both sides by $\sum_{n=1}^N b_n$ and using the fact that $\lim_{N\to\infty}\sum_{n=1}^N b_n = \infty$, we find that the quantity $\frac{\sum_{n=1}^N a_n}{\sum_{n=1}^N b_n}$ is at most $1+2\epsilon$ for all sufficiently large N. Because our hypotheses are symmetric in $\sum_n a_n$ and $\sum_n b_n$, we also have that $\frac{\sum_{n=1}^N b_n}{\sum_{n=1}^N a_n}$ is at most $1+2\epsilon$ for all suffiicently large N. It follows that

$$\lim_{N\to\infty} \frac{\sum_{n=1}^N a_n}{\sum_{n=1}^N b_n} = 1.$$

□

THEOREM 2.22. *Let $f : [1,\infty) \to \mathbb{R}$ be a positive monotone continuous function. Suppose the series $\sum_n f(n)$ diverges and that as $x \to \infty$, $f(x) \sim f(x+1)$. Then*

$$\sum_{n=1}^N f(n) \sim \int_1^N f(x)dx.$$

PROOF. Case 1: Suppose f is increasing. Then, for $n \leq x \leq n+1$, we have $f(n) \leq \int_n^{n+1} f(x)dx \leq f(n+1)$, or

$$1 \leq \frac{\int_n^{n+1} f(x)dx}{f(n)} \leq \frac{f(n+1)}{f(n)}.$$

[12]Our treatment of the next two results owes a debt to K. Conrad's *Estimating the Size of a Divergent Sum.*

By assumption we have

$$\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1,$$

so by the Squeeze Principle we have

$$\int_n^{n+1} f(x)dx \sim f(n). \tag{11}$$

Applying Lemma 2.21 with $a_n = f(n)$ and $b_n = \int_n^{n+1} f(x)dx$, we conclude

$$\int_1^{N+1} f(x)dx = \sum_{k=1}^{N} \int_k^{k+1} f(x)dx \sim \sum_{n=1}^{N} f(n).$$

Further, we have

$$\lim_{N\to\infty} \frac{\int_1^{N+1} f(x)dx}{\int_1^{N} f(x)dx} = \frac{\infty}{\infty} \overset{*}{=} \lim_{N\to\infty} \frac{f(N+1)}{f(N)} = 1,$$

where in the starred equality we have applied L'Hopital's Rule and then the Fundamental Theorem of Calculus. We conclude

$$\int_1^{N} f(x)dx \sim \int_1^{N+1} f(x)dx \sim \sum_{n=1}^{N} f(n),$$

as desired.
Case 2: Suppose f is decreasing. Then for $n \leq x \leq n+1$, we have

$$f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n),$$

or

$$\frac{f(n+1)}{f(n)} \leq \frac{\int_n^{n+1} f(x)dx}{f(n)} \leq 1.$$

Once again, by our assumption that $f(n) \sim f(n+1)$ and the Squeeze Principle we get (11), and the remainder of the proof proceeds exactly as in the previous case. $\square$

4.5. Euler Constants.

THEOREM 2.23. *(Maclaurin-Cauchy) Let $f : [1,\infty) \to \mathbb{R}$ be positive, continuous and decreasing, with $\lim_{x\to\infty} f(x) = 0$. Then we may define the **Euler constant***

$$\gamma_f := \lim_{N\to\infty} \left(\sum_{n=1}^{N} f(n) - \int_1^{N} f(x)dx \right).$$

In other words, the above limit exists.

PROOF. Put

$$a_N = \sum_{n=1}^{N} f(n) - \int_1^{N} f(x)dx,$$

so our task is to show that the sequence $\{a_N\}$ converges. As in the integral test we have that for all $n \in \mathbb{Z}^+$

$$f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n). \tag{12}$$

Using the second inequality in (12) we get

$$a_N = f(N) + \sum_{n=1}^{N-1} f(n) - \int_1^N f(x)dx \geq \sum_{n=1}^{N} (f(n) - \int_n^{n+1} f(x)dx) \geq 0,$$

and the first inequality in (12) we get

$$a_{N+1} - a_N = f(N+1) - \int_N^{N+1} f(x)dx \leq 0.$$

Thus $\{a_N\}$ is decreasing and bounded below by 0, so it converges. □

Example: Let $f(x) = \frac{1}{x}$. Then

$$\gamma = \lim_{N\to\infty} \sum_{n=1}^{N} f(n) - \int_1^N f(x)dx$$

is the **Euler-Mascheroni constant**. In the notation of the proof one has $a_1 = 1 > a_2 = 1 + 1/2 - \log 2 \approx 0.806 > a_3 = 1 + 1/2 + 1/3 - \log 3 \approx 0.734$, and so forth. My laptop computer took a couple of minutes to calculate (by sheer brute force) that

$$a_{5\times 10^4} = 0.5772256648681995272745120903\ldots$$

This shows well the limits of brute force calculation even with modern computing power since this is correct only to the first nine decimal places: in fact the tenth decimal digit of γ is 9. In fact in 1736 Euler correctly calculated the first 15 decimal digits of γ, whereas in 1790 Lorenzo Mascheroni correctly calculated the first 19 decimal digits (while incorrectly calculating several more). As of 2009, 29,844,489,545 digits of γ have been computed, by Alexander J. Yee and Raymond Chan.

The constant γ in fact plays a prominent role in classical analysis and number theory: it tends to show in asymptotic formulas in the darndest places. For instance, for a positive integer n, let $\varphi(n)$ be the number of integers k with $1 \leq k \leq n$ such that no prime number p simultaneously divides both k and n. (The classical name for φ is the **totient function**, but nowadays most people seem to call it the "Euler phi function".) It is not so hard to see that $\lim_{n\to\infty} \varphi(n)$, but function is somewhat irregular (i.e., far from being monotone) and it is of great interest to give precise lower bounds. The best lower bound I know is that for all $n > 2$,

$$\varphi(n) > \frac{n}{e^\gamma \log\log n + \frac{3}{\log\log n}}. \tag{13}$$

Note that directly from the definition we have $\varphi(n) \leq n$. On the other hand, taking n to be a product of increasingly many distinct primes, one sees that $\liminf_{n\to\infty} \frac{\varphi(n)}{n} = 0$, i.e., φ cannot be bounded below by Cn for any positive constant n. Given these two facts, (13) shows that the discrepancy between $\varphi(n)$ and n is very subtle indeed.

Remark: Whether γ is a rational or irrational number is an open question. It would be interesting to know if there are any ir/rationality results on other Euler constants γ_f.

4.6. Stirling's Formula.

5. Series With Non-Negative Terms III: Ratios and Roots

We continue our analysis of series $\sum_n a_n$ with $a_n \geq 0$ for all n. In this section we introduce two important tests based on a very simple – yet powerful – idea: if for sufficiently large n a_n is bounded above by a non-negative constant M times r^n for $0 \leq r < 1$, then the series converges by comparison to the convergent geometric series $\sum_n Mr^n$. Conversely, if for sufficiently large n a_n is bounded below by a positive constant M times r^n for $r \geq 1$, then the series diverges by comparison to the divergent geometric series $\sum_n Mr^n$.

5.1. The Ratio Test.

Theorem 2.24. *(Ratio Test) Let $\sum_n a_n$ be a series with $a_n > 0$ for all n.*
a) Suppose there exists $N \in \mathbb{Z}^+$ and $0 < r < 1$ such that for all $n \geq N$, $\frac{a_{n+1}}{a_n} \leq r$. Then the series $\sum_n a_n$ converges.
b) Suppose there exists $N \in \mathbb{Z}^+$ and $r \geq 1$ such that for all $n \geq N$, $\frac{a_{n+1}}{a_n} \geq r$. Then the series $\sum_n a_n$ diverges.
c) The hypothesis of part a) holds if $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and is less than 1.
d) The hypothesis of part b) holds if $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and is greater than 1.

Proof. a) Our assumption is that for all $n \geq N$, $\frac{a_{n+1}}{a_n} \leq r < 1$. Then $\frac{a_{n+2}}{a_n} = \frac{a_{n+2}}{a_{n+1}} \frac{a_{n+1}}{a_n} \leq r^2$. An easy induction argument shows that for all $k \in \mathbb{N}$,

$$\frac{a_{N+k}}{a_N} \leq r^k,$$

so

$$a_{N+k} \leq a_N r^k.$$

Summing these inequalities gives

$$\sum_{k=N}^{\infty} a_k = \sum_{k=0}^{\infty} a_{N+k} \leq \sum_{k=0}^{\infty} a_N r^k < \infty,$$

so the series $\sum_n a_n$ converges by comparison.
b) Similarly, our assumption is that for all $n \geq N$, $\frac{a_{n+1}}{a_n} \geq r \geq 1$. As above, it follows that for all $k \in \mathbb{N}$,

$$\frac{a_{N+k}}{a_N} \geq r^k,$$

so

$$a_{N+k} \geq a_N r^k \geq a_N > 0.$$

It follows that $a_n \nrightarrow 0$, so the series diverges by the Nth Term Test.
We leave the proofs of parts c) and d) as exercises. □

Exercise 2.24. *Prove parts c) and d) of Theorem 2.24.*

Example: Let $x > 0$. We will show that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges. (Recall we showed this earlier when $x = 1$.) We consider the quantity

$$\frac{a_{n+1}}{a_n} = \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x}{n+1}.$$

It follows that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$. Thus the series converges for any $x > 0$.

5.2. The Root Test.

In this section we give a variant of the Ratio Test. Instead of focusing on the property that the geometric series $\sum_n r^n$ has constant ratios of consecutive terms, we observe that the sequence has the property that the nth root of the $n$rth term is equal to r. Suppose now that $\sum_n$ is a series with non-negative terms with the property that $a_n^{\frac{1}{n}} \leq r$ for some $r < 1$. Raising both sides to the nth power gives $a_n \leq r^n$, and once again we find that the series converges by comparison to a geometric series.

THEOREM 2.25. *(Root Test) Let $\sum_n a_n$ be a series with $a_n \geq 0$ for all n.*
a) Suppose there exists $N \in \mathbb{Z}^+$ and $0 < r < 1$ such that for all $n \geq N$, $a_n^{\frac{1}{n}} \leq r$. Then the series $\sum_n a_n$ converges.
b) Suppose that for infinitely many positive integers n we have $a_n^{\frac{1}{n}} \geq 1$. Then the series $\sum_n a_n$ diverges.
c) The hypothesis of part a) holds if $\rho = \lim_{n\to\infty} a_n^{\frac{1}{n}}$ exists and is less than 1.
d) The hypothesis of part b) holds if $\rho = \lim_{n\to\infty} a_n^{\frac{1}{n}}$ exists and is greater than 1.

EXERCISE 2.25. *Prove Theorem 2.25.*

5.3. Ratios versus Roots.

It is a fact – somewhat folkloric among calculus students – that the Root Test is *stronger* than the Ratio Test. That is, whenever the ratio test succeeds in determining the convergence or divergence of an infinite series, the root test will also succeed.

In order to explain this result we need to make use of the limit infimum and limit supremum. First we recast the ratio and root tests in those terms.

EXERCISE 2.26. *Let $\sum_n a_n$ be a series with positive terms. Put*

$$\underline{\rho} = \liminf_{n\to\infty} \frac{a_{n+1}}{a_n},\ \overline{\rho} = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

a) Show that if $\overline{\rho} < 1$, the series $\sum_n a_n$ converges.
b) Show that if $\underline{\rho} > 1$ the series $\sum_n a_n$ diverges.

EXERCISE 2.27. *Let $\sum_n a_n$ be a series with non-negative terms. Put*

$$\overline{\theta} = \limsup_{n\to\infty} a_n^{\frac{1}{n}}.$$

a) Show that if $\overline{\theta} < 1$, the series $\sum_n a_n$ converges.
b) Show that if $\overline{\theta} > 1$, the series $\sum_n a_n$ diverges.[13]

EXERCISE 2.28. *Consider the following conditions on a real sequence $\{x_n\}_{n=1}^{\infty}$:*
(i) $\limsup_{n\to\infty} x_n > 1$.
(ii) For infinitely many n, $x_n \geq 1$.
(iii) $\limsup_{n\to\infty} x_n \geq 1$.
a) Show that (i) $\implies$ (ii) $\implies$ (iii) and that neither implication can be reversed.
b) Explain why the result of part b) of the previous Exercise is weaker than part b)

[13]This is not a typo: we really mean the limsup both times, unlike in the previous exercise.

of Theorem 2.25.
c) Give an example of a non-negative series $\sum_n a_n$ *with* $\overline{\theta} = \limsup_{n\to\infty} a_n^{\frac{1}{n}} = 1$ *such that* $\sum_n a_n = \infty$.

EXERCISE 2.29. *Let A and B be real numbers with the following property: for any real number r, if $A < r$ then $B \leq r$. Show that $B \leq A$.*

PROPOSITION 2.26. *For any series* $\sum_n a_n$ *with positive terms, we have*

$$\underline{\rho} = \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \underline{\theta} = \liminf_{n\to\infty} a_n^{\frac{1}{n}} \leq \overline{\theta} = \limsup_{n\to\infty} a_n^{\frac{1}{n}} \leq \overline{\rho} = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

PROOF. Step 1: Since for any sequence $\{x_n\}$ we have $\liminf x_n \leq \limsup x_n$, we certainly have $\underline{\theta} \leq \overline{\theta}$.
Step 2: We show that $\overline{\theta} \leq \overline{\rho}$. For this, suppose $r > \overline{\rho}$, so that for all sufficiently large n, $\frac{a_{n+1}}{a_n} \leq r$. As in the proof of the Ratio Test, we have $a_{n+k} < r^k a_n$ for all $k \in \mathbb{N}$. We may rewrite this as

$$a_{n+k} < r^{n+k}\left(\frac{a_n}{r^n}\right),$$

or

$$a_{n+k}^{\frac{1}{n+k}} < r\left(\frac{a_n}{r^n}\right)^{\frac{1}{n+k}}.$$

Now

$$\overline{\theta} = \limsup_{n\to\infty} a_n^{\frac{1}{n}} = \limsup_{k\to\infty} a_{n+k}^{\frac{1}{k}} \leq \limsup_{k\to\infty} r\left(\frac{a_n}{r^n}\right)^{\frac{1}{n+k}} = r.$$

By the preceding exercise, we conclude $\overline{\theta} \leq \overline{\rho}$.
Step 3: We must show that $\underline{\rho} \leq \underline{\theta}$. This is very similar to the argument of Step 2, and we leave it as an exercise. □

EXERCISE 2.30. *Give the details of Step 3 in the proof of Proposition 2.26.*

Now let $\sum_n a_n$ be a series which the Ratio Test succeeds in showing is convergent: that is, $\overline{\rho} < 1$. Then by Proposition 2.26, we have $\overline{\theta} \leq \overline{\rho} \leq 1$, so the Root Test also shows that the series is convegent. Now suppose that the Ratio Test succeeds in showing that the series is divergent: that is $\underline{\rho} > 1$. Then $\overline{\theta} \geq \underline{\theta} \geq \underline{\rho} > 1$, so the Root Test also shows that the series is divergent.

EXERCISE 2.31. *Consider the series* $\sum_n 2^{-n+(-1)^n}$.
a) Show that $\underline{\rho} = \frac{1}{8}$ *and* $\overline{\rho} = 2$, *so the Ratio Test fails.*
b) Show that $\underline{\theta} = \overline{\theta} = \frac{1}{2}$, *so the Root Test shows that the series converges.*

EXERCISE 2.32. *Construct further examples of series for which the Ratio Test fails but the Root Test succeeds to show either convergence or divergence.*

Warning: The sense in which the Root Test is stronger than the Ratio Test is a theoretical one. For a given relatively benign series, it may well be the case that the Ratio Test is *easier to apply* than the Root Test, even though in theory whenever the Ratio Test works the Root Test must also work.

Example: Consider again the series $\sum_{n=0}^{\infty} \frac{1}{n!}$. In the presence of factorials one should always attempt the Ratio Test first. Indeed

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Thus the Ratio Test *limit* exists (no need for liminfs or limsups) and is equal to 0, so the series converges. If instead we tried the Root Test we would have to evaluate $\lim_{n\to\infty}(\frac{1}{n!})^{\frac{1}{n}}$. This is not so bad if we keep our head – e.g. one can show that for any fixed $R > 0$ and sufficiently large n, $n! > R^n$ and thus $(\frac{1}{n!})^{\frac{1}{n}} \leq (\frac{1}{R^n})^{\frac{1}{n}} = \frac{1}{R}$. Thus the root test limit is at most $\frac{1}{R}$ for any positive R, so it is 0. But this is elaborate compared to the Ratio Test computation, which was immediate. In fact, turning these ideas around, Proposition 2.26 can be put to the following sneaky use.

COROLLARY 2.27. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Assume that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \to L \in [0,\infty]$. Then also $\lim_{n\to\infty} a_n^{\frac{1}{n}} = L$.*

PROOF. Indeed, the hypothesis gives that for the infinite series $\sum_n a_n$ we have $\rho = L$, so by Proposition 2.26 we must also have $\theta = L$![14] $\square$

EXERCISE 2.33. *Use Corollary 2.27 to evaluate the following limits:*
a) $\lim_{n\to\infty} n^{\frac{1}{n}}$.
b) For $\alpha \in \mathbb{R}$, $\lim_{n\to\infty} n^{\frac{\alpha}{n}}$.
c) $\lim_{n\to\infty}(n!)^{\frac{1}{n}}$.

5.4. Remarks.

Parts a) and b) of the Ratio Test (Theorem 2.24) are due to d'Alembert.[15] They imply parts c) and d), which is the version of the Ratio Test one meets in calculus, in which the limit is assumed to exist. As we saw above, it is equivalent to express d'Alembert's Ratio Test in terms of lim sups and lim infs: this is the approach commonly taken in contemporary undergraduate analysis texts. However, our decision to suppress the lim sups and lim infs was clinched by reading the treatment of these tests in Hardy's[16] classic text [**H**]. (He doesn't include the lim infs and lim sups, his treatment is optimal, and his analytic pedigree is beyond reproach.)

Similarly, parts a) and b) of our Root Test (Theorem 2.25) are due to Cauchy. The reader can by now appreciate why calling it "Cauchy's Test" would not be a good idea. Similar remarks about the versions with lim sups apply, except that this time, the result in part b) is actually stronger than the statement involving a lim sup, as Exercise XX explores. Our statement of this part of the Root Test is taken from [**H**, §174].

6. Series With Non-Negative Terms IV: Still More Convergence Tests

If you are taking or teaching a first course on infinite series / undergraduate analysis, I advise you to skip this section! It is rather for those who have seen the basic convergence tests of the preceding sections many times over and who have started to wonder what lies beyond.

[14] There is something decidedly strange about this argument: to show something about a sequence $\{a_n\}$ we reason in terms of the corresponding infinite series $\sum_n a_n$. But it works!

[15] Jean le Rond d'Alembert, 1717-1783

[16] Godfrey Harold Hardy, 1877-1947

6.1. The Ratio-Comparison Test.

THEOREM 2.28. *(Ratio-Comparison Test) Let $\sum_n a_n$, $\sum_n b_n$ be two series with positive terms. Suppose that there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$ we have*

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}.$$

Then:
a) If $\sum_n b_n$ converges, then $\sum_n a_n$ converges.
b) If $\sum_n a_n$ diverges, then $\sum_n b_n$ converges.

PROOF. As in the proof of the usual Ratio Test, for all $k \in \mathbb{N}$,

$$\frac{a_{N+k}}{a_N} = \frac{a_{N+k}}{a_{N+k-1}} \frac{a_{N+k-1}}{a_{N+k-2}} \cdots \frac{a_{N+1}}{a_N} \leq \frac{b_{N+k}}{b_{N+k-1}} \frac{b_{N+k-1}}{b_{N+k-2}} \cdots \frac{b_{N+1}}{b_N} = \frac{b_{N+k}}{b_N},$$

so

$$a_{N+k} \leq \left(\frac{a_N}{b_N}\right) b_{N+k}.$$

Now both parts follow by the Comparison Test. □

Warning: Some people use the term "Ratio Comparison Test" for what we (and many others) have called the Limit Comparison Test.

EXERCISE 2.34. *State and prove a **Root-Comparison Test**.*

If we apply Theorem 2.28 by taking $b_n = r^n$ for $r < 1$ and then $a_n = r^n$ for $r > 1$, we get the usual Ratio Test. It is an appealing idea to choose other series, or families of series, plug them into Theorem 2.28, and see whether we get other useful tests.

After the geometric series, perhaps our next favorite class of series with non-negative terms are the p-series $\sum_n \frac{1}{n^p}$. Let us first take $p > 1$, so that the p-series converges, and put $b_n = \frac{1}{n^p}$. Then our sufficient condition for the convergence of a series $\sum_n a_n$ with positive terms is

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} = \frac{(n+1)^{-p}}{n^{-p}} = \left(\frac{n}{n+1}\right)^p = \left(1 - \frac{1}{n+1}\right)^p.$$

Well, we have something here, but what? As is often the case for inequalities, it is helpful to weaken it a bit to get something that is simpler and easier to apply (and then perhaps come back later and try to squeeze more out of it).

LEMMA 2.29. *Let $p > 0$ and $x \in [0,1]$. Then :*
a) If $p > 1$, then $1 - px \leq (1-x)^p$.
b) If $p < 1$, then $1 - px \geq (1-x)^p$.

PROOF. For $x \in [0,1]$, put $f(x) = (1-x)^p - (1-px)$. Then f is continuous on $[0,1]$, differentiable on $[0,1)$ with $f(0) = 0$. Moreover, for $x \in [0,1)$ we have $f'(x) = p(1-(1-x)^{p-1})$. Thus:
a) If $p > 1$, $f'(x) \geq 0$ for all $x \in [0,1)$, so f is inceasing on $[0,1]$. Since $f(0) = 0$, $f(x) \geq 0$ for all $x \in [0,1]$.
b) If $0 < p < 1$, $f'(x) \leq 0$ for all $x \in [0,1)$, so f is decreasing on $[0,1]$. Since $f(0) = 0$, $f(x) \leq 0$ for all $x \in [0,1]$. □

Remark: Although it was no trouble to prove Lemma 2.29, we should not neglect the following question: how does one figure out that such an inequality should exist? The answer comes from the Taylor series expansion $(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\ldots$ – the point being that the last term is positive when $\alpha > 1$ and negative when $0 < \alpha < 1$ – which will be discussed in the following chapter.

We are now in the position to derive a classic test due to J.L. Raabe.[17]

THEOREM 2.30. *(Raabe's Test)*
a) Let $\sum_n a_n$ be a series with positive terms. Suppose that for some $p > 1$ and all sufficiently large n we have $\frac{a_{n+1}}{a_n} \leq 1 - \frac{p}{n}$. Then $\sum_n a_n < \infty$.
b) Let $\sum_n b_n$ be a series with positive terms. Suppose that for some $0 < p < 1$ and all sufficiently large n we have $1 - \frac{p}{n} \leq \frac{b_{n+1}}{b_n}$. Then $\sum_n b_n = \infty$.

PROOF. a) We have $\frac{a_{n+1}}{a_n} \leq 1-\frac{p}{n} \leq 1-\frac{p}{n+1}$ for all sufficiently large n. Applying Lemma 2.29 with $x = \frac{1}{n+1}$ gives

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{p}{n+1} \leq \left(1 - \frac{1}{n+1}\right)^p = \frac{b_{n+1}}{b_n},$$

where $b_n = \frac{1}{n^p}$. Since $\sum_n b_n < \infty$, $\sum_n a_n < \infty$ by the Ratio-Comparison Test.
b) The proof is close enough to that of part a) to be left to the reader. □

EXERCISE 2.35. *Prove Theorem 2.30b).*

Theorem 2.30b) can be strengthened, as follows.

THEOREM 2.31. *Let $\sum_n b_n$ be a series with positive terms such that for all sufficiently large n, $b_n \geq 1 - \frac{1}{n}$. Then $\sum_n b_n = \infty$.*

PROOF. We give two proofs.
First proof: Let $a_n = \frac{1}{n}$. Then $\frac{a_{n+1}}{a_n} = 1 - \frac{1}{n}$ and $\sum_n a_n = \sum_n \frac{1}{n} = \infty$, so the result follows from the Ratio-Comparison Test.
Second proof: We may assume $\frac{b_{n+1}}{b_n} \geq 1 - \frac{1}{n}$ for all $n \in \mathbb{Z}^+$. This is equivalent to $(n-1)b_n \leq nb_{n+1}$ for all $n \in \mathbb{Z}^+$, i.e., the sequence nb_{n+1} is increasing and in particular $nb_{n+1} \geq b_2 > 0$. Thus $b_{n+1} \geq \frac{b_2}{n}$ and the series $\sum_n b_{n+1}$ diverges by comparison to the harmonic series. Of course this means $\sum_n b_n = \infty$ as well. □

EXERCISE 2.36. *Show that Theorem 2.31 implies Theorem 2.30b).*

Remark: We have just seen a proof of (a stronger form of) Theorem 2.30b) which avoids the Ratio-Comparison Test. Similarly it is not difficult to give a reasonably short, direct proof of Theorem 2.30a) depending on nothing more elaborate than basic comparison. In this way one gets an independent proof of the result that the p-series $\sum_n \frac{1}{n^p}$ diverges for $p < 1$ and converges for $p > 1$. However we *do not* get a new proof of the $p = 1$ case, i.e., the divergence of the harmonic series (notice that this was *used*, not proved, in the proof of Theorem 2.31). Ratio-like tests have a lot of trouble with the harmonic series!

The merit of the present approach is rather that it puts Raabe's test into a larger context rather than just having it be one more test among many. Our treatment of Raabe's test is influenced by [**Fa03**].

[17] Joseph Ludwig Raabe, 1801-1859

6.2. Gauss's Test.

6.3. Kummer's Test.

6.4. Hypergeometric Series.

7. Absolute Convergence

7.1. Introduction to absolute convergence.

We turn now to the serious study of series with both positive and negative terms. It turns out that under one relatively mild additional hypothesis, virtually all of our work on series with non-negative terms can be usefully applied in this case. In this section we study this wonderful hypothesis: absolute convergence. (In the next section we get *really* serious by studying series when we do not have absolute convergence. As the reader will see, this leads to surprisingly delicate and intricate considerations: in practice, we very much hope that our series are absolutely convergent!)

A real series $\sum_n a_n$ is **absolutely convergent** if $\sum_n |a_n|$ converges. Since $\sum_n |a_n|$ is a series with non-negative terms, to determine absolute convergence we may use all the tools of the last three sections. A series $\sum_n a_n$ which converges but for which $\sum_n |a_n|$ diverges is said to be **nonabsolutely convergent**.[18]

The terminology *absolutely convergent* suggests that the convergence of the series $\sum_n |a_n|$ is somehow "better" than the convergence of the series $\sum_n a_n$. This is indeed the case, although it is not obvious. But the following result already clarifies matters a great deal.

Proposition 2.32. *Every absolutely convergent series is convergent.*

Proof. We shall give *two proofs* of this important result.
First Proof: Consider the three series $\sum_n a_n$, $\sum_n |a_n|$ and $\sum_n a_n + |a_n|$. Our hypothesis is that $\sum_n |a_n|$ converges. But we claim that this implies that $\sum_n a_n + |a_n|$ converges as well. Indeed, consider the expression $a_n + |a_n|$: it is equal to $2a_n = 2|a_n|$ when a_n is non-negative and 0 when a_n is negative. In particular the series $\sum_n a_n + |a_n|$ has non-negative terms and $\sum_n a_n + |a_n| \leq \sum_n 2|a_n| < \infty$. So $\sum_n a_n + |a_n|$ converges. By the Three Series Principle, $\sum_n a_n$ converges.
Second Proof: The above argument is clever – maybe too clever! Let's try something a little more fundamental: since $\sum_n |a_n|$ converges, for every $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $\sum_{n=N}^{\infty} |a_n| < \epsilon$. Therefore

$$|\sum_{n=N}^{\infty} a_n| \leq \sum_{n=N}^{\infty} |a_n| < \epsilon,$$

and $\sum_n a_n$ converges by the Cauchy criterion. □

[18] We warn the reader that the more standard terminology is **conditionally convergent**. We will later on give a separate definition for "conditionally convergent" and then it will be a *theorem* that a real series is conditionally convergent iff it is nonabsolutely convergent. The reasoning for this – which we admit will seem abstruse at best to our target audience – is that in functional analysis one studies convergence and absolute convergence of series in a more general context, such that nonabsolute converge and conditional convergence may indeed differ.

THEOREM 2.33. *(N.J. Diepeveen) For an ordered field $(F, <)$, the following are equivalent:*
(i) F is Cauchy-complete: every Cauchy sequence converges.
(ii) Every absolutely convergent series in F is convergent.

PROOF. (i) $\implies$ (ii): This was done in the Second Proof of Proposition 2.32. (ii) $\implies$ (i): Let $\{a_n\}$ be a Cauchy sequence in F. A Cauchy sequence is convergent iff it admits a convergent subsequence, so it is enough to show that $\{a_n\}$ admits a convergent subsequence. Like any sequence in an ordered set, $\{a_n\}$ admits a subsequence which is either constant, strictly increasing or strictly decreasing. A constant sequence is convergent, so we need not consider this case; moreover, a strictly decreasing sequence $\{a_n\}$ converges iff the strictly increasing sequence $\{-a_n\}$ converges, so we have reduced to the case in which $\{a_n\}$ is strictly increasing:

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots.$$

For $n \in \mathbb{Z}^+$, put

$$b_n = a_{n+1} - a_n.$$

Then $b_n > 0$, and since $\{a_n\}$ is Cauchy, $b_n \to 0$; thus we can extract a strictly *decreasing* subsequence $\{b_{n_k}\}$. For $k \in \mathbb{Z}^+$, put

$$c_k = b_{n_k} - b_{n_{k+1}}.$$

Then $c_k > 0$ for all k and

$$\sum_{k=1}^{\infty} c_k = b_{n_1}.$$

Since a_n is Cauchy, there exists for each k a positive integer m_k such that

$$0 < a_{m_{k+1}} - a_{m_k} < c_k.$$

For $k \in \mathbb{Z}^+$, put

$$d_{2k-1} = a_{m_{k+1}} - a_{m_k},\ d_{2k} = a_{m_{k+1}} - a_{m_k} - c_k.$$

We have

$$-c_k < d_{2k} < 0 < d_{2k-1} < c_k,$$

so

$$\sum_{i=1}^{\infty} |d_i| = \sum_{k=1}^{\infty} d_{2k-1} - d_{2k} = \sum_{k=1}^{\infty} c_k = b_{n_1}.$$

At last we get to use our hypothesis that absolutely convergent series are convergent: it follows that $\sum_{i=1}^{\infty} d_i$ converges and

$$\sum_{i=1}^{\infty} d_i + \sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} (d_{2k-1} + d_{2k} + c_k) = 2\left(\sum_{k=1}^{\infty} a_{m_k+1} - a_{m_k}\right).$$

Since, again, a Cauchy sequence with a convergent subsequence is itself convergent,

$$\lim_{n\to\infty} a_n = a_{m_1} + \sum_{k=1}^{\infty} (a_{m_{k+1}} - a_{m_k}) = a_{m_1} + \frac{1}{2}\left(b_{n_1} + \sum_{i=1}^{\infty} d_i\right).$$

□

For a more leisurely presentation of Theorem 2.33 and related topics, see [**CD15**].

Exercise 2.37. *Find a sequence* $\{a_n\}_{n=1}^{\infty}$ *of rational numbers such that* $\sum_{n=1}^{\infty} |a_n|$ *is a rational number but* $\sum_{n=1}^{\infty} a_n$ *is an irrational number.*

As an example of how Theorem 2.32 may be combined with the previous tests to give tests for absolute convergence, we record the following result.

Theorem 2.34. *(Ratio & Root Tests for Absolute Convergence) Let* $\sum_n a_n$ *be a real series.*
a) Assume $a_n \neq 0$ *for all* n. *If there exists* $0 \leq r < 1$ *such that for all sufficiently large* n, $|\frac{a_{n+1}}{a_n}| \leq r$, *then the series* $\sum_n a_n$ *is absolutely convergent.*
b) Assume $a_n \neq 0$ *for all* n. *If there exists* $r > 1$ *such that for all sufficiently large* n, $|\frac{a_{n+1}}{a_n}| \geq r$, *the series* $\sum_n a_n$ *is divergent.*
c) If there exists $r < 1$ *such that for all sufficiently large* n, $|a_n|^{\frac{1}{n}} \leq r$, *the series* $\sum_n a_n$ *is absolutely convergent.*
d) If there are infinitely many n *for which* $|a_n|^{\frac{1}{n}} \geq 1$, *then the series diverges.*

Proof. Parts a) and c) are immediate: applying Theorem 2.24 (resp. Theorem 2.25) we find that $\sum_n |a_n|$ is convergent – and the point is that by Theorem 2.32, this implies that $\sum_n a_n$ is convergent.

There is something to say in parts b) and d), because in general just because $\sum_n |a_n| = \infty$ does not imply that $\sum_n a_n$ diverges. (We will study this subtlety later on in detail.) But recall that whenever the Ratio or Root tests establish the divergence of a non-negative series $\sum_n b_n$, they do so by showing that $b_n \nrightarrow 0$. Thus under the hypotheses of parts b) and d) we have $|a_n| \nrightarrow 0$, hence also $a_n \nrightarrow 0$ so $\sum_n a_n$ diverges by the Nth Term Test (Theorem 2.5). $\square$

In particular, for a real series $\sum_n a_n$ define the following quantities:

$$\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| \text{ when it exists,}$$
$$\underline{\rho} = \liminf_{n\to\infty} |\frac{a_{n+1}}{a_n}|,$$
$$\overline{\rho} = \limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}|,$$
$$\theta = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \text{ when it exists,}$$
$$\overline{\theta} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}},$$

and then all previous material on Ratio and Root Tests applies to all real series.

7.2. Cauchy products II: when one series is absolutely convergent.

Theorem 2.35. *Let* $\sum_{n=0}^{\infty} a_n = A$ *and* $\sum_{n=0}^{\infty} b_n = B$ *be two absolutely convergent series, and let* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. *Then the Cauchy product series* $\sum_{n=0}^{\infty} c_n$ *is absolutely convergent, with sum* AB.

Proof. We have proved this result already when $a_n, b_n \geq 0$ for all n. We wish, of course, to reduce to that case. As far as the convergence of the Cauchy product, this is completely straightforward: we have

$$\sum_{n=0}^{\infty} |c_n| = \sum_{n=0}^{\infty} |\sum_{k=0}^{n} a_k b_{n-k}| \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k||b_{n-k}| < \infty,$$

the last inequality following from the fact that $\sum_{n=0}^{\infty}\sum_{k=0}^{n}|a_k||b_{n-k}|$ is the Cauchy product of the two non-negative series $\sum_{n=0}^{\infty}|a_n|$ and $\sum_{n=0}^{\infty}|b_n|$, hence it converges. Therefore $\sum_n |c_n|$ converges by comparison, so the Cauchy product series $\sum_n c_n$ converges.

We now wish to show that $\lim_{N\to\infty} C_N = \sum_{n=0}^{\infty} c_n = AB$. Recall the notation

$$\square_N = \sum_{0\leq i,j\leq N} a_i b_j = (a_0+\ldots+a_N)(b_0+\ldots+b_N) = A_N B_N.$$

We have

$$|C_N - AB| \leq |\square_N - AB| + |\square_N - C_N|$$
$$= |A_N B_N - AB| + |a_1 b_N| + |a_2 b_{N-1}| + |a_2 b_N| + \ldots + |a_N b_1| + \ \ldots + |a_N b_N|$$
$$\leq |A_N B_N - AB| + \left(\sum_{n=0}^{\infty}|a_n|\right)\left(\sum_{n\geq N}|b_n|\right) + \left(\sum_{n=0}^{\infty}|b_n|\right)\left(\sum_{n\geq N}|a_n|\right).$$

Fix $\epsilon > 0$; since $A_N B_N \to AB$, for sufficiently large N $|A_N B_N - AB| < \frac{\epsilon}{3}$. Put

$$\mathbb{A} = \sum_{n=0}^{\infty}|a_n|,\ \mathbb{B} = \sum_{n=0}^{\infty}|b_n|.$$

By the Cauchy criterion, for sufficiently large N we have $\sum_{n\geq N}|b_n| < \frac{\epsilon}{3\mathbb{A}}$ and $\sum_{n\geq N}|a_n| < \frac{\epsilon}{3\mathbb{B}}$ and thus $|C_N - AB| < \epsilon$. $\square$

While the proof of Theorem 2.35 may seem rather long, it is in fact a rather straightforward argument: one shows that the difference between the box product and the partial sums of the Cauchy product becomes negligible as N tends to infinity. In less space but with a bit more finesse, one can prove the following stronger result, a theorem of F. Mertens [**Me72**].[19]

THEOREM 2.36. *(Mertens' Theorem) Let $\sum_{n=0}^{\infty} a_n = A$ be an absolutely convegent series and $\sum_{n=0}^{\infty} b_n = B$ be a convergent series. Then the Cauchy product series $\sum_{n=0}^{\infty} c_n$ converges to AB.*

PROOF. (Rudin [**R**, Thm. 3.50]): define (as usual)

$$A_N = \sum_{n=0}^{N} a_n,\ B_N = \sum_{n=0}^{N} b_n,\ C_N = \sum_{n=0}^{N} c_n$$

and also (for the first time)

$$\beta_n = B_n - B.$$

Then for all $N \in \mathbb{N}$,

$$C_N = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \ldots + (a_0 b_N + \ldots + a_N b_0)$$
$$= a_0 B_N + a_1 B_{N-1} + \ldots + a_N B_0$$
$$= a_0(B+\beta_N) + a_1(B+\beta_{N-1}) + \ldots + a_N(B+\beta_0)$$
$$= A_N B + a_0\beta_N + a_1\beta_{N-1} + \ldots + a_N\beta_0 = A_N B + \gamma_N,$$

say, where $\gamma_N = a_0\beta_N + a_1\beta_{N-1} + \ldots + a_n\beta_0$. Since our goal is to show that $C_N \to AB$ and we know that $A_N B \to AB$, it suffices to show that $\gamma_N \to 0$. Now,

[19] Franz Carl Joseph Mertens, 1840-1927

put $\alpha = \sum_{n=0}^{\infty} |a_n|$. Since $B_N \to B$, $\beta_N \to 0$, and thus for any $\epsilon > 0$ we may choose $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ we have $|\beta_n| \leq \frac{\epsilon}{2\alpha}$. Put

$$M = \max_{0 \leq n \leq N_0} |\beta_n|.$$

By the Cauchy criterion, for all sufficiently large N, $M \sum_{n \geq N - N_0} |a_n| \leq \epsilon/2$. Then

$$|\gamma_N| \leq |\beta_0 a_N + \ldots + \beta_{N_0} a_{N-N_0}| + |\beta_{N_0+1} a_{N-N_0-1} + \ldots + \beta_N a_0|$$

$$\leq |\beta_0 a_N + \ldots + \beta_{N_0} a_{N-N_0}| + \frac{\epsilon}{2} \leq M \left(\sum_{n \geq N-N_0} |a_n| \right) + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

8. Non-Absolute Convergence

We say that a real seris $\sum_n a_n$ is **nonabsolutely convergent** if the series converges but $\sum_n |a_n|$ diverges, thus if it is convergent but not absolutely convergent.[20]

A series which is nonabsolutely convergent is a more delicate creature than any we have studied thus far. A test which can show that a series is convergent but nonabsolutely convergent is necessarily subtler than those of the previous section. In fact the typical undergraduate student of calculus / analysis learns exactly one such test, which we give in the next section.

8.1. The Alternating Series Test.

Consider the **alternating harmonic series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots.$$

Upon taking the absolute value of every term we get the usual harmonic series, which diverges, so the alternating harmonic series is *not* absolutely convergent. However, some computations with partial sums suggests that the alternating harmonic series *is* convergent, with sum $\log 2$. By looking more carefully at the partial sums, we can find a pattern that allows us to show that the series does indeed converge. (Whether it converges to $\log 2$ is a different matter, of course, which we will revisit much later on.)

It will be convenient to write $a_n = \frac{1}{n}$, so that the alternating harmonic series is $\sum_n \frac{(-1)^{n+1}}{n+1}$. We draw the reader's attention to three properties of this series:

(AST1) The terms alternate in sign.
(AST2) The nth term approaches 0.
(AST3) The sequence of absolute values of the terms is decreasing:

$$a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots.$$

[20] One therefore has to distinguish between the phrases "not absolutely convergent" and "nonabsolutely convergent": the former allows the possibility that the series is divergent, whereas the latter does not. In fact our terminology here is not completely standard. We defend ourselves grammatically: "nonabsolutely" is an adverb, so it must modify "convergent", i.e., it describes *how* the series converges.

These are the clues from which we will make our case for convergence. Here it is: consider the process of passing from the first partial sum $S_1 = 1$ to $S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. We have $S_3 \leq 1$, and this is no accident: since $a_2 \geq a_3$, subtracting a_2 and then adding a_3 leaves us no larger than where we started. But indeed this argument is valid in passing from any S_{2n-1} to S_{2n+1}:

$$S_{2n+1} = S_{2n-1} - a_{2n} + a_{2n+1} \leq S_{2n+1}.$$

It follows that the sequence of odd-numbered partial sums $\{S_{2n-1}\}$ is decreasing. Moreover,

$$S_{2n+1} = (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2n-1}| - |a_{2n}) + a_{2n-1} \geq 0.$$

Therefore all the odd-numbered terms are bounded below by 0. By the Monotone Sequence Lemma, the sequence $\{S_{2n+1}\}$ converges to its greatest lower bound, say S_{odd}. On the other hand, just the opposite sort of thing happens for the even-numbered partial sums:

$$S_{2n+2} = S_{2n} + a_{2n+1} - a_{2n+2} \geq S_{2n}$$

and

$$S_{2n+2} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{2n} - a_{2n+1}|) - a_{2n+2} \leq a_1.$$

Therfore the sequence of even-numbered partial sums $\{S_{2n}\}$ is increasing and bounded above by a_1, so it converges to its least upper bound, say S_{even}. Thus we have split up our sequence of partial sums into two complementary subsequences and found that each of these series converges. By X.X, the full sequence $\{S_n\}$ converges iff $S_{\text{odd}} = S_{\text{even}}$. Now the inequalities

$$S_2 \leq S_4 \leq \ldots \leq S_{2n} \leq S_{2n+1} \leq S_{2n-1} \leq \ldots \leq S_3 \leq S_1$$

show that $S_{\text{even}} \leq S_{\text{odd}}$. Moreover, for any $n \in \mathbb{Z}^+$ we have

$$S_{\text{odd}} - S_{\text{even}} \leq S_{2n+1} - S_{2n} = a_{2n+1}.$$

Since $a_{2n+1} \to 0$, we conclude $S_{\text{odd}} = S_{\text{even}} = S$, i.e., the series converges.

In fact these inequalities give something else: since for all n we have $S_{2n} \leq S_{2n+2} \leq S \leq S_{2n+1}$, it follows that

$$|S - S_{2n}| = S - S_{2n} \leq S_{2n+1} - S_{2n} = a_{2n+1}$$

and similarly

$$|S - S_{2n+1}| = S_{2n+1} - S \leq S_{2n+1} - S_{2n+2} = a_{2n+2}.$$

Thus the error in cutting off the infinite sum $\sum_{n=1}^{\infty} (-1)^{n+1} |a_n|$ after N terms is in absolute value at most the absolute value of the next term: a_{N+1}.

Of course in all this we never used that $a_n = \frac{1}{n}$ but only that we had a series satisfying (AST1) (i.e., an alternating series), (AST2) and (AST3). Therefore the preceding arguments have in fact proved the following more general result, due originally due to Leibniz.[21]

[21]Gottfried Wilhelm von Leibniz, 1646-1716

THEOREM 2.37. *(Lebniz's Alternating Series Test) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers which is decreasing and such that $\lim_{n\to\infty} a_n = 0$. Then:*
a) The associated ***alternating series*** *$\sum_n (-1)^{n+1} a_n$ converges.*
b) For $N \in \mathbb{Z}^+$, put

$$E_N = |(\sum_{n=1}^{\infty}(-1)^{n+1}a_n) - (\sum_{n=1}^{N}(-1)^{n+1}a_n)|, \tag{14}$$

the "error" obtained by cutting off the infinite sum after N terms. Then we have the ***error estimate***

$$E_N \leq a_{N+1}.$$

EXERCISE 2.38. *Let $p \in \mathbb{R}$: Show that the* ***alternating p-series*** *$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ is:*
(i) divergent if $p \leq 0$,
(ii) nonabsolutely convergent if $0 < p \leq 1$, and
(iii) absolutely convergent if $p > 1$.

EXERCISE 2.39. *Let $\frac{P(x)}{Q(x)}$ be a rational function. Give necessary and sufficient conditions for $\sum_n (-1)^n \frac{P(x)}{Q(x)}$ to be nonabsolutely convergent.*

For any convergent series $\sum_{n=1}^{\infty} a_n = S$, we may define E_N as in (14) above:

$$E_N = |S - \sum_{n=1}^{N} a_n|.$$

Then because the series converges to S, $\lim_{N\to\infty} E_N = 0$, and conversely: in other words, to say that the error goes to 0 is a rephrasing of the fact that the partial sums of the series converge to S. Each of these statements is (in the jargon of mathematicians working in this area) **soft**: we assert that a quantity approaches 0 and $N \to \infty$, so that in theory, given any $\epsilon > 0$, we have $E_N < \epsilon$ for all suffuciently large N. But as we have by now seen many times, it is often possible to show that $E_N \to 0$ without coming up with an *explicit* expression for N in terms of ϵ. But this stronger statement is exactly what we have given in Theorem 2.37b): we have given an *explicit* upper bound on E_N as a function of N. This type of statement is called a **hard** statement or an **explicit error estimate**: such statements tend to be more difficult to come by than soft statements, but also more useful to have. Here, as long as we can similarly make explicit how large N has to be in order for a_N to be less than a given $\epsilon > 0$, we get a completely explicit error estimate and can use this to actually compute the sum S to arbitrary accuracy.

Example: We compute $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ to three decimal place accuracy. (Let us agree that to "compute a number α to k decimal place accuracy means to compute it with error less than 10^{-k}. A little thought shows that this is not quite enough to guarantee that the first k decimal places of the approximation are equal to the first k decimal places of α, but we do not want to occupy ourselves with such issues here.) By Theorem 2.37b), it is enough to find an $N \in \mathbb{Z}^+$ such that $a_{N+1} = \frac{1}{N+1} < \frac{1}{1000}$.

We may take $N = 1000$. Therefore

$$|\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n}| \leq \frac{1}{1001}.$$

Using a software package, we find that

$$\sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} = 0.69264743055982030966723105 89\ldots.$$

Again, later we will show that the exact value of the sum is $\log 2$, which my software package tells me is[22]

$$\log 2 = 0.69314718055994530941723212 14.$$

Thus the *actual error* in cutting off the sum after 1000 terms is

$$E_{1000} = 0.0004997500001249997500010625033.$$

It is important to remember that this and other error estimates only give upper bounds on the error: the true error could well be much smaller. In this case we were guaranteed to have an error at most $\frac{1}{1001}$ and we see that the true error is about half of that. Thus the estimate for the error is reasonably accurate.

Note well that although the error estimate of Theorem 2.37b) is very easy to apply, if a_n tends to zero rather slowly (as in this example), it is not especially efficient for computations. For instance, in order to compute the true sum of the alternating harmonic series to six decimal place accuracy using this method, we would need to add up the first million terms: that's a lot of calculation. (Thus please be assured that this is *not* the way that a calculator or computer would compute $\log 2$.)

Example: We compute $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ to six decimal place accuracy. Thus we need to choose N such that $a_{N+1} = \frac{1}{(N+1)!} < 10^{-6}$, or equivalently such that $(N+1)! > 10^6$. A little calculation shows $9! = 362,880$ and $10! = 3,628,800$, so that we may take $N = 9$ (but not $N = 8$). Therefore

$$|\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} - \sum_{n=0}^{9} \frac{(-1)^n}{n!}| < \frac{1}{10!} < 10^{-6}.$$

Using a software package, we find

$$\sum_{n=0}^{9} \frac{(-1)^n}{n!} = 0.3678791887125220458553791887.$$

In this case the exact value of the series is

$$\frac{1}{e} == 0.3678794411714423215955237701$$

so the true error is

$$E_9 = 0.0000002524589202757401445814516374,$$

which this time is only very slightly less than the guaranteed

$$\frac{1}{10!} = 0.0000002755731922398589065255731922.$$

[22] Yes, you should be wondering how it is computing this! More on this later.

8.2. Dirichlet's Test.

What lies beyond the Alternating Series Test? We present one more result, an elegant (and useful) test due originally to Dirichlet.[23]

LEMMA 2.38. *(Summation by Parts) Let $\{a_n\}$ and $\{b_n\}$ be two sequences. Then for all $m \leq n$ we have*

$$\sum_{k=m}^{n} a_k(b_{k+1} - b_k) = (a_{n+1}b_{n+1} - a_m b_m) - \sum_{k=m}^{n} (a_{k+1} - a_k)b_{k+1}.$$

PROOF.

$$\sum_{k=m}^{n} a_k(b_{k+1} - b_k) = a_m b_{m+1} + \ldots + a_n b_{n+1} - (a_m b_m + \ldots + a_n b_n)$$

$$= a_n b_{n+1} - a_m b_m - ((a_{m+1} - a_m)b_{m+1} + \ldots + (a_n - a_{n-1})b_n)$$

$$= a_n b_{n+1} - a_m b_n - \sum_{k=m}^{n-1} (a_{k+1} - a_k)b_{k+1}$$

$$= a_n b_{n+1} - a_m b_n + (a_{n+1} - a_n)b_{n+1} - \sum_{k=m}^{n} (a_{k+1} - a_k)b_{k+1}$$

$$= a_{n+1}b_{n+1} - a_m b_m - \sum_{k=m}^{n} (a_{k+1} - a_k)b_{k+1}.$$ □

Remark: Lemma 2.38 is a discrete analogue of the familiar integration by parts formula from calculus:

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

(This deserves more elaboration than we are able to give at the moment.)

If we take $b_n = (-1)^{n+1}$, then $B_{2n+1} = 1$ for all n and $B_{2n} = 0$ for all n, so $\{b_n\}$ has bounded partial sums. Applying Dirichlet's Test with a sequence a_n which decreases to 0 and with this sequence $\{b_n\}$, we find that the series $\sum_n a_n b_n = \sum_n (-1)^{n+1} a_n$ converges. We have recovered the Alternating Series Test!

THEOREM 2.39. *(Dirichlet's Test) Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be two infinite series. Suppose that:*
(i) The partial sums $B_n = b_1 + \ldots + b_n$ are bounded.
(ii) The sequence a_n is decreasing with $\lim_{n\to\infty} a_n = 0$.
Then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

PROOF. Let $M \in \mathbb{R}$ be such that $|b_n| \leq M$ for all $n \in \mathbb{Z}^+$. Fix $\epsilon > 0$, and choose $N > 1$ such that $a_N < \frac{\epsilon}{2M}$. Then for all $n > m \geq N$, we have
(15)
$$|\sum_{k=m}^{n} a_k b_k| = |\sum_{k=m}^{n} a_k(B_k - B_{k-1})| \overset{\text{SBP}}{=} |a_{n+1}B_n - a_m B_{m-1} - \sum_{k=m}^{n} (a_{k+1} - a_k)B_k|$$

[23] Johann Peter Gustav Lejeune Dirichlet, 1805-1859

$$\leq M\left(a_{n+1} + a_m + \sum_{k=m}^{n}(a_{k+1} - a_k)\right)$$

$$= M\left(a_{n+1} + a_m + a_{n+1} - a_m\right) = 2Ma_{n+1} \leq 2Ma_N < \epsilon.$$

Therefore $\sum_n a_n b_n$ converges by the Cauchy criterion. □

EXERCISE 2.40. *a) Show: Dirichlet's Test implies the Alternating Series Test. b) Show: Dirichlet's Test implies the* ***Almost Alternating Series Test****: let $\{a_n\}$ be a sequence decreasing to 0, and for all $n \in \mathbb{Z}^+$ let $b_n \in \{\pm 1\}$ be a sign sequence which is "almost alternating" in the sense that the sequence of partial sums $B_n = b_1 + \ldots + b_n$ is bounded. Then the series $\sum_n b_n a_n$ converges.*

EXERCISE 2.41. *Show that Dirichlet's generalization of the Alternating Series Test is "as strong as possible" in the following sense: if $\{b_n\}$ is a sequence of elements, each ± 1, such that the sequence of partial sums $B_n = b_1 + \ldots + b_n$ is* unbounded*, then there is a sequence a_n decreasing to zero such that $\sum_n a_n b_n$ diverges.*

EXERCISE 2.42. *a) Use the trigonometric identity*[24]

$$\cos n = \frac{\sin(n + \frac{1}{2}) - \sin(n - \frac{1}{2})}{2\sin(\frac{1}{2})}$$

to show that the sequence $B_n = \cos 1 + \ldots + \cos n$ is bounded.
b) Apply Dirichlet's Test to show that the series $\sum_{n=1}^{\infty} \frac{\cos n}{n}$ converges.
c) Show that $\sum_{n=1}^{\infty} \frac{\cos n}{n}$ is not absolutely convergent.

EXERCISE 2.43. *Show: $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ is nonabsolutely convergent.*

Remark: Once we know about series of complex numbers and Euler's formula $e^{ix} = \cos x + i \sin x$, we will be able to give a "trigonometry-free" proof of the preceding two exercises.

Dirichlet himself applied his test to establish the convergence of a certain class of series of a mixed algebraic and number-theoretic nature. The analytic properties of these series were used to prove his celebrated theorem on prime numbers in arithmetic progressions. To give a sense of how influential this work has become, in modern terminology Dirichlet studied the analytic properties of **Dirichlet series** associated to nontrivial **Dirichlet characters**. For more information on this work, the reader may consult (for instance) [**DS**].

A real sequence $\{a_n\}$ has **bounded variation** if

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty.$$

PROPOSITION 2.40. *If the real sequence $\{a_n\}$ has bounded variation, then it is convergent.*

[24] An instance of the **sum-product identities**. Yes, I hardly remember them either.

PROOF. Fix $\epsilon > 0$, and choose $N \in \mathbb{Z}^+$ such that $\sum_{n \geq N} |a_{n+1} - a_n| < \epsilon$. Then for all $N \leq m \leq n$ we have

$$|a_n - a_m| = |(a_n - a_{n-1}) + \ldots + (a_{m+1} - a_m)| \leq \sum_{i=m}^{n-1} |a_{i+1} - a_i| \leq \sum_{n \geq N} |a_{n+1} - a_n| < \epsilon.$$

Therefore the sequence $\{a_n\}$ is Cauchy, hence convergent. □

THEOREM 2.41. *(Dedekind's Test) Let $\{a_n\}$ be a real sequence with bounded variation, and let $\sum_{n=1}^{\infty} b_n$ be a convergent real series. Then $\sum_{n=1}^{\infty} a_n b_n$ converges.*

EXERCISE 2.44. *Prove Dedekind's Test.*
(Suggestion: Proceed as in the proof of Dirichlet's Test, using (15).)

THEOREM 2.42. *(Abel's Test) Let $\{a_n\}$ be a bounded monotone real sequence, and let $\sum_{n=1}^{\infty} b_n$ be a convergent real series. Then $\sum_{n=1}^{\infty} a_n b_n$ converges.*

EXERCISE 2.45.
a) Show that a monotone sequence $\{a_n\}$ has bounded variation iff it is bounded.
b) Deduce Abel's Test from Dedekind's Test.
c) Deduce Abel's Test from Dirichlet's Test.
(Suggestion: In the notation of Abel's Test, suppose that $a_n \to a$. Replace $\{a_n\}$ with $\{a_n - a\}$ to apply Dirichlet's Test.)

EXERCISE 2.46.
a) Let $\{a_n\}$ and $\{b_n\}$ be real sequences of bounded variation, and let $\alpha, \beta \in \mathbb{R}$. Show: $\{\alpha a_n + \beta b_n\}$ has bounded variation.
b) Show: if $\sum_n a_n$ is absolutely convergent, then $\{a_n\}$ has bounded variation.
c) Show: for every real sequence $\{a_n\}$ there are increasing sequences $\{b_n\}$ and $\{c_n\}$ such that $\{a_n\} = \{b_n\} - \{c_n\}$.
d) Show: for a real sequence $\{a_n\}$, the following are equivalent:
(i) $\{a_n\}$ has bounded variation.
(ii) There are bounded increasing sequences $\{b_n\}$ and $\{c_n\}$ such that $\{a_n\} = \{b_n\} - \{c_n\}$.

8.3. A divergent Cauchy product.

Recall that we showed that if $\sum_n a_n = A$ and $\sum_n b_n = B$ are convergent series, *at least one* of which is absolutely convergent, then the Cauchy product series $\sum_n c_n$ is convergent to AB. Here we give an example – due to Cauchy! – of a Cauchy product of two nonabsolutely convergent series which fails to converge.

We will take $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$. (The series is convergent by the Alternating Series Test.) The nth term in the Cauchy product is

$$c_n = \sum_{i+j=n} (-1)^i (-1)^j \frac{1}{\sqrt{i+1}} \frac{1}{\sqrt{j+1}}.$$

The first thing to notice is $(-1)^i(-1)^j = (-1)^{i+j} = (-1)^n$, so c_n is equal to $(-1)^n$ times a sum of positive terms. We have $i, j \leq n$ so $\frac{1}{\sqrt{i+1}}, \frac{1}{\sqrt{j+1}} \geq \frac{1}{\sqrt{n+1}}$, and thus each term in c_n has absolute value at least $(\frac{1}{\sqrt{n+1}})^2 = \frac{1}{n+1}$. Since we are summing from $i = 0$ to n there are $n+1$ terms, all of the same size, we find $|c_n| \geq 1$ for all n. Thus the general term of $\sum_n c_n$ does not converge to 0, so the series diverges.

8.4. Decomposition into positive and negative parts.

For a real number r, we define its **positive part**

$$r^+ = \max(r, 0)$$

and its **negative part**

$$r^- = -\min(r, 0).$$

Exercise: Show that for any $r \in \mathbb{R}$ we have
(i) $r = r^+ - r^-$ and
(ii) $|r| = r^+ + r^-$.

For any real series $\sum_n a_n$ we have a decomposition

$$\sum_n a_n = \sum_n a_n^+ - \sum_n a_n^-,$$

at least if all three series converge. Let us call $\sum_n a_n^+$ and $\sum_n a_n^-$ the **positive part** and **negative part** of $\sum_n a_n$. Let us now suppose that $\sum_n a_n$ converges. By the Three Series Principle there are two cases:

Case 1: Both $\sum_n a_n^+$ and $\sum_n a_n^-$ converge. Hence $\sum_n |a_n| = \sum_n (a_n^+ + a_n^-)$ converges: i.e., $\sum_n a_n$ is absolutely convergent.

Case 2: Both $\sum_n a_n^+$ and $\sum_n a_n^-$ diverge. Hence $\sum_n |a_n| = \sum_n a_n^+ + a_n^-$ diverges: indeed, if it converged, then adding and subtracting $\sum_n a_n$ we would get that $2\sum_n a_n^+$ and $2\sum_n a_n^-$ converge, contradiction. Thus:

PROPOSITION 2.43. *If a series $\sum_n$ is absolutely convergent, both its positive and negative parts converge. If a series $\sum_n$ is nonabsolutely convergent, then both its positive and negative parts diverge.*

EXERCISE 2.47. *Let $\sum_n a_n$ be a real series.*
a) Show that if $\sum_n a_n^+$ converges and $\sum_n a_n^-$ diverges then $\sum_n a_n = -\infty$.
b) Show that if $\sum_n a_n^+$ diverges and $\sum_n a_n^-$ converges then $\sum_n a_n = \infty$.

Let us reflect for a moment on this state of affairs: in any nonabsolutely convergent series we have enough of a contribution from the positive terms to make the series diverge to ∞ and also enough of a contribution from the negative terms to make the series diverge to $-\infty$. Therefore if the series converges it is because of a subtle interleaving of the positive and negative terms, or, otherwise put, because lots of *cancellation* occurs between positive and negative terms. This suggests that the ordering of the terms in a nonabsolutely convergent series is rather important, and indeed in the next section we will see that changing the ordering of the terms of a nonabsolutely convergent series can have a dramatic effect.

9. Rearrangements and Unordered Summation

9.1. The Prospect of Rearrangement.

In this section we systematically investigate the validity of the "commutative law" for infinite sums. Namely, the definition we gave for convergence of an infinite series

$$a_1 + a_2 + \ldots + a_n + \ldots$$

in terms of the limit of the sequence of partial sums $A_n = a_1 + \ldots + a_n$ makes at least apparent use of the *ordering* of the terms of the series. Note that this is somewhat surprising even from the perspective of infinite sequences: the statement $a_n \to L$ can be expressed as: for all $\epsilon > 0$, there are only finitely many terms of the sequence lying outside the interval $(L-\epsilon, L+\epsilon)$, a description which makes clear that convergence to L will not be affected by any *reordering* of the terms of the sequence. However, if we reorder the terms $\{a_n\}$ of an infinite *series* $\sum_{n=1}^{\infty} a_n$, the corresponding change in the sequence A_n of partial sums is *not* simply a reordering, as one can see by looking at very simple examples. For instance, if we reorder

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots$$

as

$$\frac{1}{4} + \frac{1}{2} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots$$

Then the first partial sum of the new series is $\frac{1}{4}$, whereas every nonzero partial sum of the original series is at least $\frac{1}{2}$.

Thus there is some evidence to fuel suspicion that reordering the terms of an infinite series may not be so innocuous an operation as for that of an infinite seuqence. All of this discussion is mainly justification for our setting up the "rearrangement problem" carefully, with a precision that might otherwise look merely pedantic.

Namely, the formal notion of rearrangement of a series $\sum_{n=0}^{\infty} a_n$ begins with a permuation σ of $\mathbb{N}$, i.e., a bijective function $\sigma : \mathbb{N} \to \mathbb{N}$. We define the **rearrangement** of $\sum_{n=0}^{\infty} a_n$ by σ to be the series $\sum_{n=0}^{\infty} a_{\sigma(n)}$.

9.2. The Rearrangement Theorems of Weierstrass and Riemann.

The most basic questions on rearrangements of series are as follows.

Question 2.44. *Let $\sum_{n=0}^{\infty} a_n = S$ is a convergent infinite series, and let σ be a permutation of $\mathbb{N}$. Then:*
a) Does the rearranged series $\sum_{n=0}^{\infty} a_{\sigma(n)}$ converge?
b) If it does converge, does it converge to S?

As usual, the special case in which all terms are non-negative is easiest, the case of absolute convergence is not much harder than that, and the case of nonabsolute convergence is where all the real richness and subtlety lies.

Indeed, suppose that $a_n \geq 0$ for all n. In this case the sum $A = \sum_{n=0}^{\infty} a_n \in [0, \infty$ is simply the supremum of the set $A_n = \sum_{n=0}^{k} a_k$ of finite sums. More generally, let $S = \{n_1, \ldots, n_k\}$ be any finite subset of the natural numbers, and put $A_S = a_{n_1} + \ldots + a_{n_k}$. Now every finite subset $S \subset \mathbb{N}$ is contained in $\{0, \ldots, N\}$ for some $N \in \mathbb{N}$, so for all S, $A_S \leq A_N$ for some (indeed, for all sufficiently large) N. This shows that if we define

$$A' = \sup_S A_S$$

as S ranges over all finite subsets of $\mathbb{N}$, then $A' \leq A$. On the other hand, for all $N \in \mathbb{N}$, $A_N = a_0 + \ldots + a_N = A_{\{0,\ldots,N\}}$: in other words, each partial sum A_N arises as A_S for a suitable finite subset S. Therefore $A \leq A'$ and thus $A = A'$.

The point here is that the description $\sum_{n=0}^{\infty} a_n = \sup_S A_S$ is manifestly unchanged by rearranging the terms of the series by any permutation σ: taking $S \mapsto \sigma(S)$ gives a bijection on the set of all finite subsets of $\mathbb{N}$, and thus

$$\sum_{n=0}^{\infty} a_n = \sup_S A_S = \sup_S A_{\sigma(S)} = \sum_{n=0}^{\infty} a_{\sigma(n)}.$$

The case of absolutely convergent series follows rather easily from this.

THEOREM 2.45. *(Weierstrass) Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series with sum A. Then for every permutation σ of $\mathbb{N}$, the rearranged series $\sum_{n=0}^{\infty} a_{\sigma(n)}$ converges to A.*

PROOF. For $N \in \mathbb{Z}^+$, define

$$N_\sigma = \max_{0 \leq k < N} \sigma^{-1}(k).$$

In other words, N_σ is the least natural number such that $\sigma(\{0, 1, \ldots, N_\sigma\}) \supseteq \{0, 1, \ldots, N-1\}$. Thus for $n > N_\sigma$, $\sigma(n) \geq N$. For all $\epsilon > 0$, by the Cauchy criterion for absolute convergence, there is $N \in \mathbb{N}$ with $\sum_{n=N}^{\infty} |a_n| < \epsilon$. Then

$$\sum_{n=N_\sigma+1}^{\infty} |a_{\sigma(n)}| \leq \sum_{n=N}^{\infty} |a_n| < \epsilon,$$

and the rearranged series is absolutely convergent by the Cauchy criterion. Let $A' = \sum_{n=0}^{\infty} a_{\sigma(n)}$. Then

$$|A - A'| \leq |\sum_{n=0}^{N_\sigma} a_n - a_{\sigma(n)}| + \sum_{n > N_\sigma} |a_n| + \sum_{n > N_\sigma} |a_{\sigma(n)}| < |\sum_{n=0}^{N_\sigma} a_n - a_{\sigma(n)}| + 2\epsilon.$$

Moreover, each term a_k with $0 \leq k \leq N$ appears in both $\sum_{n=0}^{N_\sigma} a_n$ and $\sum_{n=0}^{N_\sigma} a_{\sigma(n)}$, so we may make the very crude estimate

$$|\sum_{n=0}^{N_\sigma} a_n - a_{\sigma(n)}| \leq 2 \sum_{n > N} |a_n| < 2\epsilon$$

which gives

$$|A - A'| < 4\epsilon.$$

Since ϵ was arbitrary, we conclude $A = A'$. □

EXERCISE 2.48. *Use the decomposition of $\sum_n a_n$ into its series of positive parts $\sum_n a_n^+$ and negative parts $\sum_n a_n^-$ to give a second proof of Theorem 2.45.*

THEOREM 2.46. *(Riemann Rearrangement Theorem) Let $\sum_{n=0}^{\infty} a_n$ be a non-absolutely convergent series. For any $B \in [-\infty, \infty]$, there exists a permutation σ of $\mathbb{N}$ such that $\sum_{n=0}^{\infty} a_{\sigma(n)} = B$.*

PROOF.
Step 1: Since $\sum_n a_n$ is convergent, we have $a_n \to 0$ and thus that $\{a_n\}$ is bounded, so we may choose M such that $|a_n| \leq M$ for all n. We are not going to give an explicit "formula" for σ; rather, we are going to describe σ by a certain process. For this it is convenient to imagine that the sequence $\{a_n\}$ has been sifted into a disjoint union of two subsequences, one consisting of the positive terms and one

consisting of the negative terms (we may assume without loss of generality that there $a_n \neq 0$ for all n). If we like, we may even imagine both of these subseqeunce ordered so that they are decreasing in absolute value. Thus we have two sequences

$$p_1 \geq p_2 \geq \ldots \geq p_n \geq \ldots \geq 0,$$

$$n_1 \leq n_2 \leq \ldots \leq n_n \leq \ldots \leq 0$$

so that together $\{p_n, n_n\}$ comprise the terms of the series. The key point here is Proposition 2.43 which tells us that since the convergence is nonabsolute, $\sum_n p_n = \infty$, $\sum_n n_n = -\infty$. So we may specify a rearangement as follows: we specify a choice of a certain number of positive terms – taken in decreasing order – and then a choice of a certain number of negative terms – taken in order of decreasing absolute value – and then a certain number of positive terms, and so on. As long as we include a finite, positive number of terms at each step, then in the end we will have included every term p_n and n_n eventually, hence we will get a rearrangement.

Step 2 (diverging to ∞): to get a rearrangement diverging to ∞, we proceed as follows: we take positive terms $p_1, p_2, \ldots$ in order until we arrive at a partial sum which is at least $M + 1$; then we take the first negative term n_1. Since $|n_1| \leq M$, the partial sum $p_1 + \ldots + p_{N_1} + n_1$ is still at least 1. Then we take at least one more positive term p_{N_1+1} and possibly further terms until we arrive at a partial sum which is at least $M + 2$. Then we take one more negative term n_2, and note that the partial sum is still at least 2. And we continue in this manner: after the kth step we have used at least k positive terms, at least k negative terms, and *all* the partial sums from that point on will be at least k. Therefore every term gets included eventually and the sequence of partial sums diverges to $+\infty$.

Step 3 (diverging to $-\infty$): An easy adaptation of the argument of Step 2 leads to a permutation σ such that $\sum_{n=0}^{\infty} a_{\sigma(n)} = -\infty$. We leave this case to the reader.
Step 4 (converging to $B \in \mathbb{R}$): if anything, the argument is simpler in this case. We first take positive terms $p_1, \ldots, p_{N_1}$, stopping when the partial sum $p_1 + \ldots + p_{N_1}$ is greater than B. (To be sure, we take at least one positive term, even if $0 > B$.) Then we take negative terms $n_1, \ldots, n_{N_2}$, stopping when the partial sum $p_1 + \ldots + p_{N_1} + n_1 + \ldots + n_{N_2}$ is less than B. Then we repeat the process, taking enough positive terms to get a sum strictly larger than B then enough negative terms to get a sum strictly less than B, and so forth. Because both the positive and negative parts diverge, this construction can be completed. Because the general term $a_n \to 0$, a little thought shows that the absolute value of the difference between the partial sums of the series and B approaches zero. □

Exercise 2.49. *Fill in the omitted details in the proof of Theorem 2.46. In particular:*
a) Construct a permutation σ of $\mathbb{N}$ such that $\sum_n a_{\sigma(n)} \to -\infty$.
b) Show that the rearranged series of Step 4 does indeed converge to B. (Suggestion: it may be helpful to think of the rearrangement process as a walk along the real line, taking a certain number of steps in a positive direction and then a certain number of steps in a negative direction, and so forth. The key point is that by hypothesis the step size *is approaching zero, so the amount by which we "overshoot" the limit B at each stage decreases to zero.)*

The fact that the original series $\sum_n a_n$ converges was not used very strongly in the proof. The following exercises deduce the same conclusions of Theorem 2.46 under milder hypotheses.

EXERCISE 2.50. *Let $\sum_n a_n$ be a real series such that $a_n \to 0$, $\sum_n a_n^+ = \infty$ and $\sum_n a_n^- = -\infty$. Show that the conclusion of Theorem 2.46 holds: for any $A \in [-\infty, \infty]$, there exists a permutation σ of $\mathbb{N}$ such that $\sum_{n=0}^{\infty} a_{\sigma(n)} = A$.*

EXERCISE 2.51. *Let $\sum_n a_n$ be a real series such that $\sum_n a_n^+ = \infty$.*
a) Suppose that the sequence $\{a_n\}$ is bounded. Show that there exists a permutation σ of $\mathbb{N}$ such that $\sum_n a_{\sigma(n)} = \infty$.
b) Does the conclusion of part a) hold without the assumption that the sequnece of terms is bounded?

Theorem 2.46 exposes the dark side of nonabsolutely convergent series: just by changing the order of the terms, we can make the series diverge to $\pm\infty$ or converge to any given real number! Thus nonabsolute convergence is necessarily of a more delicate and less satisfactory nature than absolute convergence. With these issues in mind, we define a series $\sum_n a_n$ to be **unconditionally convergent** if it is convergent and every rearrangement converges to the same sum, and a series to be **conditionally convergent** if it is convergent but not unconditionally convergent. Then much of our last two theorems may be summarized as follows.

THEOREM 2.47. *(Main Rearrangement Theorem) A convergent real series is unconditionally convergent iff it is absolutely convergent.*

Many texts do not use the term "nonabsolutely convergent" and instead *define* a series to be conditionally convergent if it is convergent but not absolutely convergent. Aside from the fact that this terminology can be confusing to students (especially calculus students) to whom this rather intricate story of rearrangements has not been told, it seems correct to make a distinction between the following two *a priori* different phenomena:

- $\sum_n a_n$ converges but $\sum_n |a_n|$ does not, versus
- $\sum_n a_n$ converges to A but some rearrangement $\sum_n a_{\sigma(n)}$ does not.

Now it happens that these two phenomena are equivalent for real series. However the notion of an infinite series $\sum_n a_n$, absolute and unconditional convergence makes sense in other contexts as well, for instance for series with values in an **infinite-dimensional Banach space** or with values in a **p-adic field**. In the former case it is a celebrated theorem of Dvoretsky-Rogers [**DR50**] that there exists a series which is unconditionally convergent but not absolutely convergent, whereas in the latter case one can show that *every* convergent series is unconditionally convergent whereas there exist nonabsolutely convergent series.

EXERCISE 2.52. *Let $\sum_{n=0}^{\infty} a_n$ be any nonabsolutely convergent real series, and let $-\infty \leq a \leq A \leq \infty$. Show that there exists a permutation σ of $\mathbb{N}$ such that the set of partial limits of $\sum_{n=0}^{\infty} a_{\sigma(n)}$ is the closed interval $[a, A]$.*

9.3. The Levi-Agnew Theorem.

(In this section I plan to state and prove a theorem of F.W. Levi [**Le46**] and –

later, but independently, R.P. Agnew [**Ag55**] – characterizing the permutations σ of $\mathbb{N}$ such that: for all convergent real series $\sum_{n=0}^{\infty} a_n$ the σ-rearrangeed series $\sum_{n=0}^{\infty} a_{\sigma(n)}$ converges. I also plan to discuss related results: [**Ag40**], [**Se50**], [**Gu67**], [**Pl77**], [**Sc81**], [**St86**], [**GGRR88**], [**NWW99**], [**KM04**], [**FS10**], [**Wi10**].

9.4. Unordered summation.

It is very surprising that the ordering of the terms of a nonabsolutely convergent series affects both its convergence and its sum – it seems fair to say that this phenomenon was undreamt of by the founders of the theory of infinite series.

Armed now, as we are, with the full understanding of the implications of our definiion of $\sum_{n=0}^{\infty} a_n$ as the limit of a sequence of partial sums, it seems reasonable to ask: is there an alternative definition for the sum of an infinite series, one in which the ordering of the terms is *a priori* immaterial?

The answer to this question is **yes** and is given by the theory of **unordered summation**.

To be sure to get a definition of the sum of a series which does not depend on the ordering of the terms, it is helpful to work in a more general context in which no ordering is present. Namely, let S be a nonempty set, and define an **S-indexed sequence of real numbers** to be a function $a_\bullet : S \to \mathbb{R}$. The point here is that we recover the usual definition of a sequence by taking $S = \mathbb{N}$ (or $S = \mathbb{Z}^+$) but whereas $\mathbb{N}$ and $\mathbb{Z}^+$ come equipped with a natural ordering, the "naked set" S does not.

We wish to define $\sum_{s \in S} a_s$, i.e., the "unordered sum" of the numbers a_s as s ranges over all elements of S. Here it is: for every *finite* subset $T = \{s_1, \ldots, s_N\}$ of S, we define $a_T = \sum_{s \in T} a_s = a_{s_1} + \ldots + a_{s_N}$. (We also define $a_\varnothing = 0$.) Finally, for $A \in \mathbb{R}$, we say that the unordered sum $\sum_{s \in S} a_s$ **converges to A** if: for all $\epsilon > 0$, there exists a finite subset $T_0 \subset S$ such that for all finite subsets $T_0 \subset T \subset S$ we have $|a_T - A| < \epsilon$. If there exists $A \in \mathbb{R}$ such that $\sum_{s \in S} a_s = A$, we say that $\sum_{s \in S} a_s$ is **convergent** or that the S-indexed sequence $a_\bullet$ is **summable**. (When $S = \mathbb{Z}^{\geq N}$ we already have a notion of summability, so when we need to make the distinction we will say **unordered summable**.)

Notation: because we are often going to be considering various finite subsets T of a set S, we allow ourselves the following time-saving notation: for two sets A and B, we denote the fact that A is a finite subset of B by $A \subset_f B$.

Exercise 2.53. *Suppose S is finite. Show that every S-indexed sequence $a_\bullet : S \to \mathbb{R}$ is summable, with sum $a_S = \sum_{s \in S} a_s$.*

Exercise 2.54. *If $S = \varnothing$, there is a unique function $a_\bullet : \varnothing \to \mathbb{R}$, the "empty function". Convince yourself that the most reasonable value to assign $\sum_{s \in \varnothing} a_s$ is 0.*

Exercise 2.55. *Give reasonable definitions for $\sum_{s \in S} a_s = \infty$ and $\sum_{s \in S} a_s = -\infty$.*

Confusing Remark: We say that we are doing "unordered summation", but our sequences are taking values in $\mathbb{R}$, in which the absolute value is derived from the order structure. One could also consider unordered summation of S-indexed sequences with values in an arbtirary **normed abelian group** $(G, |\ |)$.[25] A key feature of $\mathbb{R}$ is the positive-negative part decomposition, or equivalently the fact that for $M \in \mathbb{R}$, $|M| \geq A$ implies $M \geq A$ or $M \leq -A$. In other words, there are *exactly two ways* for a real number to be large in absolute value: it can be very positive or very negative. At a certain point in the theory considerations like this *must be used* in order to prove the desired results, but we delay these type of arguments for as long as possible.

The following result holds without using the positive-negative decomposition.

THEOREM 2.48. *(Cauchy Criteria for Unordered Summation) Let $a_\bullet : S \to \mathbb{R}$ be an S-index sequence, and consider the following assertions:*
(i) The S-indexed sequence $a_\bullet$ is summable.
(ii) For all $\epsilon > 0$, there exists a finite subset $T_\epsilon \subset S$ such that for all finite subsets T, T' of S containing T_ϵ,

$$|\sum_{s \in T} a_s - \sum_{s \in T'} a_s| < \epsilon.$$

(iii) For all $\epsilon > 0$, there exists $T_\epsilon \subset_f S$ such that: for all $T \subset_f S$ with $T \cap T_\epsilon = \varnothing$, we have $|a_T| = |\sum_{s \in T} a_s| < \epsilon$.
(iv) There exists $M \in \mathbb{R}$ such that for all $T \subset_f S$, $|a_T| \leq M$.

PROOF. (i) $\implies$ (ii) is immediate from the definition.
(ii) $\implies$ (i): We may choose, for each $n \in \mathbb{Z}^+$, a finite subset T_n of S such that $T_n \subset T_{n+1}$ for all n and such that for all finite subsets T, T' of S containing T_n, $|a_T - a_{T'}| < \frac{\epsilon}{2}$. It follows that the real sequence $\{a_{T_n}\}$ is Cauchy, hence convergent, say to A. We claim that $a_\bullet$ is summable to A: indeed, for $\epsilon > 0$, choose $n > \frac{2}{\epsilon}$. Then, for any finite subset T containing T_n we have

$$|a_T - A| \leq |a_T - a_{T_n}| + |a_{T_n} - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(ii) $\implies$ (iii): Fix $\epsilon > 0$, and choose $T_0 \subset_f$ as in the statement of (ii). Now let $T \subset_f S$ with $T \cap T_0 = \varnothing$, and put $T' = T \cup T_0$. Then T' is a finite subset of S containing T_0 and we may apply (ii):

$$|\sum_{s \in T} a_s| = |\sum_{s \in T'} a_s - \sum_{s \in T_0} a_s| < \epsilon.$$

(iii) $\implies$ (ii): Fix $\epsilon > 0$, and let $T_\epsilon \subset_f S$ be such that for all finite subsets T of S with $T \cap T_\epsilon = \varnothing$, $|a_T| < \frac{\epsilon}{2}$. Then, for any finite subset T' of S containing T_ϵ,

$$|a_{T'} - a_{T_\epsilon}| = |a_{T' \setminus T_\epsilon}| < \frac{\epsilon}{2}.$$

From this and the triangle inequality it follows that if T and T' are two finite subsets containing T_ϵ,

$$|a_T - a_{T'}| < \epsilon.$$

[25]Moreover, it is not totally insane to do so: such things arise naturally in functional analysis and number theory.

(iii) $\implies$ (iv): Using (iii), choose $T_1 \subset_f S$ such that for all $T' \subset_f S$ with $T_1 \cap T' = \varnothing$, $|a_{T'}| \leq 1$. Then for any $T \subset_f S$, write $T = (T \setminus T_1) \cup (T \cap T_1)$, so

$$|a_T| \leq |\sum_{s \in T \setminus T_1} a_s| + |\sum_{s \in T \cap T_1} a_s| \leq 1 + \sum_{s \in T_1} |a_s|,$$

so we may take $M = 1 + \sum_{s \in T_1} |a_s|$. □

Confusing Example: Let $G = \mathbb{Z}_p$ with its standard norm. Define $a_n : \mathbb{Z}^+ \to G$ by $a_n = 1$ for all n. Because of the non-Archimedean nature of the norm, we have for any $T \subset_f S$ $|a_T| = |\#T| \leq 1$. Therefore $a_\bullet$ satisfies condition (iv) of Theorem 2.48 above but not condition (iii): given any finite subset $T \subset \mathbb{Z}^+$, there exists a finite subset T', disjoint from T, such that $|a_{T'}| = 1$: indeed, we may take $T' = \{n\}$, where n is larger than any element of T and prime to p.

Although we have no reasonable expectation that the reader will be able to make any sense of the previous example, we offer it as motivation for delaying the proof of the implication (iv) $\implies$ (i) above, which uses the positive-negative decomposition in $\mathbb{R}$ in an essential way.

THEOREM 2.49. *An S-indexed sequence $a_\bullet : S \to \mathbb{R}$ is summable iff the finite sums ae uniformly bounded: i.e., there exists $M \in \mathbb{R}$ such that for all $T \subset_f S$, $|a_T| \leq M$.*

PROOF. We have already shown in Theorem 2.48 above that if $a_\bullet$ is summable, the finite sums are uniformly bounded. Now suppose $a_\bullet$ is not summable, so by Theorem 2.48 there exists $\epsilon > 0$ with the following property: for any $T \subset_f S$ there exists $T' \subset_f S$ with $T \cap T' = \varnothing$ and $|a_{T'}| \geq \epsilon$. Of course, if we can find such a T', we can also find a T'' disjoint from $T \cup T'$ with $|a_{T''}| \geq \epsilon$, and so forth: there will be a sequence $\{T_n\}_{n=1}^\infty$ of pairwise disjoint finite subsets of S such that for all n, $|a_{T_n}| \geq \epsilon$. But now decompose $T_n = T_n^+ \cup T_n-$, where T_n^+ consists of the elements s such that $a_s \geq 0$ and T_n^- consists of the elements s such that $a_s < 0$. It follows that

$$a_{T_n} = |a_{T_n^+}| - |a_{T_n^-}|$$

hence

$$\epsilon \leq |a_{T_n}| \leq |a_{T_n^+}| + |a_{T_n^-}|,$$

from which it follows that $\max |a_{T_n^+}, a_{T_n}-| \geq \frac{\epsilon}{2}$, so we may define for all n a subset $T_n' \subset T_n$ such that $|a_{T_n'}| \geq \frac{\epsilon}{2}$ and the sum $a_{T_n'}$ consists either entirely of non-negative elements of entirely of negative elements. If we now consider $T_1', \ldots, T_{2n-1}'$, then by the Pigeonhole Principle there must exist $1 \leq i_1 < \ldots < i_n \leq 2n-1$ such that all the terms in each T_i' are non-negative *or* all the terms in each T_i' are negative. Let $\mathcal{T}_n = \bigcup_{j=1}^n T_{i_j}'$. Then we have a disjoint union and no cancellation, so $|\mathcal{T}_n| \geq \frac{n\epsilon}{2}$: the finite sums a_T are not uniformly bounded. □

PROPOSITION 2.50. *Let $a_\bullet : S \to \mathbb{R}$ be an S-indexed sequence with $a_s \geq 0$ for all $s \in S$. Then*

$$\sum_{s \in S} a_s = \sup_{T \subset_f S} a_T.$$

PROOF. Let $A = \sup_{T \subset_f S} a_T$.

We first suppose that $A < \infty$. By definition of the supremum,, for any $\epsilon > 0$, there exists a finite subset $T \subset S$ such that $A - \epsilon < a_T \leq A$. Moreover, for any finite subset $T' \supset T$, we have $A - \epsilon a_T \leq a_{T'} \leq A$, so $a_\bullet \to A$.

Next suppose $A = \infty$. We must show that for any $M \in \mathbb{R}$, there exists a subset $T_M \subset_f S$ such that for every finite subset $T \supset T_M$, $a_T \geq M$. But the assumption $A = \infty$ implies there exists $T \subset_f S$ such that $a_T \geq M$, and then non-negativity gives $a_{T'} \geq a_T \geq M$ for all finite subsets $T' \supset T$. $\square$

THEOREM 2.51. *(Absolute Nature of Unordered Summation) Let S be any set and $a_\bullet : S \to \mathbb{R}$ be an S-indexed sequence. Let $|a_\bullet|$ be the S-indexed sequence $s \mapsto |a_s|$. Then $a_\bullet$ is summable iff $|a_\bullet|$ is summable.*

PROOF. Suppose $|a_\bullet|$ is summable. Then for any $\epsilon > 0$, there exists T_ϵ such that for all finite subsets T of S disjoint from T_ϵ, we have $||a|_T| < \epsilon$, and thus

$$|a_T| = |\sum_{s \in T} a_s| \leq |\sum_{s \in T} |a_s|| = ||a|_T| < \epsilon.$$

Suppose $|a_\bullet|$ is *not* summable. Then by Proposition 2.50, for every $M > 0$, there exists $T \subset_f S$ such that $|a|_T \geq 2M$. But as in the proof of Theorem 2.49, there must exist a subset $T' \subset T$ such that (i) $a_{T'}$ consists entirely of non-negative terms or entirely of negative terms and (ii) $|a_{T'}| \geq M$. Thus the partial sums of $a_\bullet$ are not uniformly bounded, and by Theorem 2.49 $a_\bullet$ is not summable. $\square$

THEOREM 2.52. *For $a_\bullet : \mathbb{N} \to \mathbb{R}$ an ordinary sequence and $A \in \mathbb{R}$, the following are equivalent:*
(i) The unordered sum $\sum_{n \in \mathbb{Z}^+} a_n$ is convergent, with sum A.
(ii) The series $\sum_{n=0}^\infty a_n$ is unconditionally convergent, with sum A.

PROOF. (i) $\implies$ (ii): Fix $\epsilon > 0$. Then there exists $T_\epsilon \subset_f S$ such that for every finite subset T of $\mathbb{N}$ containing T_ϵ we have $|a_T - A| < \epsilon$. Put $N = \max_{n \in T_\epsilon} n$. Then for all $n \geq N$, $\{0, \ldots, n\} \supset T_\epsilon$ so $|\sum_{k=0}^n a_k - A| < \epsilon$. It follows that the infinite series $\sum_{n=0}^\infty a_n$ converges to A in the usual sense. Now for any permutation σ of $\mathbb{N}$, the unordered sum $\sum_{n \in \mathbb{Z}^+} a_{\sigma(n)}$ is manifestly the same as the unordered sum $\sum_{n \in \mathbb{Z}^+} a_n$, so the rearranged series $\sum_{n=0}^\infty a_{\sigma(n)}$ also converges to A.
(ii) $\implies$ (i): We will prove the contrapositive: suppose the unordered sum $\sum_{n \in \mathbb{N}} a_n$ is divergent. Then by Theorem 2.49 for every $M \geq 0$, there exists $T \subset S$ with $|a_T = \sum_{s \in T} a_s| \geq M$. Indeed, as the proof of that result shows, we can choose T to be disjoint from any given finite subset. We leave it to you to check that we can therefore build a rearrangement of the series with unbounded partial sums. $\square$

EXERCISE 2.56. *Fill in the missing details of (ii) $\implies$ (i) in the proof of Theorem 2.52.*

EXERCISE 2.57. *Can one prove Theorem 2.52 without appealing to the fact that $|x| \geq M$ implies $x \geq M$ or $x \leq -M$? For instance, does Theorem 2.52 holds for S-indexed sequences with values in any Banach space? Any complete normed abelian group?*

Comparing Theorems 2.50 and 2.51 we get a second proof of the portion of the Main Rearrangement Theorem that says that a real series is unconditionally convergent iff it is absolutely convergent. Recall that our first proof of this depended on the Riemann Rearrangement Theorem, a more complicated result.

On the other hand, if we allow ourselves to use the previously derived result that unconditional convergence and absolute convergence coincide, then we can get an easier proof of (ii) $\implies$ (i): if the series $\sum_n a_n$ is unconditionally convergent, then $\sum_n |a_n| < \infty$, so by Proposition 2.48 the unordered sequence $|a_\bullet|$ is summable, hence by Theorem 2.50 the unordered sequence $a_\bullet$ is summable.

To sum up (!), when we apply the very general definition of unordered summability to the classical case of $S = \mathbb{N}$, we recover precisely the theory of absolute (= unconditional) convergence. This gives us a clearer perspective on exactly what the usual, order-dependent notion of convergence is buying us: namely, the theory of conditionally convergent series. It may perhaps be disappointing that such an elegant theory did not gain us anything new.

However when we try to generalize the notion of an infinite series in various ways, the results on unordered summability become very helpful. For instance, often in nature one encounters **biseries**

$$\sum_{n=-\infty}^{\infty} a_n$$

and **double series**

$$\sum_{m,n \in \mathbb{N}} a_{m,n}.$$

We may treat the first case as the unordered sum associated to the $\mathbb{Z}$-indexed sequence $n \mapsto a_n$ and the second as the unordered sum associated to the $\mathbb{N}\times\mathbb{N}$-indexed sequence $(m,n) \mapsto a_{m,n}$ and we are done: there is no need to set up separate theories of convergence here. Or, if we prefer, we may choose to shoehorn these more ambitiously indexed series into conventional $\mathbb{N}$-indexed series: this involves choosing a bijection b from $\mathbb{Z}$ (respectively $\mathbb{N} \times \mathbb{N}$) to $\mathbb{N}$. In both cases such bijections exist, in fact in great multitude: if S is any countably infinite set, then for any two bijections $b_1,\ b_2 : S \to \mathbb{N}$, $b_2 \circ b_1^{-1} : \mathbb{N} \to \mathbb{N}$ is a permutation of $\mathbb{N}$. Thus the discrepancy between two chosen bijections corresponds precisely to a rearrangement of the series. By Theorem 2.51, if the unordered sequence is summable, then the choice of bijection b is immaterial, as we are getting an unconditionally convergent series.

The theory of products of infinite series comes out especially cleanly in this unordered setting (which is not surprising, since it corresponds to the case of absolute convergence, where Cauchy products are easy to deal with).

EXERCISE 2.58. *Let S_1 and S_2 be two sets, and let $a_\bullet : S_1 \to \mathbb{R}$, $b_\bullet : S_2 \to \mathbb{R}$. We assume the following nontriviality condition: there exists $s_1 \in S_1$ and $s_2 \in S_2$ such that $a_{s_1} \neq 0$ and $a_{s_2} \neq 0$. We define $(a,b)_\bullet : S_1 \times S_2 \to \mathbb{R}$ by*

$$(a,b)_s = (a,b)_{(s_1,s_2)} = a_{s_1} b_{s_2}.$$

a) Show that $a_\bullet$ and $b_\bullet$ are both summable iff $(a,b)_\bullet$ is summable.
b) Assuming the equivalent conditions of part a) hold, show

$$\sum_{s \in S_1 \times S_2} (a,b)_s = \left(\sum_{s_1 \in S_1} a_{s_1} \right) \left(\sum_{s_2 \in S_2} b_{s_2} \right).$$

c) When $S_1 = S_2 = \mathbb{N}$, compare this result with the theory of Cauchy products we have already developed.

EXERCISE 2.59. *Let S be an uncountable set,*[26] *and let $a_\bullet : S \to \mathbb{R}$ be an S-indexed sequence. Show that if $a_\bullet$ is summable, then $\{s \in S \mid a_s \neq 0\}$ is countable.*

Finally we make some remarks about the provenance of unordered summation. It may be viewed as a small part of two much more general endeavors in turn of the (20th!) century real analysis. On the one hand, Moore and Smith developed a general theory of convergence which was meant to include as special cases all the various limiting processes one encounters in analysis. The key idea is that of a **net** in the real numbers, namely a **directed set** $(X, \leq)$ and a function $a_\bullet : X \to \mathbb{R}$. One says that the net $a_\bullet$ converges to $a \in \mathbb{R}$ if: for all $\epsilon > 0$, there exists $x_0 \in X$ such that for all $x \geq x_0$, $|x - x_0| < \epsilon$. Thus this is, in its way, an aggressively straightforward generalization of the definition of a convergent sequence: we just replace $(\mathbb{N}, \leq)$ by an arbitrary directed set. For instance, the notion of Riemann integrable function can be fit nicely into this context (we save the details for another exposition).

What does this have to do with unordered summation? If S is our arbitrary set, let X_S be the collection of finite subsets $T \subset S$. If we define $T_1 \leq T_2$ to mean $T_1 \subset T_2$ then X_S becomes a partially ordered set. Moreover it is directed: for any two finite subsets $T_1, T_2 \subset S$, we have $T_1 \leq T_1 \cup T_2$ and $T_2 \leq T_1 \cup T_2$. Finally, to an S-indexed sequence $a_\bullet : S \to \mathbb{R}$ we associate a net $a_\bullet : X_S \to \mathbb{R}$ by $T \mapsto a_T = \sum_{s \in T} a_s$ (note that we are using the same notation for the sequence and the net, which seems reasonable and agrees with what we have done before). Then the reader may check that our definition of summability of the S-indexed sequence $a_\bullet$ is precisely that of convergence of the net $a_\bullet : X_S \to \mathbb{R}$. This particular special case of a net obtained by starting with an arbitrary (unordered!) set S and passing to the (directed!) set X_S of finite subsets of S was given special prominence in a 1940 text of Tukey[27] [**T**]: he called the directed set X_S a **stack** and the function $a_\bullet : X_S \to \mathbb{R}$ a **phalanx**. But in fact, for reasons that we had better not get into here, Tukey's work was well regarded at the time but did not stick, and I doubt that one contemporary mathematician out of a hundred could define a "phalanx".[28]

Note that above we mentioned in passing the connection between unordered summation and the Riemann integral. This connection was taken up, in a much different way, by Lebesgue.[29] Namely he developed an entirely new theory of integration, now called **Lebesgue integration** This has two advantages over the Riemann integral: (i) in the classical setting there are many more functions which are Lebesgue integrable than Riemann integrable; for instance, essentially any bounded function $f : [a, b] \to \mathbb{R}$ is Lebesgue integrable, even functions which are discontinuous at every point; and (ii) the Lebesgue integral makes sense in the context of real-valued functions on a **measure space** $(S, \mathcal{A}, \mu)$. It is not our business to say what the

[26] Here we are following our usual convention of allowing individual exercises to assume knowledge that we do not want to assume in the text itself. Needless to say, there is no need to attempt this exercise if you do not already know and care about uncountable sts.

[27] John Wilder Tukey, 1915-2000

[28] On the other hand, ask a contemporary mathematician – *the right kind* of mathematician, to be sure, but for instance any of several people here at the University of Georgia – what a "stack" is, and she will light up and start rapidly telling you an exciting, if impenetrable, story. At some point you will probably realize that what she means by a stack is completely different from a stack in Tukey's sense, and when I say "completely different" I mean **completely different**.

[29] Henri Léon Lebesgue, 1875-1941

latter is...but, for any set S, there is an especially simple kind of measure, the **counting measure**, which associates to any finite subset its cardinality (and associates to any infinite subset the extended real number ∞). It turns out that the integral of a function $f : S \to \mathbb{R}$ with respect to counting measure is nothing else than the unordered sum $\sum_{s \in S} f(s)$! The general case of a Lebesgue integral – and even the case where $S = [a, b]$ and μ is the "standard Lebesgue measure" – is significantly more technical and generally delayed to graduate-level courses, but one can see a nontrivial part of it in the theory of unordered summation. Most of all, Exercise X.X on products is a very special case of the celebrated **Fubini**[30] **Theorem** which evaluates double integrals in terms of iterated integrals. Special cases of this involving the Riemann integral are familiar to students of multivariable calculus.

10. Power Series I: Power Series as Series

10.1. Convergence of Power Series.

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers. Then a series of the form $\sum_{n=0}^{\infty} a_n x^n$ is called a **power series**. Thus, for instance, if we had $a_n = 1$ for all n we would get the *geometric series* $\sum_{n=0}^{\infty} x^n$ which converges iff $x \in (-1, 1)$ and has sum $\frac{1}{1-x}$.

The nth partial sum of a power series is $\sum_{k=0}^{n} a_k x^k$, a **polynomial** in x. One of the major themes of Chapter three will be to try to view power series as "infinite polynomials": in particular, we will regard x as a variable and be interested in the propeties – continuity, differentiability, integrability, and so on – of the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ defined by a power series.

However, if we want to regard the series $\sum_{n=0}^{\infty} a_n x^n$ as a function of x, what is its domain? The natural domain of a power series is the set of all values of x for which the series converges. Thus the basic question about power series that we will answer in this section is the following.

Question 2.53. *For a sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers, for which values of $x \in \mathbb{R}$ does the power series $\sum_{n=0}^{\infty} a_n x^n$ converge?*

There is one value of x for which the answer is trivial. Namely, if we plug in $x = 0$ to our general power series, we get

$$\sum_{n=0}^{\infty} a_n 0^n = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 = a_0.$$

So every power series converges at least at $x = 0$.

Example 1: Consider the power series $\sum_{n=0}^{\infty} n! x^n$. We apply the Ratio Test:

$$\lim_{n \to \infty} \frac{(n+1)! x^{n+1}}{n! x^n} = \lim_{n \to \infty} (n+1)|x|.$$

The last limit is 0 if $x = 0$ and otherwise is $+\infty$. Therefore the Ratio Test shows that (as we already knew!) the series converges absolutely at $x = 0$ and diverges at every nonzero x. So it is indeed possible for a power series to converge *only* at $x = 0$. Note that this is disappointing if we are interesteted in $f(x) = \sum_n a_n x^n$ as

[30] Guido Fubini, 1879-1943

a function of x, since in this case it is just the function from $\{0\}$ to $\mathbb{R}$ which sends 0 to a_0. There is nothing interesting going on here.

Example 2: consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. We apply the Ratio Test:

$$\lim_{n\to\infty} |\frac{x^{n+1}}{(n+1)!}||\frac{n!}{x^n}| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

So the power series converges for all $x \in \mathbb{R}$ and defines a function $f : \mathbb{R} \to \mathbb{R}$.

Example 3: Fix $R \in (0, \infty)$ and consider the power series $\sum_{n=0}^{\infty} \frac{1}{R^n} x^n$. This is a geometric series with geometric ratio $\rho = \frac{x}{R}$, so it converges iff $|\rho| = |\frac{x}{R}| < 1$, i.e., iff $x \in (-R, R)$.

Example 4: Fix $R \in (0, \infty)$ and consider the power series $\sum_{n=1}^{\infty} \frac{1}{nR^n} x^n$. We apply the Ratio Test:

$$\lim_{n\to\infty} \frac{nR^n}{(n+1)R^{n+1}} \frac{|x|^{n+1}}{|x|^n} = |x| \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{1}{R} = \frac{|x|}{R}.$$

Therefore the series converges absolutely when $|x| < R$ and diverges when $|x| > R$. We must look separately at the case $|x| = R$ – i.e., when $x = \pm R$. When $x = R$, the series is the harmonic series $\sum_n \frac{1}{n}$, hence divergent. But when $x = -R$, the series is the alternating harmonic series $\sum_n \frac{(-1)^n}{n}$, hence (nonabsolutely) convergent. So the power series converges for $x \in [-R, R)$.

Example 5: Fix $R \in (0, \infty)$ and consider the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{nR^n} x^n$. We may rewrite this series as $\sum_{n=1}^{\infty} \frac{1}{nR^n}(-x)^n$, i.e., the same as in Example 4 but with x replaced by $-x$ throughout. Thus the series converges iff $-x \in [-R, R)$, i.e., iff $x \in (-R, R]$.

Example 6: Fix $R \in (0, \infty)$ and consider the power series $\sum_{n=1}^{\infty} \frac{1}{n^2 R^n} x^n$. We apply the Ratio Test:

$$\lim_{n\to\infty} \frac{n^2 R^n}{(n+1)^2 R^{n+1}} \frac{|x|^{n+1}}{|x|^n} = |x| \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{R} = \frac{|x|}{R}.$$

So once again the series converges absolutely when $|x| < R$, diverges when $|x| > R$, and we must look separately at $x = \pm R$. This time plugging in $x = R$ gives $\sum_n \frac{1}{n^2}$ which is a convergent p-series, whereas plugging in $x = -R$ gives $\sum_n \frac{(-1)^n}{n^2}$: since the p-series with $p = 2$ is convergent, the alternating p-series with $p = 2$ is absolutely convergent. Therefore the series converges (absolutely, in fact) on $[-R, R]$.

Thus the convergence set of a power series can take any of the following forms:

- the single point $\{0\} = [0, 0]$.
- the entire real line $\mathbb{R} = (-\infty, \infty)$.
- for any $R \in (0, \infty)$, an open interval $(-R, R)$.
- for any $R \in (0, \infty)$, a half-open interval $[-R, R)$ or $(-R, R]$
- for any $R \in (0, \infty)$, a closed interval $[-R, R]$.

In each case the set of values is an interval containing 0 and with a certain **radius**, i.e., an extended real number $R \in [0, \infty)$ such that the series definitely converges for all $x \in (-R, R)$ and definitely diverges for all x outside of $[-R, R]$. Our goal is to show that this is the case for *any* power series.

This goal can be approached at various degrees of sophistication. At the calculus level, it seems that we have already said what we need to: namely, we use the Ratio Test to see that the convergence set is an interval around 0 of a certain radius R. Namely, taking a general power series $\sum_n a_n x^n$ and applying the Ratio Test, we find

$$\lim_{n\to\infty} \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = |x| \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

So if $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, the Ratio Test tells us that the series converges when $|x|\rho < 1$ – i.e., iff $|x| < \frac{1}{\rho}$ – and diverges when $|x|\rho > 1$ – i.e., iff $|x| > \frac{1}{\rho}$. That is, the radius of convergence R is precisely the reciprocal of the Ratio Test limit ρ, with suitable conventions in the extreme cases, i.e., $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$.

So what more is there to say or do? The issue here is that we have *assumed* that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists. Although this is usually the case in simple examples of interest, it is certainly does not happen in general (we ask the reader to revisit §$X.X$ for examples of this). This we need to take a different approach in the general case.

LEMMA 2.54. *Let $\sum_n a_n x^n$ be a power series. Suppose that for some $A > 0$ we have $\sum_n a_n A^n$ is convergent. Then the power series converges absolutely at every $x \in (-A, A)$.*

PROOF. Let $0 < B < A$. It is enough to show that $\sum_n a_n B^n$ is absolutely convergent, for then so is $\sum_n a_n(-B)^n$. Now, since $\sum_n a_n A^n$ converges, $a_n A^n \to 0$: by omitting finitely many terms, we may assume that $|a_n A^n| \leq 1$ for all n. Then, since $0 < \frac{B}{A} < 1$, we have

$$\sum_n |a_n B^n| = \sum_n |a_n A^n| \left(\frac{B}{A}\right)^n \leq \sum_n \left(\frac{B}{A}\right)^n < \infty.$$

□

THEOREM 2.55. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series.*
a) There exists $R \in [0, \infty]$ such that:
(i) For all x with $|x| < R$, $\sum_n a_n x^n$ converges absolutely and
(ii) For all x with $|x| > R$, $\sum_n a_n x^n$ diverges.
b) If $R = 0$, then the power series converges only at $x = 0$.
c) If $R = \infty$, the power series converges for all $x \in \mathbb{R}$.
d) If $0 < R < \infty$, the convergence set of the power series is either $(-R, R)$, $[-R, R)$, $(-R, R]$ or $[-R, R]$.

PROOF. a) Let R be the least upper bound of the set of $x \geq 0$ such that $\sum_n a_n x^n$ converges. If y is such that $|y| < R$, then there exists A with $|y| < A < R$ such that $\sum_n a_n A^n$ converges, so by Lemma 2.54 the power series converges absolutely on $(-A, A)$, so in particular it converges absolutely at y. Thus R satisfies property (i). Similarly, suppose there exists y with $|y| > R$ such that $\sum_n a_n y^n$ converges. Then there exists A with $R < A < |y|$ such that the power series converges on $(-A, A)$, contradicting the definition of R.

We leave the proof of parts b) through d) to the reader as a straightforward exercise. $\square$

EXERCISE 2.60. *Prove parts) b), c) and d) of Theorem 2.55.*

EXERCISE 2.61. *Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two power series with positive radii of convergence R_a and R_b. Let $R = \min(R_a, R_b)$. Put $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Show that the "formal identity"*

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

is valid for all $x \in (-R, R)$. (Suggestion: this is really a matter of figuring out which previously established results on absolute convergence and Cauchy products to apply.)

The drawback of Theorem 2.55 is that it does not give an explicit description of the radius of convergence R in terms of the coefficients of the power series, as is the case when the ratio test limit $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists. In order to get an analogue of this is in the general case, we need to appeal to the Root Test instead of the Ratio Test and make use of the limit supremum. The following elegant result is generally attributed to the eminent turn of the century mathematician Hadamard,[31] who published it in 1888 [**Ha88**] and included it in his 1892 PhD thesis. This seems remarkably late in the day for a result which is so closely linked to (Cauchy's) root test. It turns out that the result was indeed established by our most usual suspect: it was first proven by Cauchy in 1821 [**Ca21**] but apparently had been all but forgotten.

THEOREM 2.56. *(Cauchy-Hadamard Formula) Let $\sum_n a_n x^n$ be a power series and put*

$$\overline{\theta} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

Then the radius of convergence of the power series is $R = \frac{1}{\overline{\theta}}$: that is, the series converges absolutely for $|x| < R$ and diverges for $|x| > R$.

PROOF. We have $\limsup_{n\to\infty} |a_n x^n|^{\frac{1}{n}} = |x| \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = |x|\overline{\theta}$. Put $R = \frac{1}{\overline{\theta}}$. If $|x| < R$, choose A such that $|x| < A < R$ and then A' such that

$$\overline{\theta} = \frac{1}{R} < A' < \frac{1}{A}.$$

Then for all sufficiently large n, $|a_n x^n|^{\frac{1}{n}} \leq A'A < 1$, so the series converges absolutely by the Root Test. Similarly, if $|x| > R$, choose A such that $R < |x| < A$ and then A' such that

$$\frac{1}{A} < A' < \frac{1}{R} = \overline{\theta}.$$

Then there are infinitely many non-negative integers n such that $|a_n x^n|^{\frac{1}{n}} \geq A'A > 1$, so the series $\sum_n a_n x^n$ diverges: indeed $a_n x^n \nrightarrow 0$. $\square$

The following result gives a natural criterion for the radius of convergence of a power series to be 1.

[31] Jacques Salomon Hadamard, 1865-1963

COROLLARY 2.57. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers, and let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$.*
a) If $\{a_n\}$ is bounded, then $R \geq 1$.
b) If $a_n \nrightarrow 0$, then $R \leq 1$.
c) Thus if $\{a_n\}$ is bounded but not convergent to zero, $R = 1$.

EXERCISE 2.62. *Prove Corollary 2.57.*

The following result will be useful in Chapter 3 when we consider power series as functions and wish to differentiate and integrate them termwise.

THEOREM 2.58. *Let $\sum_n a_n x^n$ be a power series with radius of convergence R. Then, for any $k \in \mathbb{Z}$, the radius of convergence of the power series $\sum_{n \to \infty} n^k a_n x^n$ is also R.*

PROOF. Since $\lim_{n\to\infty} \frac{(n+1)^k}{n^k} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^k = 1$, by Corollary 2.27 we have
$$\lim_{n\to\infty} (n^k)^{1/n} = \lim_{n\to\infty} n^{k/n} = 1.$$
(Alternately, one can of course compute this limit by the usual methods of calculus: take logarithms and apply L'Hôpital's Rule.) Therefore
$$\limsup_{n\to\infty} (n^k|a_n|)^{\frac{1}{n}} = \left(\lim_{n\to\infty} (n^k)^{\frac{1}{n}}\right)\left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right) = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.$$
The result now follows from Hadamard's Formula. □

Remark: For the reader who is less than comfortable with limits infimum and supremum, we recommend simply assuming that the Ratio Test limit $\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ exists and proving Theorem 2.58 under that additional assumption using the Ratio Test. This will be good enough for most of the power series encountered in practice.

EXERCISE 2.63. *By Theorem 2.58, the radii of convergence of $\sum_n a_n x^n$ and $\sum_n n a_n x^n$ are equal, say both equal to R. Show that the interval of convergence of $\sum_n n a_n x^n$ is contained in the interval of convergence of $\sum_n a_n x^n$, and give an example where a containment is proper. In other words, passage from $\sum_n a_n x^n$ to $\sum_n n a_n x^n$ does not change the radius of convergence, but convergence at one or both of the endpoints may be lost.*

10.2. Recentered Power Series.

10.3. Abel's Theorem.

THEOREM 2.59. *(Abel's Theorem) Let $\sum_{n=0}^{\infty} a_n$ be a convergent series. Then*
$$\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n.$$

PROOF. [**R**, Thm. 8.2] Since $\sum_n a_n$ converges, the sequence $\{a_n\}$ is bounded so by Corollary 2.57 the radius of convergence of $f(x) = \sum_n a_n x^n$ is at least one. As usual, we put $A_0 = 0$, $A_n = a_1 + \ldots + A_n$ and $A = \lim_{n\to\infty} A_n = \sum_{n=0}^{\infty} a_n$. Then
$$\sum_{n=0}^{n} a_n x^n = \sum_{n=0}^{n} (A_n - A_{n-1}) x^n = (1-x) \sum_{n=0}^{m-1} A_n x^n + A_m x^m.$$

For $|x| < 1$, we let $m \to \infty$ and get

$$f(x) = (1-x)\sum_{n=0}^{\infty} A_n x^n.$$

Now fix $\epsilon > 0$, and choose N such that $n \geq N$ implies $|A - A_n| < \frac{\epsilon}{2}$. Then, since

$$(1-x)\sum_{n=0}^{\infty} x^n = 1 \tag{16}$$

for all $|x| < 1$, we get

$$|f(x) - A| = |f(x) - 1 \cdot A| = |(1-x)\sum_{n=0}^{\infty}(A_n - A)x^n|$$

$$= |(1-x)\sum_{n=0}^{\infty}(A_n - A)x^n| = (1-x)\sum_{n=0}^{N}|A_n - A||x|^n + \left(\frac{\epsilon}{2}\right)(1-x)\sum_{n>N}|x|^n.$$

By (16), for any $x \in (0,1)$, the last term in right hand side of the above equation is $\frac{\epsilon}{2}$. Moreover the limit of the first term of the right hand side as x approaches 1 from the left is zero. We may therefore choose $\delta > 0$ such that for all $x > 1 - \delta$, $|(1-x)\sum_{n=0}^{\infty}(A_n - A)x^n| < \frac{\epsilon}{2}$ and thus for such x,

$$|f(x) - A| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

The rest of this section is an extended exercise in "Abel's Theorem appreciation". First of all, we think it may help to restate the result in a form which is slightly more general and moreover makes more clear exactly what has been established.

THEOREM 2.60. *(Abel's Theorem Mark II) Let $f(x) = \sum_n a_n(x-c)^n$ be a power series with radius of convergence $R > 0$, hence convergent at least for all $x \in (c-R, c+R)$.*
a) Suppose that the power series converges at $x = c + R$. Then the function $f : (c-R, c+R] \to \mathbb{R}$ is continuous at $x = c+R$: $\lim_{x\to(c+R)^-} f(x) = f(c+R)$.
b) Suppose that the power series converges at $x = c - R$. Then the function $f : [c-R, c+R) \to \mathbb{R}$ is continuous at $x = c-R$: $\lim_{x\to(c-R)^+} f(x) = f(c-R)$.

EXERCISE 2.64. *Prove Theorem 2.60.*

EXERCISE 2.65. *Consider $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, which converges for all $x \in (-1,1)$. Show that $\lim_{x\to -1^+} f(x)$ exists and thus f extends to a continuous function on $[-1,1)$. Nevertheless $f(-1) \neq \lim_{x\to -1^+} f(x)$. Why does this not contradict Abel's Theorem?*

In the next chapter it will be a major point of business to study $f(x) = \sum_n a_n(x-c)^n$ as a function from $(c-R, c+R)$ to $\mathbb{R}$ and we will prove that it has many wonderful properties: e.g. more than being continuous it is in fact **smooth**, i.e., possesses derivatives of all orders at every point. But studying power series "at the boundary" is a much more delicate affair, and the comparatively modest continuity assured by Abel's Theorem is in fact extremely powerful and useful.

As our first application, we round out our treatment of Cauchy products by showing that the Cauchy product never "wrongly converges".

THEOREM 2.61. *Let $\sum_{n=0}^{\infty} a_n$ be a series converging to A and $\sum_{n=0}^{\infty} b_n$ be a series converging to B. As usual, we define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ and the Cauchy product series $\sum_{n=0}^{\infty} c_n$. Suppose that $\sum_{n=0}^{\infty} c_n$ converges to C. Then $C = AB$.*

PROOF. Rather remarkably, we "reduce" to the case of power series! Namely, put $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $g(x) = \sum_{n=0}^{\infty} b_n x^n$ and $h(x) = \sum_{n=0}^{\infty} c_n x^n$. By assumption, $f(x)$, $g(x)$ and $h(x)$ all converge at $x = 1$, so by Lemma 2.54 the radii of convergence of $\sum_n a_n x^n$, $\sum_n b_n x^n$ and $\sum_n c_n x^n$ are all at least one. Now all we need to do is apply Abel's Theorem:

$$C = h(1) \overset{\text{AT}}{=} \lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} f(x) g(x)$$

$$= \left(\lim_{x \to 1^-} f(x) \right) \left(\lim_{x \to 1^-} g(x) \right) \overset{\text{AT}}{=} f(1) g(1) = AB.$$

$\square$

Abel's Theorem gives rise to a **summability method**, or a way to extract numerical values out of certain divergent series $\sum_n a_n$ "as though they converged". Instead of forming the sequence of partial sums $A_n = a_0 + \ldots + a_n$ and taking the limit, suppose instead we look at $\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_x x^n$. We say the series $\sum_n a_n$ is **Abel summable** if this limit exists, in which case we write it as $\mathcal{A} \sum_{n=0}^{\infty} a_n$, the **Abel sum** of the series. The point of this is that by Abel's theorem, if a series $\sum_{n=0}^{\infty} a_n$ is actually convergent, say to A, then it is also Abel summable and its Abel sum is also equal to A. However, there are series which are divergent yet Abel summable.

Example: Consider the series $\sum_{n=0}^{\infty} (-1)^n$. As we saw, the partial sums alternate between 0 and 1 so the series does not diverge. We mentioned earlier that (the great) L. Euler believed that nevertheless the right number to attach to the series $\sum_{n=0}^{\infty} (-1)^n$ is $\frac{1}{2}$. Since the two partial limits of the sequence of partial sums are 0 and 1, it seems vaguely plausible to split the difference.

Abel's Theorem provides a much more convincing argument. The power series $\sum_n (-1)^n x^n$converges for all x with $|x| < 1$, and moreover for all such x we have

$$\sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1 - (-x)} = \frac{1}{1 + x},$$

and thus

$$\lim_{x \to 1^-} \sum_{n=0}^{\infty} (-1)^n x^n = \lim_{x \to 1^-} \frac{1}{1 + x} = \frac{1}{2}.$$

That is, the series $\sum_n (-1)^n$ is divergent but Abel summable, with Abel sum $\frac{1}{2}$. So Euler's idea was better than we gave him credit for.

EXERCISE 2.66. *Suppose that $\sum_{n=0}^{\infty} a_n$ is a series with non-negative terms. Show that in this case the converse of Abel's Theorem holds: if* $\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = L$*, then $\sum_{n=0}^{\infty} a_n = L$.*

We end with one final example of the uses of Abel's Theorem. Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, which is easily seen to be nonabsolutely convergent. Can we by any chance evaluate its sum exactly?

Here is a crazy argument: consider the function

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n},$$

which converges for $x \in (-1,1)$. Actually the series defining f is geometric with $r = -x^2$, so we have

$$f(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}. \tag{17}$$

Now the antiderivative F of f with $F(0) = 0$ is $F(x) = \arctan x$. Assuming we can integrate power series termwise as if they were polynomials (!), we get

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}. \tag{18}$$

This latter series converges for $x \in (-1,1]$ and evaluates at $x = 1$ to the series we started with. Thus our guess is that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = F(1) = \arctan 1 = \frac{\pi}{4}.$$

There are at least two gaps in this argument, which we will address in a moment. But before that we should probably look at some numerics. Namely, summing the first 10^4 terms and applying the Alternating Series Error Bound, we find that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \approx 0.78542316\ldots,$$

with an error of at most $\frac{1}{2\cdot 10^4}$. On the other hand, my software package tells me that

$$\frac{\pi}{4} = 0.78539816339744830961566084581\ldots.$$

The difference between these two numerical approximations is $0.000249975\ldots$, or about $\frac{1}{40000}$. This is fairly convincing evidence that indeed $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$.

So let's look back at the gaps in the argument. First of all there is this wishful business about integrating a power series termwise as though it were a polynomial. I am happy to tell you that such termwise integration is indeed justified for all power series on the interior of their interval of convergence: this and other such pleasant facts are discussed in §3.2. But there is another, much more subtle gap in the above argument (too subtle for many calculus books, in fact: check around). The power series expansion (17) of $\frac{1}{1+x^2}$ is valid only for $x \in (-1,1)$: it certainly is not valid for $x = 1$ because the power series doesn't even converge at $x = 1$. On the other hand, we want to evaluate the antiderivative F at $x = 1$. This seems problematic, but there is an out...Abel's Theorem. Indeed, the theory of termwise integration of power series will tell us that (18) holds on $(-1,1)$ and then we have:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \stackrel{\text{AT}}{=} \lim_{x\to 1^-} \frac{(-1)^n x^{2n+1}}{2n+1} = \lim_{x\to 1^-} \arctan x = \arctan 1.$$

where the first equality holds by Abel's Theorem and the last equality holds by continuity of the arctangent function. And finally, of course, $\arctan 1 = \frac{\pi}{4}$. Thus Abel's theorem can be used (and should be used!) to justify a number of calculations in which we are apparently playing with fire by working at the boundary of the interval of convergence of a power series.

EXERCISE 2.67. *In §8.1, we showed that the alternating harmonic series converged and alluded to the identity* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} = \log 2$. *Try to justify this along the lines of the example above.*

CHAPTER 3

Sequences and Series of Functions

1. Pointwise and Uniform Convergence

All we have to do now is take these lies and make them true somehow. – G. Michael[1]

1.1. Pointwise convergence: cautionary tales.

Let I be an interval in the real numbers. A **sequence of real functions** is a sequence $f_0, f_1, \ldots, f_n, \ldots$, with each f_n a function from I to $\mathbb{R}$.

For us the following example is all-important: let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. So f may be viewed as a function $f : (-R, R) \to \mathbb{R}$. Put $f_n = \sum_{k=0}^{n} a_k x^k$, so each f_n is a polynomial of degree at most n; therefore f_n makes sense as a function from $\mathbb{R}$ to $\mathbb{R}$, but let us restrict its domain to $(-R, R)$. Then we get a sequence of functions $f_0, f_1, \ldots, f_n, \ldots$.

As above, our stated goal is to show that the function f has many desirable properties: it is continuous and indeed infinitely differentiable, and its derivatives and antiderivatives can be computed term-by-term. Since the functions f_n have all these properties (and more – each f_n is a polynomial), it seems like a reasonable strategy to define some sense in which the sequence $\{f_n\}$ **converges** to the function f, in such a way that this converges process *preserves* the favorable properties of the f_n's.

The previous description perhaps sounds overly complicated and mysterious, since in fact there is an evident sense in which the sequence of functions f_n converges to f. Indeed, to say that x lies in the open interval $(-R, R)$ of convergence is to say that the sequence $f_n(x) = \sum_{k=0}^{n} a_k x^k$ converges to $f(x)$.

This leads to the following definition: if $\{f_n\}_{n=1}^{\infty}$ is a sequence of real functions defined on some interval I and $f : I \to \mathbb{R}$ is another function, we say f_n **converges to f pointwise** if for all $x \in I$, $f_n(x) \to f(x)$. (We also say f is the **pointwise limit** of the sequence $\{f_n\}$.) In particular the sequence of partial sums of a power series converges pointwise to the power series on the interval I of convergence.

Remark: There is similarly a notion of an infinite series of functions $\sum_{n=0}^{\infty} f_n$ and of pointwise convergence of this series to some limit function f. Indeed, as in the case of just one series, we just define $S_n = f_0 + \ldots + f_n$ and say that $\sum_n f_n$ converges pointwise to f if the sequence S_n converges pointwise to f.

[1]George Michael, 1963–

The great mathematicians of the 17th, 18th and early 19th centuries encountered many sequences and series of functions (again, especially power series and Taylor series) and often did not hesitate to assert that the pointwise limit of a sequence of functions having a certain nice property itself had that nice property.[2] The problem is that statements like this unfortunately need not be true!

Example 1: Define $f_n = x^n : [0,1] \to \mathbb{R}$. Clearly $f_n(0) = 0^n = 0$, so $f_n(0) \to 0$. For any $0 < x \leq 1$, the sequence $f_n(x) = x^n$ is a geometric sequence with geometric ratio x, so that $f_n(x) \to 0$ for $0 < x < 1$ and $f_n(1) \to 1$. It follows that the sequence of functions $\{f_n\}$ has a pointwise limit $f : [0,1] \to \mathbb{R}$, the function which is 0 for $0 \leq x < 1$ and 1 at $x = 1$. Unfortunately the limit function is discontinuous at $x = 1$, despite the fact that each of the functions f_n are continuous (and are polynomials, so really as nice as a function can be). Therefore **the pointwise limit of a sequence of continuous functions need not be continuous**.

Remark: Example 1 was chosen for its simplicity, not to exhibit maximum pathology. It is possible to construct a sequence $\{f_n\}_{n=1}^{\infty}$ of polynomial functions converging pointwise to a function $f : [0,1] \to \mathbb{R}$ that has infinitely many discontinuities! (On the other hand, it turns out that it is not possible for a pointwise limit of continuous functions to be discontinuous at *every* point. This is a theorem of R. Baire. But we had better not talk about this, or we'll get distracted from our stated goal of establishing the wonderful properties of power series.)

One can also find assertions in the math papers of old that if f_n converges to f pointwise on an interval $[a,b]$, then $\int_a^b f_n dx \to \int_a^b f dx$. To a modern eye, there are in fact two things to establish here: first that if each f_n is Riemann integrable, then the pointwise limit f must be Riemann integrable. And second, that *if* f is Riemann integrable, its integral is the limit of the sequence of integrals of the f_n's. In fact *both* of these are false!

Example 2: Define a sequence $\{f_n\}_{n=0}^{\infty}$ with common domain $[0,1]$ as follows. Let f_0 be the constant function 1. Let f_1 be the function which is constantly 1 except $f(0) = f(1) = 0$. Let f_2 be the function which is equal to f_1 except $f(1/2) = 0$. Let f_3 be the function which is equal to f_2 except $f(1/3) = f(2/3) = 0$. And so forth. To get from f_n to f_{n+1} we change the value of f_n at the finitely many rational numbers $\frac{a}{n}$ in $[0,1]$ from 1 to 0. Thus each f_n is equal to 1 except at a finite set of points: in particular it is bounded with only finitely many discontinuities, so it is Riemann integrable. The functions f_n converges pointwise to a function f which is 1 on every irrational point of $[0,1]$ and 0 on every rational point of $[0,1]$. Since every open interval (a,b) contains both rational and irrational numbers, the function f is not Riemann integrable: for any partition of $[0,1]$ its upper sum is 1 and its lower sum is 0. Thus a pointwise limit of Riemann integrable functions need not be Riemann integrable.

[2]This is an exaggeration. The precise definition of convergence of real sequences did not come until the work of Weierstrass in the latter half of the 19th century. Thus mathematicians spoke of functions f_n "approaching" or "getting infinitely close to" a fixed function f. Exactly what they meant by this – and indeed, whether even they knew exactly what they meant (presumably some did better than others) is a matter of serious debate among historians of mathematics.

Example 3: We define a sequence of functions $f_n : [0,1] \to \mathbb{R}$ as follows: $f_n(0) = 0$, and $f_n(x) = 0$ for $x \geq \frac{1}{n}$. On the interval $[0, \frac{1}{n}]$ the function forms a "spike": $f(\frac{1}{2n}) = 2n$ and the graph of f from $(0,0)$ to $(\frac{1}{2n}, 2n)$ is a straight line, as is the graph of f from $(\frac{1}{2n}, 2n)$ to $(\frac{1}{n}, 0)$. In particular f_n is piecewise linear hence continuous, hence Riemann integable, and its integral is the area of a triangle with base $\frac{1}{n}$ and height $2n$: $\int_0^1 f_n dx = 1$. On the other hand this sequence converges pointwise to the zero function f. So

$$\lim_{n\to\infty} \int_0^1 f_n = 1 \neq 0 = \int_0^1 \lim_{n\to\infty} f_n.$$

Example 4: Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded differentiable function such that $\lim_{n\to\infty} g(n)$ does not exist. (For instance, we may take $g(x) = \sin(\frac{\pi x}{2})$.) For $n \in \mathbb{Z}^+$, define $f_n(x) = \frac{g(nx)}{n}$. Let M be such that $|g(x)| \leq M$ for all $x \in \mathbb{R}$. Then for all $x \in \mathbb{R}$, $|f_n(x)| \leq \frac{M}{n}$, so f_n conveges pointwise to the function $f(x) \equiv 0$ and thus $f'(x) \equiv 0$. In particular $f'(1) = 0$. On the other hand, for any fixed nonzero x, $f_n'(x) = \frac{ng'(nx)}{n} = g'(nx)$, so

$$\lim_{n\to\infty} f_n'(1) = \lim_{n\to\infty} g'(n) \text{ does not exist.}$$

Thus

$$\lim_{n\to\infty} f_n'(1) \neq (\lim_{n\to\infty} f_n)'(1).$$

A common theme in all these examples is the **interchange of limit operations**: that is, we have some other limiting process corresponding to the condition of continuity, integrability, differentiability, integration or differentiation, and we are wondering whether it changes things to perform the limiting process on each f_n individually and then take the limit versus taking the limit first and then perform the limiting process on f. As we can see: in general it does matter! This is not to say that the interchange of limit operations is something to be systematically avoided. On the contrary, it is an essential part of the subject, and in "natural circumstances" the interchange of limit operations is probably valid. But we need to develop theorems to this effect: i.e., under *some specific additional hypotheses*, interchange of limit operations is justified.

1.2. Consequences of uniform convergence. It turns out that the key hypothesis in most of our theorems is the notion of *uniform convergence*.

Let $\{f_n\}$ be a sequence of functions with domain I. We say f_n **converges uniformly** to f and write $f_n \stackrel{u}{\to} f$ if for all $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$ and all $x \in I$, $|f_n(x) - f(x)| < \epsilon$.

How does this definition differ from that of pointwise convergence? Let's compare: $f_n \to f$ pointwise if for all $x \in I$ and all $\epsilon > 0$, there exists $n \in \mathbb{Z}^+$ such that for all $n \geq N$, $|f_n(x) - f(x)|\epsilon$. The only difference is in the order of the quantifiers: in pointwise convergence we are first given ϵ and x and then must find an $N \in \mathbb{Z}^+$: that is, the N is allowed to depend both on ϵ and the point $x \in I$. In the definition of uniform convergence, we are given $\epsilon > 0$ and must find an $N \in \mathbb{Z}^+$ which works simultaneously (or "uniformly") for all $x \in I$. Thus uniform convergence is a stronger condition than pointwise convergence, and in particular if f_n converges to f uniformly, then certainly f_n converges to f pointwise.

EXERCISE 3.1. *Show that there is a Cauchy Criterion for uniform convergence, namely:* $f_n \stackrel{u}{\to} f$ *on an interval* I *iff for all* $\epsilon > 0$, *there exists* $N \in \mathbb{Z}^+$ *such that for all* $m, n \geq N$ *and all* $x \in I$, $|f_m(x) - f_n(x)| < \epsilon$.

The following result is the most basic one fitting under the general heading "uniform convegence justifies the exchange of limiting operations."

THEOREM 3.1. *Let* $\{f_n\}$ *be a sequence of functions with common domain* I, *and let* c *be a point of* I. *Suppose that for all* $n \in \mathbb{Z}^+$, $\lim_{x \to c} f_n = L_n$. *Suppose moreover that* $f_n \stackrel{u}{\to} f$. *Then the sequence* $\{L_n\}$ *is convergent,* $\lim_{x \to c} f(x)$ *exists and we have equality:*

$$\lim_{n \to \infty} L_n = \lim_{n \to \infty} \lim_{x \to c} f_n(x) = \lim_{x \to c} f(x) = \lim_{x \to c} \lim_{n \to \infty} f_n(x).$$

PROOF. Step 1: We show that the sequence $\{L_n\}$ is convergent. Since we don't yet have a real number to show that it converges *to*, it is natural to try to use the Cauchy criterion, hence to try to bound $|L_m - L_n|$. Now comes the trick: for all $x \in I$ we have

$$|L_m - L_n| \leq |L_m - f_m(x)| + |f_m(x) - f_n(x)| + |f_n(x) - L_n|.$$

By the Cauchy criterion for uniform convergence, for any $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that for all $m, n \geq N$ and all $x \in I$ we have $|f_m(x) - f_n(x)| < \frac{\epsilon}{3}$. Moreover, the fact that $f_m(x) \to L_m$ and $f_n(x) \to L_n$ give us bounds on the first and last terms: there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|L_n - f_n(x)| < \frac{\epsilon}{3}$ and $|L_m - f_m(x)| < \frac{\epsilon}{3}$. Combining these three estimates, we find that by taking $x \in (c - \delta, c + \delta)$, $x \neq c$ and $m, n \geq N$, we have

$$|L_m - L_n| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

So the sequence $\{L_n\}$ is Cauchy and hence convergent, say to the real number L.
Step 2: We show that $\lim_{x \to c} f(x) = L$ (so in particular the limit exists!). Actually the argument for this is very similar to that of Step 1:

$$|f(x) - L| \leq |f(x) - f_n(x)| + |f_n(x) - L_n| + |L_n - L|.$$

Since $L_n \to L$ and $f_n(x) \to f(x)$, the first and last term will each be less than $\frac{\epsilon}{3}$ for sufficiently large n. Since $f_n(x) \to L_n$, the middle term will be less than $\frac{\epsilon}{3}$ for x sufficiently close to c. Overall we find that by taking x sufficiently close to (but not equal to) c, we get $|f(x) - L| < \epsilon$ and thus $\lim_{x \to c} f(x) = L$. $\square$

COROLLARY 3.2. *Let* f_n *be a sequence of continuous functions with common domain* I *and suppose that* $f_n \stackrel{u}{\to} f$ *on* I. *Then* f *is continuous on* I.

Since Corollary 3.2 is somewhat simpler than Theorem 3.1, as a service to the student we include a separate proof.

PROOF. Let $x \in I$. We need to show that $\lim_{x \to c} f(x) = f(c)$, thus we need to show that for any $\epsilon > 0$ there exists $\delta > 0$ such that for all x with $|x - c| < \delta$ we have $|f(x) - f(c)| < \epsilon$. The idea – again! – is to trade this one quantity for three quantities that we have an immediate handle on by writing

$$|f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|.$$

By uniform convergence, there exists $n \in \mathbb{Z}^+$ such that $|f(x) - f_n(x)| < \frac{\epsilon}{3}$ for all $x \in I$: in particular $|f_n(c) - f(c)| = |f(c) - f_n(c)| < \frac{\epsilon}{3}$. Further, since $f_n(x)$ is continuous, there exists $\delta > 0$ such that for all x with $|x - c| < \delta$ we have $|f_n(x) - f_n(c)| < \frac{\epsilon}{3}$. Consolidating these estimates, we get

$$|f(x) - f(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

□

EXERCISE 3.2. *Consider again $f_n(x) = x^n$ on the interval $[0,1]$. We saw in Example 1 above that f_n converges pointwise to the discontinuous function f which is 0 on $[0,1)$ and 1 at $x = 1$.*
a) Show directly from the definition that the convergence of f_n to f is not *uniform.*
b) Try to pinpoint exactly where the proof of Theorem 3.1 breaks down when applied to this non-uniformly convergent sequence.

EXERCISE 3.3. *Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of functions. Show the following are equivalent:*
(i) $f_n \stackrel{u}{\to} f$ on $[a,b]$.
(ii) $f_n \stackrel{u}{\to} f$ on $[a,b)$ and $f_n(b) \to f(b)$.

THEOREM 3.3. *Let $\{f_n\}$ be a sequence of Riemann integrable functions with common domain $[a,b]$. Suppose that $f_n \stackrel{u}{\to} f$. Then f is Riemann integrable and*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n = \int_a^b f.$$

PROOF. Since we have not covered the Riemann integral in these notes, we are not in a position to give a full proof of Theorem 3.3. For this see [**R**, Thm. 7.16] or my McGill lecture notes. We will content ourselves with the special case in which each f_n is continuous, hence by Theorem 3.1 so is f. All continuous functions are Riemann integrable, so certainly f is Riemann integrable: what remains to be seen is that it is permisible to interchange the limit and the integral.

To see this, fix $\epsilon > 0$, and let $N \in \mathbb{Z}^+$ be such that for all $n \geq N$, $f(x) - \frac{\epsilon}{b-a} < f_n(x) \leq f(x) + \frac{\epsilon}{b-a}$. Then

$$(\int_a^n f) - \epsilon = \int_a^b (f - \frac{\epsilon}{b-a}) \leq \int_a^b f_n \leq \int_a^b (f + \frac{\epsilon}{b-a}) = (\int_a^b f) + \epsilon.$$

That is, $|\int_a^b f_n - \int_a^b f| < \epsilon$ and therefore $\int_a^b f_n \to \int_a^b f$. □

EXERCISE 3.4. *It follows from Theorem 3.3 that the sequences in Examples 2 and 3 above are not uniformly convergent. Verify this directly.*

COROLLARY 3.4. *Let $\{f_n\}$ be a sequence of continuous functions defined on the interval $[a,b]$ such that $\sum_{n=0}^{\infty} f_n \stackrel{u}{\to} f$. For each n, let $F_n : [a,b] \to \mathbb{R}$ be the unique function with $F_n' = f_n$ and $F_n(a) = 0$, and similarly let $F : [a,b] \to \mathbb{R}$ be the unique function with $F' = f$ and $F(a) = 0$. Then $\sum_{n=0}^{\infty} F_n \stackrel{u}{\to} F$.*

EXERCISE 3.5. *Prove Corollary 3.4.*

Our next order of business is to discuss differentiation of sequences of functions. But let us take a look back at Example 4, which was of a bounded function $g : \mathbb{R} \to \mathbb{R}$ such that $\lim_{n\to\infty} g(x)$ does not exist and $f_n(x) = \frac{g(nx)}{n}$. Let M be such that $|g(x)| \leq M$ for all $\mathbb{R}$. Then for all $x \in \mathbb{R}$, $|f_n(x)| \leq \frac{M}{n}$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, for any $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $|f_n(x)| \leq \frac{M}{n} < \epsilon$. Thus $f_n \overset{u}{\to} 0$. In other words, we have the somewhat distressing fact that *uniform convergence of f_n to f does not imply that f_n' converges.*

Well, don't panic. What we want is true *in practice*; we just need suitable hypotheses. We will give a relatively simple result sufficient for our coming applications. Before stating and proving it, we include the following quick calculus refresher.

THEOREM 3.5. *(Fundamental Theorem of Calculus) Let $f : [a, b] \to \mathbb{R}$ be a continuous function.*
(FTCa) We have $\frac{d}{dx} \int_a^x f(t)dt = f(x)$.
(FTCb) If $F : [a, b] \to \mathbb{R}$ is such that $F' = f$, then $\int_a^b f = F(b) - F(a)$.

THEOREM 3.6. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions with common domain $[a, b]$. We suppose:*
(i) Each f_n is continuously differentiable, i.e., f_n' exists and is continuous,
(ii) The functions f_n converge pointwise on $[a, b]$ to some function f, and
(iii) The functions f_n' converge uniformly on $[a, b]$ to some function g.
Then f is differentiable and $f' = g$, or in other words

$$(\lim_{n\to\infty} f_n)' = \lim_{n\to\infty} f_n'.$$

PROOF. Let $x \in [a, b]$. Since $f_n' \overset{u}{\to} g$ on $[a, b]$, certainly $f_n' \overset{u}{\to} g$ on $[a, x]$. Since each f_n' is assumed to be continuous, by 3.1 g is also continuous. Now applying Theorem 3.3 and (FTCb) we have

$$\int_a^x g = \lim_{n\to\infty} \int_a^x f_n' = \lim_{n\to\infty} f_n(x) - f_n(a) = f(x) - f(a).$$

Differentiating both sides and applying (FTCa), we get

$$g = (f(x) - f(a))' = f'.$$

□

COROLLARY 3.7. *Let $\sum_{n=0}^\infty f_n(x)$ be a series of functions converging pointwise to $f(x)$. Suppose that each f_n' is continuously differentiable and $\sum_{n=0}^\infty f_n'(x) \overset{u}{\to} g$. Then f is differentiable and $f' = g$:*

$$(\sum_{n=0}^\infty f_n)' = \sum_{n=0}^\infty f_n'. \tag{19}$$

EXERCISE 3.6. *Prove Corollary 3.7.*

When for a series $\sum_n f_n$ it holds that $(\sum_n f_n)' = \sum_n f_n'$, we say that the series can be differentiated **termwise** or **term-by-term**. Thus Corollary 3.7 gives a condition under which a series of functions can be differentiated termwise.

Although Theorem 3.6 (or more precisely, Corollary 3.7) will be sufficient for our needs, we cannot help but record the following stronger version.

THEOREM 3.8. *Let $\{f_n\}$ be differentiable functions on the interval $[a,b]$ such that $\{f_n(x_0)\}$ is convergent for some $x_0 \in [a,b]$. If there is $g : [a,b] \to \mathbb{R}$ such that $f_n' \xrightarrow{u} g$ on $[a,b]$, then there is $f : [a,b] \to \mathbb{R}$ such that $f_n \xrightarrow{u} f$ on $[a,b]$ and $f' = g$.*

PROOF. [**R**, pp.152-153]
Step 1: Fix $\epsilon > 0$, and choose $N \in \mathbb{Z}^+$ such that $m, n \geq N$ implies $|f_m(x_0) - f_n(x_0)|\frac{\epsilon}{2}$ and $|f_m'(t) - f_n'(t)| < \frac{\epsilon}{2(b-a)}$ for all $t \in [a,b]$. The latter inequality is telling us that the derivative of $g := f_m - f_n$ is small on the entire interval $[a,b]$. Applying the Mean Value Theorem to g, we get a $c \in (a,b)$ such that for all $x, t \in [a,b]$ and all $m, n \geq N$,

$$|g(x) - g(t)| = |x - t||g'(c)| \leq |x - t| \left(\frac{\epsilon}{2(b-a)} \right) \leq \frac{\epsilon}{2}. \tag{20}$$

It follows that for all $x \in [a,b]$,

$$|f_m(x) - f_n(x)| = |g(x)| \leq |g(x) - g(x_0)| + |g(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

By the Cauchy criterion, f_n is uniformly convergent on $[a,b]$ to some function f.
Step 2: Now **fix** $x \in [a,b]$ and define

$$\varphi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}$$

and

$$\varphi(t) = \frac{f(t) - f(x)}{t - x},$$

so that for all $n \in \mathbb{Z}^+$, $\lim_{x \to t} \varphi_n(t) = f_n'(x)$. Now by (20) we have

$$|\varphi_m(t) - \varphi_n(t)| \leq \frac{\epsilon}{2(b-a)}$$

for all $m, n \geq N$, so once again by the Cauchy criterion φ_n converges uniformly for all $t \neq x$. Since $f_n \xrightarrow{u} f$, we get $\varphi_n \xrightarrow{u} \varphi$ for all $t \neq x$. Finally we apply Theorem 3.1 on the interchange of limit operations:

$$f'(x) = \lim_{t \to x} \varphi(t) = \lim_{t \to x} \lim_{n \to \infty} \varphi_n(t) = \lim_{n \to \infty} \lim_{t \to x} \varphi_n(t) = \lim_{n \to \infty} f_n'(x). \qquad \square$$

1.3. A criterion for uniform convergence: the Weierstrass M-test.

We have just seen that uniform convergence of a sequence of functions (and possibly, of its derivatives) has many pleasant consequences. The next order of business is to give a useful general criterion for a sequence of functions to be uniformly convergent.

For a function $f : I \to \mathbb{R}$, we define

$$||f|| = \sup_{x \in I} |f(x)|.$$

In (more) words, $||f||$ is the least $M \in [0, \infty]$ such that $|f(x)| \leq M$ for all $x \in I$.

THEOREM 3.9. *(Weierstrass M-Test) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on an interval I. Let $\{M_n\}_{n=1}^{\infty}$ be a non-negative sequence such that $||f_n|| \leq M_n$ for all n and $M = \sum_{n=1}^{\infty} M_n < \infty$. Then $\sum_{n=1}^{\infty} f_n$ is uniformly convergent.*

PROOF. Put $S_n(x) = \sum_{k=1}^n f_k(x)$. Since the series $\sum_n M_n$ is convergent, it is Cauchy: for all $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$ and $m \geq 0$ we have $M_{n+m} - M_n = \sum_{k=n+1}^{n+m} M_k < \epsilon$. But then for all $x \in I$ we have

$$|S_{n+m}(x) - S_n(x)| = |\sum_{k=n+1}^{n+m} f_k(x)| \leq \sum_{k=n+1}^{n+m} |f_k(x)| \leq \sum_{n=k+1}^{n+m} ||f_k|| \leq \sum_{k=n+1}^{n+m} M_k < \epsilon.$$

Therefore the series is uniformly convergent by the Cauchy criterion. □

1.4. Another criterion for uniform convergence: Dini's Theorem.

There is another criterion for uniform convergence which is sometimes useful, due originally to Dini.[3] To state it we need a little terminology: let $f_n : I \to \mathbb{R}$ be a sequence of functions. We say that we have an **increasing sequence** if for all $x \in I$ and all n, $f_n(x) \leq f_{n+1}(x)$.

Warning: An "increasing sequence of functions" is *not the same* as a "sequence of increasing functions"! In the former, what is increasing is the values $f_n(x)$ for any fixed x as we increase the index n, whereas the in the latter, what is increasing is the values $f_n(x)$ for any fixed n as we we increase x. For instance, $f_n(x) = \sin x - \frac{1}{n}$ is an increasing sequence of functions but not a sequence of increasing functions.

Evidently we have the allied definition of a **decreasing sequence of functions**, that is, a sequence of functions $f_n : I \to \mathbb{R}$ such that for all $x \in I$ and all n, $f_{n+1}(x) \leq f_n(x)$. Note that $\{f_n\}$ is an increasing sequence of functions iff $\{-f_n\}$ is a decreasing sequence of functions.

In order to prove the desired theorem we need a technical result borrowed (bordering on stolen!) from real analysis, a special case of the celebrated **Heine-Borel Theorem**. And in order to state the result we need some setup.

Let $[a, b]$ be a closed interval with $a \leq b$. A collection $\{I_i\}$ of intervals **covers** $[a, b]$ if each $x \in [a, b]$ is contained in I_i for at least on i, or more succinctly if

$$[a, b] \subset \bigcup_i I_i.$$

LEMMA 3.10. *(Compactness Lemma) Let $[a, b]$ be a closed interval and $\{I_i\}$ a collection of intervals covering $[a, b]$. Assume moreover that either*
(i) each I_i is an open interval, or
(ii) Each I_i is contained in $[a, b]$ and is either open or of the form $[a, c)$ for $a < c \leq b$ or of the form $(d, b]$ for $a \leq d < b$.
Then there exists a finite set of indices $i_1, \ldots, i_k$ such that

$$[a, b] = I_{i_1} \cup \ldots \cup I_{i_k}.$$

PROOF. Let us first assume hypothesis (i) that each I_i is open. At the end we will discuss the minor modifications necessary to deduce the conclusion from hypothesis (ii).

We define a subset $S \subset [a, b]$ as follows: it is the set of all $c \in [a, b]$ such that

[3] Ulisse Dini, 1845-1918

the closed interval $[a, c]$ can be covered by finitely many of the open intervals $\{I_i\}$. This set is nonempty because $a \in S$: by hypothesis, there is at least one i such that $a \in I_i$ and thus, quite trivially, $[a, a] = \{a\} \subset I_i$. let $C = \sup S$, so $a \leq C \leq b$. If $a = b$ then we are done already, so assume $a < b$.
CLAIM: We cannot have $C = a$. Indeed, choose an interval I_i containing a. Then I_i, being open, contains $[a, a + \epsilon]$ for some $\epsilon > 0$, so $C \geq a + \epsilon$.
CLAIM: $C \in S$. That is, if for every $\epsilon > 0$ it is possible to cover $[a, C - \epsilon]$ by finitely many I_i's, then it is also possible to cover $[a, C]$ by finitely many I_i's. Indeed, since the I_i's cover $[a, b]$, there exitst at least one interval I_{i_0} containing the point C, and once again, since I_{i_0} is open, it must contain $[C - \delta, C]$ for some $\delta > 0$. Choosing $\epsilon < \delta$, we find that by adding if necessary one more interval I_{i_0} to our finite family, we can cover $[a, C]$ with finitely many of the intervals I_i.
CLAIM: $C = b$. For if not, $a < C < b$ and $[a, C]$ can be covered with finitely many intervals $I_{i_1} \cup \ldots \cup I_{i_k}$. But once again, whichever of these intervals contains C must, being open, contain all points of $[C, C + \delta]$ for some $\delta > 0$, contradicting the fact that C is the largest element of S. So we must have $C = b$.
Finally, assume (ii). We can easily reduce to a situation in which hypothesis (i) applies and use what we have just proved. Namely, given a collection $\{I_i\}$ of intervals satisfying (ii), we define a new collection $\{J_i\}$ of intervals as follows: if I_i is an *open subinterval* of $[a, b]$, put $J_i = I_i$. If I_i is of the form $[a, c)$, put $J_i = (a - 1, c)$. If I_i is of the form $(d, b]$, put $J_i = (d, b + 1)$. Thus for all i, J_i is an open interval containing I_i, so since the collection $\{I_i\}$ covers $[a, b]$, so does $\{J_i\}$. Applying what we just proved, there exist $i_1, \ldots, i_k$ such that $[a, b] = J_{i_1} \cup \ldots J_{i_k}$. But since $J_i \cap [a, b] = I_i$ – or, in words, in expanding from I_i to J_i we have only added points which lie outside $[a, b]$ so it could not turn a noncovering subcollection into a covering subcollection – we must have $[a, b] = I_{i_1} \cup \ldots I_{i_n}$, qed. □

THEOREM 3.11. *(Dini's Theorem) Let $\{f_n\}$ be a sequence of functions defined on a closed interval $[a, b]$. We suppose that:*
(i) Each f_n is continuous on $[a, b]$.
(ii) The sequence $\{f_n\}$ is either increasing or decreasing.
(iii) f_n converges pointwise to a continuous function f.
Then f_n converges to f uniformly on $[a, b]$.

PROOF.
Step 1: The sequence $\{f_n\}$ is decreasing iff $\{-f_n\}$ is decreasing, and $f_n \to f$ pointwise (resp. uniformly) iff $-f_n \to -f$ pointwise (resp. uniformly), so without loss of generality we may assume the sequence is decreasing. Similarly, f_n is continuous, decreasing and converges to f pointwise (resp. uniformly) iff $f_n - f$ is decreasing, continuous and converges to $f - f = 0$ pointwise (resp. uniformly). So we may as well assume that $\{f_n\}$ is a decreasing sequence of functions converging pointwise to the zero function. Note that under these hypotheses we have $f_n(x) \geq 0$ for all $n \in \mathbb{Z}^+$ and all $x \in [a, b]$.
Step 2: Fix $\epsilon > 0$, and let $x \in [a, b]$. Since $f_n(x) \to 0$, there exists $N_x \in \mathbb{Z}^+$ such that $0 \leq f_{N_x}(x) < \frac{\epsilon}{2}$. Since f_{N_x} is continuous at x, there is a relatively open interval I_x containing x such that for all $y \in I_x$, $|f_{N_x}(y) - f_{N_x}(x)| < \frac{\epsilon}{2}$ and thus $f_{N_x}(y) = |f_{N_x}(y)| \leq |f_{N_x}(y) - f_{N_x}(x)| + |f_{N_x}(x)| < \epsilon$. Since the sequence of functions is decreasing, it follows that for all $n \geq N_x$ and all $y \in I_x$ we have $0 \leq f_n(y) < \epsilon$.
Step 3: Applying part (ii) of the Compactness Lemma, we get a finite set $\{x_1, \ldots, x_k\}$

such that $[a,b] = I_{x_1} \cup \ldots \cup I_{x_k}$. Let $N = \max_{1 \leq i \leq k} N_{x_i}$. Then for all $n \geq N$ and all $x \in [a,b]$, there exists some i such that $x \in I_{x_i}$ and thus $0 \leq f_n(x) < \epsilon$. that is, for all $x \in [a,b]$ and all $n \geq N$ we have $|f_n(x) - 0| < \epsilon$, so $f_n \stackrel{u}{\to} 0$. $\square$

Example: Our very first example of a sequence of functions, namely $f_n(x) = x^n$ on $[0,1]$, is a pointwise convergent decreasing sequence of continuous functions. However the limit function f is discontinuous so the convergence cannot be uniform. In this case all of the hypotheses of Dini's Theorem are satisfied *except* the continuity of the limit function, which is therefore necessary. In this regard Dini's Theorem may be regarded as a partial converse of Theorem 3.1: under certain additional hypotheses, the continuity of the limit function becomes sufficient as well as necessary for uniform convergence.

Remark: I must say that I cannot think of any really snappy application of Dini's Theorem. If you find one, please let me know!

2. Power Series II: Power Series as (Wonderful) Functions

THEOREM 3.12. *(Wonderful Properties of Power Series) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. Consider $f(x) = \sum_{n=0}^{\infty} a_n x^n$ as a function $f : (-R, R) \to \mathbb{R}$. Then:*
a) f is continuous.
b) f is differentiable. Morever, its derivative may be computed termwise*:*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

c) Since the power series f' has the same radius of convergence $R > 0$ as f, f is in fact infinitely differentiable.
d) For all $n \in \mathbb{N}$ we have $f^{(n)}(0) = (n!)a_n$.

PROOF.
a) Let $0 < A < R$, so f defines a function from $[-A, A]$ to $\mathbb{R}$. We claim that the series $\sum_n a_n x^n$ converges to f uniformly on $[-A, A]$. Indeed, as a function on $[-A, A]$, we have $||a_n x^n|| = |a_n| A^n$, and thus $\sum_n ||a_n x^n|| = \sum_n |a_n| A^n < \infty$, because power series converge absolutely on the interior of their interval of convergence. Thus by the Weierstrass M-test f is the uniform limit of the sequence $S_n(x) = \sum_{k=0}^{n} a_k x^k$. But each S_n is a polynomial function, hence continuous and infinitely differentiable. So by Theorem 3.1 f is continuous on $[-A, A]$. Since any $x \in (-R, R)$ lies in $[-A, A]$ for some $0 < A < R$, f is continuous on $(-R, R)$.
b) According to Corollary 3.7, in order to show that $f = \sum_n a_n x^n = \sum_n f_n$ is differentiable and the derivative may be compuited termwise, it is enough to check that (i) each f_n is continuously differentiable and (ii) $\sum_n f_n'$ is uniformly convergent. But (i) is trivial, since $f_n = a_n x^n$ – of course monomial functions are continuously differentiable. As for (ii), we compute that $\sum_n f_n' = \sum_n (a_n x^n) = \sum_n n a_{n-1} x^{n-1}$. By X.X, this power series also has radius of convergence R, hence by the result of part a) it is uniformly convergent on $[-A, A]$. Therefore Corollary 3.7 applies to show $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$.
c) We have just seen that for a power series f convergent on $(-R, R)$, its derivative f' is also given by a power series convergent on $(-R, R)$. So we may continue in this way: by induction, derivatives of all orders exist.

d) The formula $f^{(n)}(0) = (n!)a_n$ is simply what one obtains by repeated termwise differentiation. We leave this as an exercise to the reader. $\square$

EXERCISE 3.7. *Prove Theorem 3.12d).*

EXERCISE 3.8. *Show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$, then $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ is an anti-derivative of f.*

The following exercise drives home that uniform convergence of a sequence or series of functions on all of $\mathbb{R}$ is a very strong condition, often too much to hope for.

EXERCISE 3.9. *Let $\sum_n a_n x^n$ be a power series with infinite radius of convergence, hence defining a function $f : \mathbb{R} \to \mathbb{R}$. Show that the following are equivalent:*
(i) The series $\sum_n a_n x^n$ is uniformly convergent on $\mathbb{R}$.
(ii) We have $a_n = 0$ for all sufficiently large n.

EXERCISE 3.10. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with $a_n \geq 0$ for all n. Suppose that the radius of convergence is 1, so that f defines a function on $(-1, 1)$. Show that the following are equivalent:*
(i) $f(1) = \sum_n a_n$ converges.
(ii) The power series converges uniformly on $[0, 1]$.
(iii) f is bounded on $[0, 1)$.

The fact that for any power series $f(x) = \sum_n a_n x^n$ with positive radius of convergence we have $a_n = \frac{f^{(n)}(0)}{n!}$ yields the following important result.

COROLLARY 3.13. *(Uniqueness Theorem) Let $f(x) = \sum_n a_n x^n$ and $g(x) = \sum_n b_n x^n$ be two power series with radii of convergence R_a and R_b with $0 < R_a \leq R_b$, so that both f and g are infinitely differentiable functions on $(-R_a, R_a)$. Suppose that for some δ with $0 < \delta \leq R_a$ we have $f(x) = g(x)$ for all $x \in (-\delta, \delta)$. Then $a_n = b_n$ for all n.*

EXERCISE 3.11. *Suppose $f(x) = \sum_n a_n x^n$ and $g(x) = \sum_n b_n x^n$ are two power series each converging on some open interval $(-A, A)$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements of $(-A, A) \setminus \{0\}$ such that $\lim_{n\to\infty} x_n = 0$. Suppose that $f(x_n) = g(x_n)$ for all $n \in \mathbb{Z}^+$. Show that $a_n = b_n$ for all n.*

The upshot of Corollary 3.13 is that the only way that two power series can be equal as functions – even in some very small interval around zero – is if all of their coefficients are equal. This is not obvious, since in general $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n$ does not imply $a_n = b_n$ for all n. Another way of saying this is that the only power series a function can be equal to on a small interval around zero is its Taylor series, which brings us to the next section.

3. Taylor Polynomials and Taylor Theorems

Recall that a function $f : I \to \mathbb{R}$ is **infinitely differentiable** if all of its higher derivatives $f', f'', f''', f^{(4)}, \ldots$ exist. When we speak of differentiability of f at a point $x \in I$, we will tacitly assume that x is not an endpoint of I, although it would not be catastrophic if it were (we would need to speak of right-hand derivatives at a left endpoint and left-hand derivatives at a right endpoint). For $k \in \mathbb{N}$, we say that a function $f : I \to \mathbb{R}$ is C^k if its kth derivative exists and is continuous. (Since by convention the 0th derivative of f is just f itself, a C^0 function is a continuous function.)

3.1. Taylor's Theorem (Without Remainder).

For $n \in \mathbb{N}$ and $c \in I$ (not an endpoint), we say that two functions $f, g : I \to \mathbb{R}$ **agree to order n at c** if

$$\lim_{x \to c} \frac{f(x) - g(x)}{(x-c)^n} = 0.$$

Exercise: If $0 \leq m \leq n$ and f and g agree to order n at c, then f and g agree to order m at c.

Example 0: We claim that two continuous functions f and g agree to order 0 at c iff $f(c) = g(c)$. Indeed, suppose that f and g agree to order 0 at c. Since f and g are continuous, we have

$$0 = \lim_{x \to c} \frac{f(x) - g(x)}{(x-c)^0} = \lim_{x \to c} f(x) - g(x) = f(c) - g(c).$$

The converse, that if $f(c) = g(c)$ then $\lim_{x \to c} f(x) - g(x) = 0$, is equally clear.

Example 1: We claim that two differentiable functions f and g agree to order 1 at c iff $f(c) = g(c)$ and $f'(c) = g'(c)$. Indeed, by Exercise X.X, both hypotheses imply $f(c) = g(c)$, so we may assume that, and then we find

$$\lim_{x \to c} \frac{f(x) - g(x)}{x - c} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} - \frac{g(x) - g(c)}{x - c} = f'(c) - g'(c).$$

Thus assuming $f(c) = g(c)$, f and g agree to order 1 at c if and only $f'(c) = g'(c)$.

The following result gives the expected generalization of these two examples. It is generally attributed to Taylor,[4] probably correctly, although special cases were known to earlier mathematicians. Note that **Taylor's Theorem** often refers to a later result (Theorem 3.15) that we call "Taylor's Theorem With Remainder", even though if I am not mistaken it is Theorem 3.14 and not Theorem 3.15 that was actually proved by Brook Taylor.

THEOREM 3.14. *(Taylor) Let $n \in \mathbb{N}$ and $f, g : I \to \mathbb{R}$ be two n times differentiable functions. Let c be an interior point of I. The following are equivalent:*
(i) We have $f(c) = g(c), f'(c) = g'(c), \ldots, f^{(n)}(c) = g^{(n)}(c)$.
(ii) f and g agree to order n at c.

PROOF. Set $h(x) = f(x) - g(x)$. Then (i) holds iff

$$h(c) = h'(c) = \ldots = h^{(n)}(c) = 0$$

and (ii) holds iff

$$\lim_{x \to c} \frac{h(x)}{(x-c)^n} = 0.$$

So we may work with h instead of f and g. We may also assume that $n \geq 2$, the cases $n = 0$ and $n = 1$ having been dealt with above.
(i) $\implies$ (ii): $L = \lim_{x \to c} \frac{h(x)}{(x-c)^n}$ is of the form $\frac{0}{0}$, so L'Hôpital's Rule gives

$$L = \lim_{x \to c} \frac{h'(x)}{n(x-c)^{n-1}},$$

[4] Brook Taylor, 1685 - 1731

provided the latter limit exists. By our assumptions, this latter limit is still of the form $\frac{0}{0}$, so we may apply L'Hôpital's Rule again. We do so iff $n > 2$. In general, we apply L'Hôpital's Rule $n-1$ times, getting

$$L = \lim_{x \to c} \frac{h^{(n-1)}(x)}{n!(x-c)} = \frac{1}{n!} \left(\lim_{x \to c} \frac{h^{(n-1)}(x) - h^{(n-1)}(c)}{x-c} \right),$$

provided the latter limit exists. But the expression in parentheses is nothing else than the derivative of the function $h^{(n-1)}(x)$ at $x = c$ – i.e., it is $h^{(n)}(c) = 0$ (and, in particular the limit exists; only now have the $n-1$ applications of L'Hôpital's Rule been unconditionally justified), so $L = 0$. Thus (ii) holds.
(ii) $\implies$ (i):
CLAIM There is a polynomial $P(x) = \sum_{k=0}^{n} a_k(x-c)^k$ of degree at most n such that $P(c) = h(c)$, $P'(c) = h'(c), \ldots, P^{(n)}(c) = h^{(n)}(c)$. We will take up this easy – but important! – fact in the following section. Taking $f(x) = h(x)$, $g(x) = P(x)$, hypothesis (i) is satisfied, and thus by the already proven implication (i) $\implies$ (ii), we know that $h(x)$ and $P(x)$ agree to order n at $x = c$:

$$\lim_{x \to c} \frac{h(x) - P(x)}{(x-c)^n} = 0.$$

Moreover, by assumption $h(x)$ agrees to order n with the zero function:

$$\lim_{x \to c} \frac{h(x)}{(x-c)^n} = 0.$$

Subtracting these limits gives

$$\lim_{x \to c} \frac{P(x)}{(x-c)^n} = 0. \tag{21}$$

Now it is easy to see – e.g. by L'Hôpital's Rule – that (21) can only hold if

$$a_0 = a_1 = \ldots = a_n = 0,$$

i.e., $P = 0$. Then for all $0 \leq k \leq n$, $h^{(k)}(c) = P^{(k)}(c) = 0$: (i) holds. □

Remark: Above we avoided a subtle pitfall: we applied L'Hôpital's Rule $n-1$ times to $\lim_{x \to c} \frac{h(x)}{(x-c)^n}$, but the final limit we got was still of the form $\frac{0}{0}$ – so why not apply L'Hôpital one more time? The answer is if we do we get that

$$L = \lim_{x \to c} \frac{h^{(n)}(x)}{n!},$$

assuming this limit exists. But to assume this last limit exists and is equal to $h^{(n)}(0)$ is to assume that nth derivative of h is *continuous* at zero, which is slightly more than we want (or need) to assume.

3.2. Taylor Polynomials.

Recall that we still need to establish the CLAIM made in the proof of Theorem 3.14. This is in fact more important than the rest of Theorem 3.14! So let

$$P(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n$$

be a polynomial of degree at most n, let $f : I \to \mathbb{R}$ be a function which is n times differentiable at c, and let us see whether and in how many ways we may choose the coefficients $a_0, \ldots, a_n$ such that $f^{(k)}(c) = P^{(k)}(c)$ for all $0 \leq k \leq n$.

There is much less here than meets the eye. For instance, since $P(c) = a_0 + 0 + \ldots = a_0$, clearly we have $P(c) = f(c)$ iff

$$a_0 = f(c).$$

Moreover, since $P'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \ldots + na_n(x-c)^{n-1}$, we have $P'(c) = a_1$ and thus $P'(c) = f(c)$ iff

$$a_1 = f'(c).$$

Since $P''(x) = 2a_2 + 3\cdot 2a_3(x-c) + 4\cdot 3a_4(x-c) + \ldots + n(n-1)a_n x^{n-2}$, we have $P''(c) = 2a_2$ and thus $P''(c) = f''(c)$ iff $a_2 = \frac{f''(c)}{2}$. And it proceeds in this way. (Just) a little thought shows that $P^{(k)}(c) = k!a_k$ – after differentiating k times the term $a_k(x-c)^k$ becomes the constant term – all higher terms vanish when we plug in $x = c$ – and since we have applied the power rule k times we pick up altogether a factor of $k \cdot (k-1) \cdots 1 = k!$. Therefore we must have

$$a_k = \frac{f^{(k)}(c)}{k!}.$$

In other words, no matter what the values of the derivatives of f at c are, there is a **unique** polynomial of degree at most k satisfying them, namely

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)(x-c)^k}{k!}.$$

$T_n(x)$ is called the **degree n Taylor polynomial for f at c**.

Exercise 3.12. *Fix $c \in \mathbb{R}$. Show that every polynomial $P(x) = b_0 + b_1 x + \ldots + b_n x^n$ can be written in the form $a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n$ for unique $a_0, \ldots, a_n$. (Hint: $P(x+c)$ is also a polynomial.)*

For $n \in \mathbb{N}$, a function $f : I \to \mathbb{R}$ **vanishes to order n** at c if $\lim_{x\to c} \frac{f(x)}{(x-c)^n} = 0$. Note that this concept came up prominently in the proof of Theorem 3.14 in the form: f and g agree to order n at c iff $f - g$ vanishes to order n at c.

Exercise 3.13. *Let f be a function which is n times differentiable at $x = c$, and let T_n be its degree n Taylor polynomial at $x = c$. Show that $f - T_n$ vanishes to order n at $x = c$. (This is just driving home a key point of the proof of Theorem 3.14 in our new terminology.)*

Exercise 3.14. *a) Show that for a function $f : I \to \mathbb{R}$, the following are equivalent:*
(i) f is differentiable at c.
(ii) We may write $f(x) = a_0 + a_1(x-c) + g(x)$ for a function $g(x)$ vanishing to order 1 at c.
b) Show that if the equivalent conditions of part a) are satisfied, then we must have $a_0 = f(c)$, $a_1 = f'(c)$ and thus the expression of a function differentiable at c as the sum of a linear function and a function vanishing to first order at c is unique.

Exercise 3.15. *(D. Piau) Let $a, b \in \mathbb{Z}^+$, and consider the following function $f_{a,b} : \mathbb{R} \to \mathbb{R}$:*

$$f_{a,b}(x) = \begin{cases} x^a \sin\left(\frac{1}{x^b}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

a) Show that $f_{a,b}$ vanishes to order $a-1$ at 0 but does not vanish to order a at 0.
b) Show that $f_{a,b}$ is differentiable at $x=0$ iff $a \geq 2$, in which case $f'_{a,b}(0)=0$.
c) Show that $f_{a,b}$ is twice differentiable at $x=0$ iff $a \geq b+3$, in which case $f''_{a,b}(0)=0$.
d) Deduce in particular that for any $n \geq 2$, $f_{n,n}$ vanishes to order n at $x=0$ but is not twice differentiable – hence not n times differentiable – at $x=0$.
e) Exactly how many times differentiable is $f_{a,b}$?

3.3. Taylor's Theorem With Remainder.

To state the following theorem, it will be convenient to make a convention: real numbers a,b, by $|[a,b]|$ we will mean the interval $[a,b]$ if $a \leq b$ and the interval $[b,a]$ if $b<a$. So $|[a,b]|$ is the set of real numbers lying between a and b.

THEOREM 3.15. *(Taylor's Theorem With Remainder) Let $n \in \mathbb{N}$ and $f : [a,b] \rightarrow \mathbb{R}$ be an $n+1$ times differentiable function. Let $T_n(x)$ be the degree n Taylor polynomial for f at c, and let $x \in [a,b]$.*
a) There exists $z \in |[c,x]|$ such that

$$f(x) = T_n(x) + \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}. \tag{22}$$

b) We have

$$R_n(x) = |f(x)-T_n(x)| \leq \frac{||f^{(n+1)}||}{(n+1)!}|x-c|^{n+1},$$

where $||f^{(n+1)}||$ is the supremum of $|f^{(n+1)}|$ on $|[c,x]|$.

PROOF. a) [**R**, Thm. 5.15] Put

$$M = \frac{f(x)-T_n(x)}{(x-c)^{n+1}},$$

so

$$f(x) = T_n(x) + M(x-c)^{n+1}.$$

Thus our goal is to show that $(n+1)!M = f^{(n+1)}(z)$ for some $z \in |[c,x]|$. To see this, we define an auxiliary function g: for $a \leq t \leq b$, put

$$g(t) = f(t) - T_n(t) - M(t-c)^{n+1}.$$

Differentiating $n+1$ times, we get that for all $t \in (a,b)$,

$$g^{(n+1)}(t) = f^{(n+1)}(t) - (n+1)!M.$$

Therefore it is enough to show that there exists $z \in |[c,x]|$ such that $g^{(n+1)}(z)=0$. By definition of T_n and g, we have $g^{(k)}(c)=0$ for all $0 \leq k \leq n$. Moreover, by definition of M we have $g(x)=0$. So in particular we have $g(c)=g(x)=0$ and **Rolle's Theorem** applies to give us $z_1 \in |[c,x]|$ with $g'(z_1)=0$ for some $z_1 \in |[c,x]|$. Now we iterate this argument: since $g'(c)=g'(z_1)=0$, by Rolle's Theorem there exists $z_2 \in |[x,z_1]|$ such that $(g')'(z_2)=g''(z_2)=0$. Continuing in this way we get a sequence of points $z_1,z_2,\ldots,z_{n+1} \in |[c,x]|$ such that $g^{(k)}(z_k)=0$, so finally that $g^{(n+1)}(z_{n+1})=0$ for some $z_{n+1} \in |[c,x]|$. Taking $z=z_{n+1}$ completes

the proof of part a).
Part b) follows immediately: we have $|f^{(n+1)}(z)| \leq ||f^{(n+1)}||$, so

$$|f(x) - T_n(x)| = |\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}| \leq \frac{||f^{(n+1)}||}{(n+1)!}|x-c|^{n+1}.$$

$\square$

Remark: There are in fact several different versions of "Taylor's Theorem With Remainder" corresponding to different ways of expressing the remainder $R_n(x) = |f(x) - T_n(x)|$. The particular expression derived above is due to Lagrange.[5]

Exercise 3.16. *Show that Theorem 3.15 (Taylor's Theorem With Remainder) immediately implies Theorem 3.14 (Taylor's Theorem) under the additional hypothesis that $f^{(n+1)}$ exists on the interval $|[c, x]|$.*

4. Taylor Series

Let $f : I \to \mathbb{R}$ be an infinitely differentiable function, and let $c \in I$. We define the **Taylor series** of f at c to be

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!}.$$

Thus by definition, $T(x) = \lim_{n\to\infty} T_n(x)$, where T_n is the degree n Taylor polynomial of x at c. In particular $T(x)$ is a power series, so all of our prior work on power series applies.

Just as with power series, it is no real loss of generality to assume that $c = 0$, in which case our series takes the simpler form

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!},$$

since to get from this to the general case one merely has to make the change of variables $x \mapsto x - c$. It is somewhat traditional to call Taylor series centered around $c = 0$ **Maclaurin series**. But there is no good reason for this – Taylor series were introduced by Taylor in 1721, whereas Colin Maclaurin's *Theory of fluxions* was not published until 1742 and in this work explicit attribution is made to Taylor's work.[6] Using separate names for Taylor series centered at 0 and Taylor series centered at an arbitrary point c often suggests – misleadingly! – to students that there is some conceptual difference between the two cases. So we will not use the term "Maclaurin series" here.

Exercise 3.17. *Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{\frac{-1}{x^2}}$ for $x \neq 0$ and $f(0) = 0$. Show that f is infinitely differentiable and in fact $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.*

When dealing with Taylor series there are two main issues.

[5]Joseph-Louis Lagrange, 1736-1813

[6]For that matter, special cases of the Taylor series concept were well known to Newton and Gregory in the 17th century and to the Indian mathematician Madhava of Sangamagrama in the 14th century.

QUESTION 3.16. *Let $f : I \to \mathbb{R}$ be an infinitely differentiable function and $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$ be its Taylor series.*
a) For which values of x does $T(x)$ converge?
b) If for $x \in I$, $T(x)$ converges, do we have $T(x) = f(x)$?

Notice that Question 3.16a) is simply asking for which values of $x \in \mathbb{R}$ a power series is convergent, a question to which we worked out a very satisfactory answer in $\S X.X$. Namely, the set of values x on which a power series converges is an interval of radius $R \in [0, \infty]$ centered at 0. More precisely, in theory the value of R is given by Hadamard's Formula $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$, and in practice we expect to be able to apply the Ratio Test (or, if necessary, the Root Test) to compute R.

If $R = 0$ then $T(x)$ only converges at $x = 0$ and there we certainly have $T(0) = f(0)$: this is a trivial case. Henceforth we assume that $R \in (0, \infty]$ so that f converges (at least) on $(-R, R)$. Fix a number A, $0 < A \leq R$ such that $(-A, A) \subset I$. We may then move on to Question 3.16b): must $f(x) = T(x)$ for all $x \in (-A, A)$?

In fact the answer is **no**. Indeed, consider the function $f(x)$ of Exercise 3.17. $f(x)$ is infinitely differentiable and has $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, so its Taylor series is $T(x) = \sum_{n=0}^{\infty} \frac{0x^n}{n!} = \sum_{n=0}^{\infty} 0 = 0$, i.e., it converges for all $x \in \mathbb{R}$ to the zero function. Of course $f(0) = 0$ (every function agrees with its Taylor series at $x = 0$), but for $x \neq 0$, $f(x) = e^{\frac{-1}{x^2}} \neq 0$. Therefore $f(x) \neq T(x)$ in any open interval around $x = 0$.

There are plenty of other examples. Indeed, in a sense that we will not try to make precise here, "most" infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ are not equal to their Taylor series expansions in any open interval about any point. That's the bad news. However, one could interpret this to mean that we are not really interested in "most" infinitely differentiable functions: the **special functions** one meets in calculus, advanced calculus, physics, engineering and analytic number theory are almost invariably equal to their Taylor series expansions, at least in some small interval around any given point $x = c$ in the domain.

In any case, if we wish to *try* to show that a $T(x) = f(x)$ on some interval $(-A, A)$, we have a tool for this: Taylor's Theorem With Remainder. Indeed, since $R_n(x) = |f(x) - T_n(x)|$, we have

$$f(x) = T(x) \iff f(x) = \lim_{n\to\infty} T_n(x)$$

$$\iff \lim_{n\to\infty} |f(x) - T_n(x)| = 0 \iff \lim_{n\to\infty} R_n(x) = 0.$$

So it comes down to being able to give upper bounds on $R_n(x)$ which tend to zero as $n \to \infty$. According to Taylor's Theorem with Remainder, this will hold whenever we can show that the norm of the nth derivative $||f^{(n)}||$ does not grow too rapidly.

Example: We claim that for all $x \in \mathbb{R}$, the function $f(x) = e^x$ is equal to its Taylor series expansion at $x = 0$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

First we compute the Taylor series expansion. Although there are some tricks for this, in this case it is really no trouble to figure out exactly what $f^{(n)}(0)$ is for all non-negative integers n. Indeed, $f^{(0)}(0) = f(0) = e^0 = 1$, and $f'(x) = e^x$, hence every derivative of e^x is just e^x again. We conclude that $f^{(n)}(0) = 1$ for all n and thus the Taylor series is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, as claimed. Next note that this power series converges for all real x, as we have already seen: just apply the Ratio Test. Finally, we use Taylor's Theorem with Remainder to show that $R_n(x) \to 0$ for each fixed $x \in \mathbb{R}$. Indeed, Theorem 3.15 gives us

$$R_n(x) \leq \frac{||f^{(n+1)}||}{(n+1)!}|x-c|^{n+1},$$

where $||f^{(n+1)}||$ is the supremum of the the absolute value of the $(n+1)$st derivative on the interval $|[0,x]|$. But – lucky us – in this case $f^{(n+1)}(x) = e^x$ for all n and the maximum value of e^x on this interval is e^x if $x \geq 0$ and 1 otherwise, so in either way $||f^{(n+1)}|| \leq e^{|x|}$. So

$$R_n(x) \leq e^{|x|}\left(\frac{x^{n+1}}{(n+1)!}\right).$$

And now we win: the factor inside the parentheses approaches zero with n and is being multiplied by a quantity which is *independent of* n, so $R_n(x) \to 0$. In fact a moment's thought shows that $R_n(x) \to 0$ *uniformly* on any bounded interval, say on $[-A, A]$, and thus our work on the general properties of uniform convergence of power series (in particular the M-test) is not needed here: everything comes from Taylor's Theorem With Remainder.

Example continued: we use Taylor's Theorem With Remainder to compute $e = e^1$ accurate to 10 decimal places.

A little thought shows that the work we did for $f(x) = e^x$ carries over verbatim under somewhat more general hypotheses.

THEOREM 3.17. *Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a smooth function. Suppose that for all $A \in [0, \infty)$ there exists a number M_A such that for all $x \in [-A, A]$ and all $n \in \mathbb{N}$,*

$$|f^{(n)}(x)| \leq M_A.$$

Then:
a) The Taylor series $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$ converges absolutely for all $x \in \mathbb{R}$.
b) For all $x \in \mathbb{R}$ we have $f(x) = T(x)$: that is, f is equal to its Taylor series expansion at 0.

EXERCISE 3.18. *Prove Theorem 3.17.*

EXERCISE 3.19. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function with* ***periodic derivatives****: there exists some $k \in \mathbb{Z}^+$ such that $f = f^{(k)}$. Show that f satisfies the hypothesis of Theorem 3.17 and therefore is equal to its Taylor series expansion at $x = 0$ (or in fact, about any other point $x = c$).*

Example: Let $f(x) = \sin x$. Then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x = f(x)$, so f has periodic derivatives. In particular the sequence of

nth derivatives evaluated at 0 is $\{0, 1, 0, -1, 0, \ldots\}$. By Exercise X.X, we have

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

for all $x \in \mathbb{R}$. Similarly, we have

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

EXERCISE 3.20. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function.*
*a) Suppose f is **odd**: $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Then the Taylor series expansion of f is of the form $\sum_{n=0}^{\infty} a_n x^{2n+1}$, i.e., only odd powers of x appear.*
*b) Suppose f is **even**: $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Then the Taylor series expansion of f is of the form $\sum_{n=0}^{\infty} a_n x^{2n}$, i.e., only even powers of x appear.*

Example: Let $f(x) = \log x$. Then f is defined and smooth on $(0, \infty)$, so in seeking a Taylor series expansion we must pick a point other than 0. It is traditional to set $c = 1$ instead. Then $f(1) = 0$, $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = (-1)^2 2! x^{-3}$, and in general $f^{(n)}(x) = (-1)^{n-1}(n-1)! x^{-n}$. Therefore the Taylor series expansion about $c = 1$ is

$$T(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!(x-1)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n.$$

This power series is convergent when $-1 < x-1 \leq 1$ or $0 < x \leq 2$. We would like to show that it is actually equal to $f(x)$ on $(0, 2)$. Fix $A \in (0, 1)$ and $x \in [1-A, 1+A]$. The functions $f^{(n)}$ are decreasing on this interval, so the maximum value of $|f^{(n+1)}|$ occurs at $1 - A$: $||f^{(n+1)}|| = n!(1-A)^{-n}$. Therefore, by Theorem 3.15 we have

$$R_n(x) \leq \frac{||f^{(n+1)}||}{(n+1)!}|x-c|^{n+1} = \frac{|x-1|^{n+1}}{(1-A)^n(n+1)} \leq \frac{A}{n+1}\left(\frac{A}{1-A}\right)^n.$$

But now when we try to show that $R_n(x) \to 0$, we are in for a surprise: the quantity $\frac{A}{n+1}\left(\frac{A}{1-A}\right)^n$ tends to 0 as $n \to \infty$ iff $\frac{A}{1-A} \leq 1$ iff $A \leq \frac{1}{2}$. Thus we have shown that $\log x = T(x)$ for $x \in [\frac{1}{2}, \frac{3}{2}]$ only!

In fact $f(x) = T(x)$ for all $x \in (0, 2)$, but we need a different argument. Namely, we know that for all $x \in (0, 2)$ we have

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty}(1-x)^n.$$

As always for power series, the convergence is uniform on $[1-A, 1+A]$ for any $0 < A < 1$, so by Corollary 3.4 we may integrate termwise, getting

$$\log x = \sum_{n=0}^{\infty} \frac{-(1-x)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n.$$

There is a clear moral here: even if we can find an exact expression for $f^{(n)}$ and for $||f^{(n)}||$, the error bound given by Theorem 3.15b) may not be good enough to show that $R_n(x) \to 0$, even for rather elementary functions. This does not in itself imply that $T_n(x)$ does not converge to $f(x)$ on its interval of convergence: we may simply need to use less direct means. As a general rule, we try to exploit the Uniqueness Theorem for power series, which says that if we can – by any means necessary! –

express $f(x)$ as a power series $\sum_n a_n(x-c)^n$ convergent in an interval around c of positive radius, then this power series must be the Taylor series expansion of f, i.e., $a_n = \frac{f^{(n)}(c)}{n!}$.

EXERCISE 3.21. *Let* $f(x) = \arctan x$. *Show:* $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1}$ *for all* $x \in (-1,1]$. *(For convergence at the right endpoint, use Abel's Theorem as in* §2.10.*)*

5. The Binomial Series

Even for familiar, elementary functions, using Taylor's Theorem to show $R_n(x) \to 0$ may require nonroutine work. We give a case study: the **binomial series**.

Let $\alpha \in \mathbb{R}$. For $x \in (-1,1)$, we define

$$f(x) = (1+x)^{\alpha}.$$

Case 1: Suppose $\alpha \in \mathbb{N}$. Then f is just a polynomial; in particular f is defined and infinitely differentiable for all real numbers.
Case 2: Suppose α is positive but not an integer. Depending on the value of α, f may or may not be defined for $x < -1$ (e.g. it is for $\alpha = \frac{2}{3}$ and it is not for $\alpha = \frac{3}{2}$), but in any case f is only $\langle \alpha \rangle$ times differentiable at $x = -1$.
Case 3: Suppose $\alpha < 0$. Then $\lim_{x \to -1^+} f(x) = \infty$.

The upshot of this discussion is that if α is not a positive integer, then f is defined and infinitely differentiable on $(-1,\infty)$ and on no larger interval than this.

For $n \in \mathbb{Z}^+$, $f^{(n)}(x) = (\alpha)(\alpha-1)\cdots(\alpha-(n-1))(1+x)^{\alpha-n}$, so $f^{(n)}(0) = (\alpha)(\alpha-1)\cdots(\alpha-(n-1))$. Of course we have $f^{(0)}(0) = f(0) = 1$, so the Taylor series to f at $c = 0$ is

$$T(x) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)(\alpha-1)\cdots(\alpha-(n-1))}{n!} x^n.$$

If $\alpha \in \mathbb{N}$, we recognize the nth Taylor series coefficient as the *binomial coefficient* $\binom{\alpha}{n}$, and this ought not to be surprising because for $\alpha \in \mathbb{N}$, expanding out $T(x)$ simply gives the binomial theorem:

$$\forall \alpha \in \mathbb{N},\ (1+x)^{\alpha} = \sum_{n=0}^{\alpha} \binom{\alpha}{n} x^n.$$

So let's extend our definition of binomial coefficients: for *any* $\alpha \in \mathbb{R}$, put

$$\binom{\alpha}{0} = 1,$$

$$\forall n \in \mathbb{Z}^+, \binom{\alpha}{n} = \frac{(\alpha)(\alpha-1)\cdots(\alpha-(n-1))}{n!}.$$

EXERCISE 3.22. *For any* $\alpha \in \mathbb{R}, n \in \mathbb{Z}^+$, *show*

$$\binom{\alpha}{n} = \binom{\alpha-1}{n-1} + \binom{\alpha-1}{n}. \tag{23}$$

Finally, we rename the Taylor series to $f(x)$ as the **binomial series**

$$B(\alpha, x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

The binomial series is as old as calculus itself, having been studied by Newton in the 17th century.[7] It remains one of the most important and useful of all power series. For us, our order of business is the usual one when given a Taylor series: first, for each fixed α we wish to find the interval I on which the series $B(\alpha, x)$ converges. Second, we would like to show – if possible! – that for all $x \in I$, $B(\alpha, x) = (1+x)^\alpha$.

THEOREM 3.18. *Let $\alpha \in \mathbb{R} \setminus \mathbb{N}$, and consider the* ***binomial series***

$$B(\alpha, x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = 1 + \sum_{n=1}^{\infty} \binom{\alpha}{n} x^n.$$

a) For all such α, the radius of convergence of $B(\alpha, x) = 1$.
b) For all $\alpha > 0$, the series $B(\alpha, 1)$ and $B(\alpha, -1)$ are absolutely convergent.
c) If $\alpha \in (-1, 0)$, the series $B(\alpha, 1)$ is nonabsolutely convergent.
d) If $\alpha \leq -1$, then $B(\alpha, -1)$ and $B(\alpha, 1)$ are divergent.

PROOF. a) We apply the Ratio Test:

$$\rho = \lim_{n\to\infty} \left| \frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}} \right| = \lim_{n\to\infty} \left| \frac{\alpha - n}{n+1} \right| = 1,$$

so the radius of convergence is $\frac{1}{\rho} = 1$.
b) Step 0: Let $a > 1$ be a real number and $m \in \mathbb{Z}^+$. Then we have

$$\left(\frac{a}{a-1}\right)^m = \left(1 + \frac{1}{a-1}\right)^m \geq 1 + \frac{m}{a-1} > 1 + \frac{m}{a} = \frac{a+m}{a} > 0,$$

where in the first inequality we have just taken the first two terms of the usual (finite!) binomial expansion. Taking reciprocals, we get

$$\left(\frac{a-1}{a}\right)^m < \frac{a}{a+m}.$$

Step 1: Suppose $\alpha \in (0, 1)$. Choose an integer $m \geq 2$ such that $\frac{1}{m} < \alpha$. Then

$$|\binom{\alpha}{n}| = \frac{\alpha(1-\alpha)\cdots(n-1-\alpha)}{n!} < 1(1 - \frac{1}{m})\cdots(n-1-\frac{1}{m})\frac{1}{n!}$$

$$= \frac{m-1}{m}\frac{2m-1}{2m}\cdots\frac{(n-1)m-1}{(n-1)m}\frac{1}{n} = a_n \frac{1}{n},$$

say. Using Step 0, we get

$$a_n^{m-1} < \frac{m}{2m-1}\frac{2m}{3m-1}\cdots\frac{(n-1)m}{nm-1}$$

$$= \frac{m}{m-1}\cdots\frac{2m}{2m-1}\cdots\frac{(n-1)m}{(n-1)m-1}\frac{m-1}{nm-1} \leq \frac{1}{a_n}\frac{1}{n}$$

[7]In fact it is older. For an account of the early history of the binomial series, see [**Co49**].

It follows that $a_n < \frac{1}{n^{\frac{1}{m}}}$, so $|\binom{\alpha}{n}| < \frac{1}{n^{1+\frac{1}{m}}}$, so

$$\sum_n |\binom{\alpha}{n}| \leq \sum_n \frac{1}{n^{1+\frac{1}{m}}} < \infty.$$

This shows that $B(\alpha, 1)$ is absolutely convergent; since $|\binom{\alpha}{n}(-1)^n| = |\binom{\alpha}{n}|$, it also shows that $B(\alpha, -1)$ is absolutely convergent.
Step 2: Using the identity (23), we find

$$S(\alpha, x) = 1 + \sum_{n=1}^{\infty} \binom{\alpha}{n} x^n = 1 + \sum_{n=1}^{\infty} \left(\binom{\alpha - 1}{n-1} + \binom{\alpha - 1}{n} \right) x^n = (1+x) S(\alpha - 1, x).$$

Thus for any fixed x, if $S(\alpha - 1, x)$ (absolutely) converges, so does $S(\alpha, x)$. By an evident induction argument, if $S(\alpha, x)$ (absolutely) converges, so does $S(\alpha + n, x)$ for all $n \in \mathbb{N}$. Since $S(\alpha, -1)$ and $S(\alpha, 1)$ are absolutely convergent for all $\alpha \in (0, 1)$, they are thus absolutely convergent for all non-integers $\alpha > 0$.
c) If $\alpha \in (-1, 0)$ and $n \in \mathbb{N}$, then

$$\binom{\alpha}{n+1} / \binom{\alpha}{n} = \frac{\alpha - n}{n+1} \in (-1, 0);$$

this shows simultaneously that the sequence of terms of $B(\alpha, 1) = \sum_{n=0}^{\infty} \binom{\alpha}{n}$ is decreasing in absolute value and alternating in sign. Further, write $\alpha = \beta - 1$, so that $\beta \in (0, 1)$. Choose an integer $m \geq 2$ such that $\frac{1}{m} < \beta$. Then

$$|\binom{\alpha}{n}| = \frac{(1-\beta)(2-\beta)\cdots(n-1-\beta)}{(n-1)!} \frac{n-\beta}{n} = b_n \frac{n-\beta}{n}.$$

Arguing as in Step 1 of part b) shows that $b_n < \frac{1}{n^{\frac{1}{m}}}$, and hence

$$\lim_{n\to\infty} |\binom{\alpha}{n}| = \lim_{n\to\infty} b_n \cdot \lim_{n\to\infty} \frac{n-\beta}{n} = 0 \cdot 1 = 0.$$

Therefore the Alternating Series Test applies to show that $S(\alpha, 1)$ converges.
d) The absolute value of the nth term of both $B(\alpha, -1)$ and $B(\alpha, 1)$ is $|\binom{\alpha}{n}|$. If $\alpha \leq -1$, then $|\alpha - n| \geq n + 1$ and thus

$$|\binom{\alpha}{n+1} / \binom{\alpha}{n}| = |\frac{\alpha - n}{n+1}| \geq 1,$$

and thus $\binom{\alpha}{n} \not\to 0$. By the Nth term test, $S(\alpha, -1)$ and $S(\alpha, 1)$ diverge. $\square$

EXERCISE 3.23. *Show that for $\alpha \in (-1, 0)$, the binomial series $B(\alpha, -1)$ diverges.*

Remark: As the reader has surely noted, the convergence of the binomial series $S(\alpha, x)$ at $x = \pm 1$ is a rather delicate and tricky enterprise. In fact most texts at this level – even [**S**] – do not treat it. We have taken Step 1 of part b) from [**Ho66**].

Remark: There is an extension of the Ratio Test due to J.L. Raabe which simplifies much the of the above analysis, including the preceding exercise.

THEOREM 3.19. *Let $\alpha \in \mathbb{R} \setminus \mathbb{N}$; let $f(x) = (1+x)^\alpha$, and consider its Taylor series at zero, the* ***binomial series***

$$B(\alpha, x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

a) For all $x \in (-1, 1)$, $f(x) = B(\alpha, x)$.
b) If $\alpha > -1$, $f(1) = B(\alpha, 1)$.
c) If $\alpha > 0$, $f(-1) = B(\alpha, -1)$.

PROOF. [**La**] Let $T_{n-1}(x)$ be the $(n-1)$st Taylor polynomial for f at 0, so

$$B(\alpha, x) = \lim_{n\to\infty} T_{n-1}(x)$$

is the Taylor series expansion of f at zero. As usual, put $R_{n-1}(x) = f(x) - T_{n-1}(x)$.
a) By Theorem 3.15b),

$$R_{n-1}(x) = \int_0^x \frac{f^n(t)(x-t)^{n-1}dt}{(n-1)!} = \frac{1}{(n-1)!}\int_0^x \alpha(\alpha-1)\cdots(\alpha-n+1)(1+t)^{\alpha-n}(x-t)^{n-1}dt.$$

By the Mean Value Theorem for Integrals, there is $\theta \in (0, 1)$ such that

$$R_{n-1}(x) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}(1+\theta x)^{\alpha-n}(x-\theta x)^{n-1}(x-0).$$

Put

$$t = \frac{1-\theta}{1+\theta x}, c_n(s) = \binom{\alpha-1}{n-1} s^{n-1}.$$

Then

$$(1+s)^{\alpha-1} = \sum_{n=1}^{\infty} c_n(s)$$

and

$$R_{n-1}(x) = c_n(xt)\alpha x(1+\theta x)^{\alpha-1}.$$

Since $x \in (-1, 1)$, we have $t \in (0, 1)$, so $|xt| < 1$. It follows that $\sum_{n=1}^{\infty} c_n(xt)$ converges, so by the nth term test $c_n(xt) \to 0$ as $n \to \infty$ and thus $R_{n-1}(x) \to 0$.
b) The above argument works verbatim if $x = 1$ and $\alpha > -1$.
c) If $\alpha > 0$, then by Theorem 3.18b), $S(\alpha, -1)$ is convergent. Moreover, $\alpha - 1 > -1$, so $\sum_{n=1}^{\infty} c_n(1)$ converges and thus $c_n(1) \to 0$. But $|c_n(-1)| = |c_n(1)|$, so also $c_n(-1) \to 0$ and thus $R_{n-1}(-1) \to 0$. □

6. The Weierstrass Approximation Theorem

6.1. Statement of Weierstrass Approximation.

THEOREM 3.20. *(Weierstrass Approximation Theorem) Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\epsilon > 0$ be any positive number. Then there exists a polynomial $P = P(\epsilon)$ such that for all $x \in [a, b]$, $|f(x) - P(x)| < \epsilon$. In other words, any continuous function defined on a closed interval is the uniform limit of a sequence of polynomials.*

EXERCISE 3.24. *For each $n \in \mathbb{Z}^+$, let $P_n : \mathbb{R} \to \mathbb{R}$ be a polynomial function. Suppose that there is $f : \mathbb{R} \to \mathbb{R}$ such that $P_n \xrightarrow{u} f$ on all of $\mathbb{R}$. Show that the sequence of functions $\{P_n\}$ is eventually constant: there exists $N \in \mathbb{Z}^+$ such that for all $m, n \geq N$, $P_n(x) = P_m(x)$ for all $x \in \mathbb{R}$.*

It is interesting to compare Theorem 3.20 with Taylor's theorem, which gives conditions for a function to be equal to its Taylor series. Note that any such function must be C^{∞} (i.e., it must have derivatives of all orders), whereas in the Weierstrass Approximation Theorem we can get *any* continuous function. An important difference is that the Taylor polynomials $T_N(x)$ have the property that $T_{N+1}(x) = T_N(x) + a_{N+1}x^n$, so that in passing from one Taylor polynomial to the next, we are not changing any of the coefficients from 0 to N but only adding a higher order term. In contrast, for the sequence of polynomials $P_n(x)$ uniformly converging to f in Theorem 1, $P_{n+1}(x)$ is not required to have any simple algebraic relationship to $P_n(x)$.

Theorem 3.20 was first established by Weierstrass in 1885. To this day it is one of the most central and celebrated results of mathematical analysis. Many mathematicians have contributed novel proofs and generalizations, notably S.J. Bernstein [**Be12**] and M.H. Stone [**St37**], [**St48**]. But – more than any result of undergraduate mathematics I can think of except the **quadratic reciprocity law** – the passage of time and the advancement of mathematical thought have failed to single out any one preferred proof. We have decided to follow an argument given by Noam Elkies.[8] This argument is reasonably short and reasonably elementary, although as above, not definitively more so than certain other proofs. However it unfolds in a logical way, and every step is of some intrinsic interest. Best of all, at a key stage we get to apply our knowledge of Newton's binomial series!

6.2. Piecewise Linear Approximation.

A function $f : [a, b] \to \mathbb{R}$ is **piecewise linear** if it is a continuous function made up out of finitely many straight line segments. More formally, there exists a partition $P = \{a = x_0 < x_1 \ldots < x_n = b\}$ such that for $1 \leq i \leq n$, the restriction of f to $[x_{i-1}, x_i]$ is a linear function. For instance, the absolute value function is piecewise linear. In fact, the general piecewise can be expressed in terms of absolute values of linear functions, as follows.

LEMMA 3.21. *Let $f : [a, b] \to \mathbb{R}$ be a piecewise linear function. Then there is $n \in \mathbb{Z}^+$ and $a_1, \ldots, a_n, m_1, \ldots, m_n, b \in \mathbb{R}$ such that*

$$f(x) = b + \sum_{i=1}^{n} m_i |x - a_j|.$$

PROOF. We leave this as an elementary exercise. Some hints:
(i) If $a_j \leq a$, then as functions on $[a, b]$, $|x - a_j| = x - a_j$.
(ii) The following identities may be useful:

$$\max(f, g) = \frac{|f + g|}{2} + \frac{|f - g|}{2}$$
$$\min(f, g) = \frac{|f + g|}{2} - \frac{|f - g|}{2}.$$

(iii) One may, for instance, go by induction on the number of "corners" of f. □

Now every continuous function $f : [a, b] \to \mathbb{R}$ may be uniformly approximated by piecewise linear functions, and moreover *this is very easy to prove.*

[8] http://www.math.harvard.edu/~elkies/M55b.10/index.html

PROPOSITION 3.22. *(Piecewise Linear Approximation) Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\epsilon > 0$ be any positive number. Then there exists a piecewise function P such that for all $x \in [a, b]$, $|f(x) - P(x)| < \epsilon$.*

PROOF. Since every continuous function on $[a, b]$ is uniformly continuous, there exists a $\delta > 0$ such that whenever $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$. Choose n large enough so that $\frac{b-a}{n} < \delta$, and consider the uniform partition $P_n = \{a < a + \frac{b-a}{n} < \ldots < b\}$ of $[a, b]$. There is a piecewise linear function P such that $P(x_i) = f(x_i)$ for all x_i in the partition, and is defined in between by "connecting the dots" (more formally, by linear interpolation). For any i and for all $x \in [x_{i-1}, x_i]$, we have $f(x) \in (f(x_{i-1}) - \epsilon, f(x_{i-1}) + \epsilon)$. The same is true for $P(x)$ at the endpoints, and one of the nice properties of a linear function is that it is either increasing or decreasing, so its values are always in between its values at the endpoints. Thus for all x in $[x_{i-1}, x_i]$ we have $P(x)$ and $f(x)$ both lie in an interval of length 2ϵ, so it follows that $|P(x) - f(x)| < 2\epsilon$ for all $x \in [a, b]$. □

6.3. A Very Special Case.

LEMMA 3.23. *(Elkies) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series. We suppose:*
(i) The sequence of signs *of the coefficients a_n is eventually constant.*
(ii) The radius of convergence is 1*.*
(iii) $\lim_{x \to 1^-} f(x) = L$ *exists.*
Then $\sum_{n=0}^{\infty} a_n = L$, and the convergence of the series to the limit function is uniform on $[0, 1]$*.*

PROPOSITION 3.24. *For any $\alpha > 0$, the function $f(x) = |x|$ on $[-\alpha, \alpha]$ can be uniformly approximated by polynomials.*

PROOF. Step 1: Suppose that for all $\epsilon > 0$, there is a polynomial function $P : [-1, 1] \to \mathbb{R}$ such that $|P(x) - |x|| < \epsilon$ for all $x \in [-1, 1]$. Put $x = \frac{y}{\alpha}$. Then for all $y \in [-\alpha, \alpha]$ we have

$$|\alpha P(\frac{y}{\alpha}) - \alpha|\frac{y}{\alpha}|| = |Q(y) - |y|| < \alpha\epsilon,$$

where $Q(y) = \alpha P(\frac{y}{\alpha})$ is a polynomial function of y. Since $\epsilon > 0$ was arbitrary: if $x \mapsto |x|$ can be uniformly approximated by polynomials on $[-1, 1]$, then it can be uniformly approximated by polynomials on $[-\alpha, \alpha]$. So we are reduced to $\alpha = 1$.
Step 2: Let $T_N(y) = \sum_{n=0}^{N} \binom{\frac{1}{2}}{n} y^n$ be the degree N Taylor polynomial for the function $\sqrt{1-y}$. The radius of convergence is 1, and since

$$\lim_{y \to 1^-} T(y) = \lim_{y \to 1^-} \sqrt{1-y} = 0,$$

by Lemma 3.23, $T_N(y) \overset{u}{\to} \sqrt{1-y}$ on $[0, 1]$. Now for all $x \in [-1, 1]$, $y = 1 - x^2 \in [0, 1]$, so making this substitution we find that on $[-1, 1]$,

$$T_N(1 - x^2) \overset{u}{\to} \sqrt{1 - (1 - x^2)} = \sqrt{x^2} = |x|.$$

□

6.4. Proof of the Weierstrass Approximation Theorem.

It will be convenient to first introduce some notation. For $a < b \in \mathbb{R}$, let $\mathcal{C}[a, b]$ be the set of all continuous functions $f : [a, b] \to \mathbb{R}$, and let $\mathcal{P}$ be the set of all polynomial functions $f : [a, b] \to \mathbb{R}$. Let $\mathrm{PL}([a, b])$ denote the set of piecewise linear functions $f : [a,b] \to \mathbb{R}$.

For a subset $S \subset \mathcal{C}[a,b]$, we define the **uniform closure** of S to be the set of all functions $f \in \mathcal{C}[a,b]$ which are uniform limits of sequences in S: precisely, for which there is a sequence of functions $f_n : [a,b] \to \mathbb{R}$ with each $f_n \in S$ and $f_n \stackrel{u}{\to} f$.

LEMMA 3.25. *For any subset $S \subset \mathcal{C}[a,b]$, we have $\overline{\overline{S}} = \overline{S}$.*

PROOF. Simply unpacking the notation is at least half of the battle here. Let $f \in \overline{\overline{S}}$, so that there is a sequence of functions $g_i \in \overline{S}$ with $g_i \stackrel{u}{\to} f$. Similarly, since each $g_i \in \overline{S}$, there is a sequence of continuous functions $f_{ij} \stackrel{u}{\to} g_i$. Fix $k \in \mathbb{Z}^+$: choose n such that $||g_n - f|| < \frac{1}{2k}$ and then j such that $||f_{nj} - g_n|| < \frac{1}{2k}$; then

$$||f_{nj} - f|| \leq ||f_{nj} - g_n|| + ||g_n - f|| < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}.$$

Thus if we put $f_k = f_{nj}$, then for all $k \in \mathbb{Z}^+$, $||f_k - f|| < \frac{1}{k}$ and thus $f_k \stackrel{u}{\to} f$. $\square$

Observe that the Piecewise Linear Approximation Theorem is

$$\overline{\mathrm{PL}[a,b]} = \mathcal{C}[a,b],$$

whereas the Weierstrass Approximation Theorem is

$$\overline{\mathcal{P}} = \mathcal{C}[a,b].$$

Finally we can reveal our PROOF STRATEGY: it is enough to show that every piecewise linear function can be uniformly approximated by polynomial functions, for then $\overline{\mathcal{P}} \supset \mathrm{PL}[a,b]$, so

$$\overline{\mathcal{P}} = \overline{\overline{\mathcal{P}}} \supset \overline{\mathrm{PL}[a,b]} = \mathcal{C}[a,b].$$

As explained above, the following result completes the proof of Theorem 3.20.

PROPOSITION 3.26. *We have $\overline{\mathcal{P}} \supset \mathrm{PL}[a,b]$.*

PROOF. Let $f \in \mathrm{PL}[a,b]$. By Lemma 3.21, we may write

$$f(x) = b + \sum_{i=1}^{n} m_i |x - a_i|.$$

We may assume $m_i \neq 0$ for all i. Choose real numbers $A < B$ such that for all $1 \leq i \leq b$, if $x \in [a,b]$, then $x - a_i \in [A,B]$. For each $1 \leq i \leq n$, by Lemma 3.24 there is a polynomial P_i such that for all $x \in [A,B]$, $|P_i(x) - |x|| < \frac{\epsilon}{n|m_i|}$. Let $P : [a,b] \to \mathbb{R}$ by $P(x) = b + \sum_{i=1}^{n} m_i P_i(x - a_i)$. Then $P \in \mathcal{P}$ and for all $x \in [a,b]$,

$$|f(x) - P(x)| = |\sum_{i=1}^{n} m_i (P_i(x - a_i) - |x - a_i|)| \leq \sum_{i=1}^{n} |m_i| \frac{\epsilon}{n|m_i|} = \epsilon. \qquad \square$$

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