NOTES ON INFINITE SERIES VII: POWER SERIES AND ABEL'S THEOREM

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7. Power series and Abel's theorem

A series of the form $\sum_{n=0}^{\infty} a_n x^n$ is called a **power series**. Notice that this is a generalization of the geometric series, in which $a_n = 1$ for all n. Note that any partial sum of a power series is of the form $\sum_{n=0}^{N} a_n x^n$ so is just a **polynomial** in x. Later on in the course a major theme will be to try to view power series as "infinite polynomials"; in particular, we will regard x as a variable and be interested in the properties (continuity, differentiability, and so on) of the function $f(x) = \sum_{n=0}^{\infty}$ defined by a power series.

However, if we want to regard the series $\sum_{n=0}^{\infty} a_n x^n$ as a function of x we must first answer the following question: what is its domain? The natural domain of a power series is the set of values of x for which the series converges. Thus the basic question about power series we must answer in this section is:

Question: For a fixed sequence $(a_n)_{n=0}^{\infty}$, for which values of x does the series $\sum_n a_n x^n$ converge?

Note that if we plug in zero, we get the series

$$a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + \ldots = a_0 + 0 + 0 + \ldots = 0,$$

so every power series at least converges at $x = 0$.

On the other hand, we answered the question for the geometric series $\sum_{n=0}^{\infty} x^n$: it converges if and only if $|x| < 1$, i.e., on the interval $(-1, 1)$.

Theorem 1. *(Interval of convergence theorem) Let $\sum_n a_n x^n$ be a power series. Recall the root test quantity $\overline{\theta} = \limsup_n |a_n|^{\frac{1}{n}}$. Put $R := \frac{1}{\overline{\theta}}$; if $\overline{\theta} = \infty$, we put $R = 0$, and if $\overline{\theta} = 0$, we put $R = \infty$. Then:*
a) For any x with $|x| < R$, the series $\sum_n a_n x^n$ converges absolutely.
b) For any x with $|x| > R$, the series $\sum_n a_n x^n$ is divergent.
c) At the values $x = \pm R$, the series may converge absolutely, may converge conditionally or may diverge.

Proof: In fact this result is an immediate corollary of the root test: applying it to the power series $\sum_n a_n x^n$ we get $\limsup_n |a_n x^n|^{\frac{1}{n}} = |x| \limsup_n |a_n|^{\frac{1}{n}} = |x|\overline{\theta}$. By the root test, the power series is absolutely convergent if $|x|\overline{\theta} < 1$, divergent when $|x|\overline{\theta} > 1$, and no conclusion can be drawn when $|x| = \frac{1}{\overline{\theta}} = R$. This is exactly the desired result.

That is, the domain of a power series is always an interval centered at zero and with radius R which is determined by the root test (or the ratio test, when the ratio test limit exists). The issue of convergence at the endpoints $x = \pm R$ is too delicate for the root test, and must be tested on a case-by-case basis just by plugging in the two values of x. Let us give a few examples:

Example: $\sum_n \frac{1}{n!}x^n$. In practice we should try the ratio test first, since it's often easier. In this case we have $\rho = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{n+1} = 0$. Thus $\theta = 0$ and $R = \infty$: this series converges for all real x.[1]

Exercise 34: Show that the power series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ and $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ have $R = \infty$, i.e., they converge for all real x.

Example: $\sum_n n^n x^n$. This is evidently a job for the root test: $|n^n|^{\frac{1}{n}} = n \to \infty$. Therefore $R = 0$ and the series converges only for $x = 0$.

Example: $\sum_{n=1}^{\infty} \frac{x^n}{n}$. Applying the ratio test we get $\rho = \lim_{n\to\infty} \frac{n}{n+1} = 1$, so $R = \frac{1}{\rho} = 1$. We must test the endpoints separately: at $x = 1$ we get the harmonic series, which diverges. At $x = -1$ we get the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges by the alternating series test.

Exercise 35: Suupose that $(a_n)_{n=0}^{\infty}$ is a sequence which is bounded but does not converge to zero. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$.

Remark: If (a_n) is any sequence of positive terms, then evaluating the power series $\sum_n a_n x^n$ at $x = -1$ yields an alternating series. If $\limsup |a_n|^{\frac{1}{n}} = 1$ (e.g. under the circumstances of Exercise 35) then this is a boundary point of the interval of convergence so gets special consideration. This explains why alternating series arise "in nature."

Exercise 36: Suppose $\sum_{n=0}^{\infty} a_n x^n$ is a power series with a finite nonzero radius of convergence R. We are interested in the issue of convergence/divergence at both boundary points $x = \pm R$. Given that at each boundary point the power series could conceivably i) converge absolutely, ii) converge conditionally, or iii) diverge, and that there are two boundary points, there are altogether nine possibilities for the behavior of a power series at the boundary. How many of these nine possibilities actually occur? For each, give either an example or a proof that it cannot occur. (Hint: $|a_n R^n| = |a_n(-R)^n|$.)

7.1. **Algebra of power series.** For any two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, convergent or not, it makes sense to add, subtract and multiply them formally. The sum is defined as for any series, $\sum_n a_n x^n + \sum_n b_n x^n = \sum_n (a_n + b_n)x^n$; similarly, of course for subtraction.

On the other hand, viewing power series as "infinitely long polynomials" suggests

[1] Later in the course we will see that this power series defines the exponential function.

a formula for the product:

$$(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots) \cdot (b_0 + b_1x + b_2x^2 + \ldots),$$

namely we multiply out all possible products and collect coefficients of x^n. This gives us the power series

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \ldots.$$

In other words, we get $\sum_n c_nx^n$, where $c_n = \sum_{i+j=n} a_ib_j$ – this is precisely the Cauchy product!

We can use our theory of Cauchy products to understand convergence of the product power series, and conversely, with one additional tool, we can use power series to give a proof of a result that we promised earlier (at least, in the notes), namely that the Cauchy product never converges to the wrong sum.

Proposition 2. *Let $f(x) = \sum_n a_nx^n$ and $g(x) = \sum_n b_nx^n$ be power series with radii of convergence R_a and R_b. Then the power series $h(x) = \sum_n c_nx^n$, where $c_n = \sum_{i+j=n} a_ib_j$, has radius of convergence at least $R := \min(R_a, R_b)$. Moreover, whenever $|x| < R$, $h(x) = f(x)g(x)$.*

Proof: When $|x| < R$, both series $\sum_n a_nx^n$ and $\sum_n b_nx^n$ are absolutely convergent by Theorem 25. Thus the result follows from Theorem 15.

Exercise 37: Take $f(x) = 1 - x$ and $g(x) = \sum_{n=0}^{\infty} x^n$. What is the product power series $h(x) = f(x)g(x)$? Is its radius of convergence equal to the minimum of the radii of convergence of f and g?

Exercise 38*: Suppose that $f(x) = \sum_{n=0} a_nx^n$ is a power series with $a_0 \neq 0$.
a)Show that there is a unique power series $g(x) = \sum_{n=0}^{\infty} b_nx^n$ such that the Cauchy product power series $f(x)g(x)$ is $1 + \sum_{n=1}^{\infty} 0x^n = 1$. In other words, $g(x) = \frac{1}{f(x)}$. (Hint: Postulate a power series $g(x) = \sum_{n=0} b_nx^n$, and solve the equations $a_0b_0 = 1$, $a_0b_1 + a_1b_0 = 0$, $a_0b_2 + a_1b_1 + a_2b_0 = 0, \ldots$ for the b_n's.)
b) More precisely, show that for all $n \geq 1$,

$$b_n = \frac{-1}{a_0} \cdot \sum_{k=1}^{n} \left(\frac{a_{n-k}}{a_0}\right)^k.$$

c) Show that if the radius of convergence of $f(x)$ is positive, then so is the radius of convergence of $g(x)$. (Hint: Because the radius of convergence is positive, there exists a constant C such that for all $n \geq 1$, $|a_n| \leq C^n$.)

Remark: The treatment of power series we have given is "centered at $x = 0$." If we wanted a theory of power series in which an arbitrary number c plays the role of the central point (later it will become clear why we would want to do this), we need only perform the change of variables $x \mapsto x - c$: in other words, we may still refer to series of the form $\sum_{n=0}^{\infty} a_n(x - c)^n$ as power series.

Exercise 39: Find a power series $\sum_{n=0}^{\infty} a_n(x - c)^n$ whose domain of convergence is $(e, \pi]$.

Remark: This is just the beginning of the study of power series. Later on, when we have the tools to study the properties that a power series has as a function of x, we will become much more motivated to do power series manipulations. In particular, under the right conditions one can *compose* power series to get another power series; this will be very important in building up more complicated power series out of simpler ones.

7.2. **Abel's theorem.** It is as follows:

Theorem 3. *(Abel's theorem) Suppose that $\sum_{n=0}^{\infty} a_n$ is a convergent series. Then*

$$\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n.$$

Proof:[2] Since $\sum_n a_n$ converges, (a_n) is bounded, so the radius of convergence of $f(x) = \sum_n a_n x^n$ is at least 1. Write $S_n = a_1 + \ldots + a_n$, $S_1 = 0$, $S = \lim_n S_n = \sum_n a_n$. We have

$$\sum_{n=0}^{m} a_n x^n = \sum_{n=0}^{m} (S_n - S_{n-1})x^n = (1-x)\sum_{n=0}^{m-1} S_n x^n + S_m x^m.$$

For $|x| < 1$, we let $m \to \infty$ and get

$$f(x) = (1-x)\sum_{n=0}^{\infty} S_n x^n.$$

Fix $\epsilon > 0$ and choose N such that $n \geq N$ implies $|S - S_n| < \frac{\epsilon}{2}$. Then, since

$$(1-x)\sum_{n=0}^{\infty} x^n = 1 \tag{1}$$

for $|x| < 1$, we get

$$|f(x) - S| = |(1-x)\sum_{n=0}^{\infty}(S_n - S)x^n|.$$

Suppose that $x > 1 - \delta$ for some sufficiently small $\delta > 0$. Then we have

$$(1-x)\sum_{n=0}^{\infty}(S_n - S)x^n| = (1-x)\sum_{n=0}^{N} |S_n - S||x|^n + (1-x)\sum_{n>N} \frac{\epsilon}{2}|x|^n.$$

By equation (1), the second term in the last expression is equal to $\frac{\epsilon}{2}$ for any x in $(0,1)$; moreover, the limit of the first expression as $x \to 1^-$ is zero. Putting these two facts together we get that for sufficiently small δ, $|f(x) - S| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, and since this is so for any $\epsilon > 0$ we get the desired result.

Abel's Theorem is precisely what we need to prove the theorem about Cauchy products, namely: suppose that $\sum_n a_n = A$ and $\sum_n b_n = B$ are two convergent series and the Cauchy product $\sum_n c_n$ converges to C. All we need to do is regard this as a problem of evaluating power series at $x = 1$: put $f(x) = \sum_n a_n x^n$, $g(x) = \sum_n b_n x^n$ and $h(x) = \sum_n c_n x^n$. (Since the series converge, (a_n) and (b_n) are bounded sequences, so the radii of convergence of the two power series is at

[2] From Rudin's *Principles of Mathematical Analysis*, pp. 174-175.

least one.) We showed above that for $|x| < 1$, we have $h(x) = f(x)g(x)$, and the assumption is that $f(1)$, $g(1)$ and especially $h(1)$ all exist. Thus, by Abel's theorem we conclude

$$C = h(1) = \lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} f(x)g(x)$$
$$= \lim_{x \to 1^-} f(x) \lim_{x \to 1^-} g(x) = f(1)g(1) = AB.$$

Is this the only point of Abel's Theorem? No, it is much more interesting than this. For one thing, it gives an alternate and more permissive theory of summation of series. Instead of forming the sequence of partial sums and taking the limit, suppose we look at $\lim_{x \to 1^-} a_n x^n$. If this limit exists, let us call it $A \sum_{n=0}^{\infty} a_n$, the **Abel sum** of the series. Then Abel's theorem just says that if a series is convergent, it is "Abel summable" and the Abel sum is the usual sum. However, the converse is not true:

Example: Consider the series whose nth term is $a_n = (-1)^n$; of course, this is divergent according to our scheme. We mentioned a while ago that Euler believed that the sum of the series was "in some sense" equal to $\frac{1}{2}$: since the two limit points of the sequence of partial sums are 0 and 1, this seems vaguely plausible. Abel's summation gives a precise meaning to this: for all x with $|x| < 1$ we have

$$\sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1 - (-x)} = \frac{1}{1 + x}.$$

The Abel sum is the limit of this expression as x approaches 1 from the left, which is indeed $\frac{1}{1+1} = \frac{1}{2}$.

Exercise 40*: Suppose that $\sum_n a_n$ is a series with positive terms. Show that in this case the converse of Abel's theorem holds: namely, if $\lim_{x \to 1^-} \sum_n a_n x^n = L$, then $\sum_n a_n = L$.

We end our discussion of numerical series with the following example: the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ is convergent, albeit only conditionally, by the Alternating Series Test. What is its sum? The following is a crazy formal argument: start with the function $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, which converges for x in $(-1, 1)$. But actually the series is geometric with $r = (-x^2)$, so we have $f(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$. Now the integral F of f (with $F(0) = 0$) is $F(x) = \arctan x$. Assuming that we can integrate power series term-by-term like polynomials (!), we get

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Note that this series converges for x in $(-1, 1]$ and at $x = 1$ is the series we want: thus, our guess is that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} = F(1) = \arctan 1 = \frac{\pi}{4}.$$

But not so fast! We have used first the notion of an integral of a function, which we will explore in the second unit of the course. Second, we have used a fact about integrating a series of functions by integrating each of the terms and adding them together, a naive hope which is not always true but is true under certain conditions and in particular true for power series on the *interior* of the interval of convergence. Determining which properties of functions (continuity, differentiability, integrability...) are preserved under passage to the limit will be the task of the third unit of the course. Finally, notice that we are only integrating a function defined on the open interval $(-1,1)$, whereas the point at which we want to evaluate the antiderivative F is the endpoint $x=1$. This seems problematic, but it turns out it is okay by Abel's Theorem: the power series expansion of the arctangent will be valid for x in $(-1,1)$ and the power series continues to converge at $x=1$, so $\arctan 1 = \lim_{x \to 1^-} \arctan x = \lim_{x \to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ by Abel's Theorem! Thus Abel's theorem is useful in justifying a large number of seemingly dubious calculations.