NOTES ON INFINITE SERIES VI: NONABSOLUTE CONVERGENCE

PETE L. CLARK

6. Nonabsolutely convergent series

6.1. **Leibniz' Alternating Series Test.** Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Since upon taking absolute values we get the harmonic series, this series is *not* absolutely convergent. However, a computation with partial sums suggests that the series converges to $\log 2$. By looking more precisely at the partial sums, we can find a pattern that allows us to see that the series converges.

There are three properties of the series to notice: first, the general term approaches zero; otherwise the series would certainly diverge. Second, the terms alternate in sign:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \dots.$$

Third, in absolute value the terms of the series are monotonically decreasing. Now consider the process of passing from $S_1 = 1$ to $S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Clearly $S_3 < 1$, and this must be the case because $|a_2| > |a_3|$, therefore subtracting $|a_2|$ and adding $|a_3|$ leaves us with a smaller partial sum than we started with. But indeed this argument is valid in passing from any S_{2n-1} to S_{2n+1}: $S_{2n+1} = S_n - |a_{2n}| + |a_{2n+1}| < S_{2n+1}$. Thus the series of odd-numbered partial sums S_{2n-1} is decreasing. On the other hand, in passing from S_2 to S_4 we have $S_4 = S_2 + \frac{1}{3} - \frac{1}{4}$, so that $S_4 > S_2$ and similarly $S_{2n+2} > S_{2n}$ for all n. Very similar reasoning shows that every odd-numbered partial sum S_{2n+1} with $n \geq 1$ is greater than S_2 (or, if you like, every odd-numbered partial sum is greater than $S_0 = 0$), because we can regroup these (finite!) sums to be S_2 plus a sum of positive terms. Thus, since the odd-numbered partial sums (S_{2n-1}) form a decreasing sequence which is bounded below, it must converge to some number S_{odd}. Similarly, one has that $S_{2n} < S_2$ for all n, so the even-numbered partial sums form a bounded increasing sequence, converging say to S_{even}. In order for a sequence to converge, it is necessary and sufficient for the subsequences of even and odd-numbered terms to converge to the same value. But the difference $|S_{2n} - S2n - 1| = |a_{2n}| \to 0$, so indeed $S_{odd} = S_{even} = S$ is the sum of the series.

The above argument is obviously not just applicable to the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$; rather, we get the following very important result due to Leibniz.

Theorem 1. *(Alternating Series Test) Let $\sum_{n=1}^{\infty} a_n$ be a series satisfying the following three conditions:*
i) The terms a_n are alternating in sign: for all n, $a_n \cdot a_{n+1} < 0$.
ii) The sequence $(|a_n|)$ is monotonically decreasing.

iii) $\lim_{n\to\infty} a_n = 0$.
Then the series $\sum_{n=1}^{\infty}$ *is convergent.*

In fact, the proof of the theorem gives an explicit and useful bound on the error $E_N = |\sum_{n=1}^{\infty} - \sum_{n=1}^{N} a_n|$ obtained in cutting off an alternating series after the Nth term.

Corollary 2. *(Alternating series error estimate) Let* $\sum_{n=1}^{\infty} a_n$ *satisfy the hypotheses of the alternating series test. Then, for any* N*, the sum of the series lies between* S_N *and* S_{N+1}*. In particular,* $E_N \leq |a_{N+1}|$.

Exercise 26: Prove the alternating series error estimate.

This makes it very easy to approximate sums of series. For instance, in order to get three place accuracy in the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, we need to choose n such that $|a_n| < 10^{-3}$, so it will do to take $S_{999} = 0.6936\ldots$.[1]

Example: An important family of nonabsolutely convergent series are the alternating p-series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ for $0 < p < 1$.

6.2. **Dirichlet's Test.** One of the problems with nonabsolutely convergent series is that there are many fewer tests which can apply to show that they are convergent. The following test, which gives a generalization of the Alternating Series Test, is one of the few exceptions:

Theorem 3. *(Dirichlet's Test) Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two infinite series. Suppose that:*
i) The partial sums of $\sum_{n=1}^{\infty} a_n$ *are bounded.*
ii) The terms b_n *decrease monotonically to zero.*
Then the series $\sum_{n=1}^{\infty} a_n b_n$ *is convergent.*

Proof:[2] Let S_n be the sequence of partial sums for $\sum_{k=1}^{\infty} a_k$, which is bounded, say, by M. For an $\epsilon > 0$, choose a positive integer $N > 1$ such that $b_N < \frac{\epsilon}{2M}$. For $n > m \geq N$, we have

$$|\sum_{k=m}^{n} a_k b_k| = \sum_{k=m}^{n} (s_k - s_{k-1}) b_k| =$$

$$|\sum_{k=m}^{n} s_k b_k - \sum_{k=m-1}^{n-1} s_k b_{k+1}| =$$

$$|\sum_{k=m}^{n-1} s_k (b_k - b_{k+1} + s_n b_n - s_{m-1} b_m|$$

$$\leq \sum_{k=m}^{n-1} |s_k||b_k - b_{k+1}||s_n||b_n| + |s_{m-1}|b_m|$$

[1]This is easy in the sense that it is very little work to say how large a partial sum we need in order to get a predetermined accuracy. On the other hand, in order to get k decimal place accuracy in this example we need to compute 10^k terms, which means that the convergence is actually quite slow compared to examples obtained by comparison with a geometric series.

[2]From p. 232 of your text.

$$\leq M\left(\sum_{k=m}^{n-1} |b_k - b_{k+1}| + |b_n| + |b_m|\right)$$

$$= M(\sum_{k=m}^{n-1} (b_k - b_{k+1}) + b_n + b_m)$$

$$= M(b_m - b_n + b_n + b_m) = 2Mb_m \leq 2Mb_N < \epsilon.$$

By the Cauchy criterion for convergence, the series $\sum_{n=1}^{\infty} a_n b_n$ converges. This completes the proof.

Notice that Leibniz's alternating series test follows as a corollary: we take $a_n = (-1)^n$ and b_n to be a sequence decreasing monotonically to zero. Indeed, we get a more general version of Leibniz's test: if we start with a sequence (b_n) decreasing monotonically to zero and insert ± 1 signs $a_n \in \{\pm 1\}$ which are "approximately alternating" in the sense that the partial sums $a_1 + \ldots + a_n$ are bounded, then Dirichlet's test says that the series $\sum_n a_n b_n$ still converges.

Exercise 27*: Show that Dirichlet's generalization of the alternating series test the strongest possible in the following sense: if (a_n) is a sequence of elements in $\{\pm 1\}$ such that the partial sums $S_n = a_1 + \ldots + a_n$ form an *unbounded* sequence, then there exists a sequence (b_n) decreasing monotonically to zero such that $\sum_n a_n b_n$ diverges.

Exercise 28: Show that if the hypothesis that the terms are monotonically decreasing in absolute value cannot be omitted from the alternating series test: i.e., there exist alternating series whose general term approaches zero but which are divergent.

Your textbook presents (Theorem 6.18 on p. 232) an interesting version of Dirichlet's test in which the terms of the sequence (a_n) are allowed to be **complex numbers**, and the hypotheses of the theorem are still the same. Our policy in this course is that we do not "officially speaking" consider complex-valued series and functions. However, it is the case that most of our results have analogues in the complex numbers and the proofs are all but identical to the case of the real numbers. (We single out the case of absolute convergence. A series of complex numbers $\sum_n z_n$, where $z_n = a_n + ib_n$ is said to be convergent if both its real and imaginary parts $\sum_n a_n$ and $\sum_n b_n$ are convergent. It is said to be absolutely convergent if the series $\sum_n |z_n|$ converges, where $|z_n| = |a_n + ib_n| = \sqrt{a_n^2 + b_n^2}$ is the usual complex absolute value – i.e., the distance from (a_n, b_n) to the origin in the complex plane. It is still true that absolutely convergent series are convergent, and the second proof of the proposition that absolute convergence implies convergence goes through verbatim in the complex case.)

Occasionally however it will be enlightening for us to consider the complex case as an aid to understanding the real case.[3] The following is a nice example of this:

[3] *The shortest path between two truths on the real line passes through the complex plane.* – Jacques Hadamard

Example:[4] We will show that the series $\sum_n \frac{\sin n}{n}$ and $\sum_n \frac{\cos n}{n}$ are both convergent. For this we assume the identity $e^{ix} = \cos x + i \sin x$, so the convergence of both series is equivalent to the convergece of the complex series $\sum_n \frac{e^{in}}{n}$. Writing the general term as $a_n \cdot b_n$, where $a_n = e^{in}$ and $b_n = \frac{1}{n}$, we can apply Dirichlet's test if we verify that the partial sums $\sum_{n=0}^{N} e^{in}$ are uniformly bounded. But since $e^{in} = (e^i)^n$, this is actually a finite geometric series, so the sum is $\frac{1-(e^i)^{N+1}}{1-e^i}$. Now $|e^{ix}| = 1$ for all real x, and $e^{ix} = 1$ if and only if x is an integer multiple of 2π. In particular $e^i = e^{i \cdot 1} \neq 1$, so the sum is well-defined, and $|(e^i)^{N+1}| = 1$, so an application of the triangle inequality gives $|\frac{1-e^{i(N+1)}}{1-e^i}| \leq \frac{2}{|1-e^i|}$, gives a bound independent of N. Therefore Dirichlet's test applies.

Exercise 29: Show that neither $\sum_n \frac{\sin n}{n}$ nor $\sum_n \frac{\cos n}{n}$ is absolutely convergent.
Suggestion: The idea is that both $|\sin n|$ and $|\cos n|$, which are obviously somewhere between 0 and 1, are actually bounded away from zero plenty of the time. The task is to make this precise. A hint for one way to do it: what is the largest length of an interval on which either $|\sin x|$ or $|\cos x|$ is at most $\frac{1}{2}$? So how many consecutive values of $|\sin n|$ (or $|\cos n|$) can be less than or equal to $\frac{1}{2}$?

6.3. **A divergent Cauchy product.** Recall that above we showed that if $\sum_n a_n = A$ and $\sum_n b_n = B$ were convergent series, *at least one* of which is absolutely convergent, then the Cauchy product series $\sum_n c_n$ is convergent to AB, where $c_n = \sum_{i+j=n} a_i b_j$. Here we give an example, due to Cauchy, of a situation in which the Cauchy product of two nonabsolutely convergent series fails to converge.

We will take both series equal to the alternating p-series $\sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}}$. The nth term in the Cauchy product is

$$c_n = \sum_{i+j=n} (-1)^i (-1)^j \frac{1}{\sqrt{i+1}} \frac{1}{\sqrt{j+1}}.$$

The first thing to notice is that $(-1)^i(-1)^j = (-1)^{i+j} = (-1)^n$, so c_n equals $(-1)^n$ times a sum of positive terms. Since for all i, $\frac{1}{\sqrt{i+1}} \geq \frac{1}{\sqrt{n+1}}$, and similarly for j, each term in c_n is at most $(\frac{1}{\sqrt{n+1}})^2 = \frac{1}{n+1}$. Since we are summing from $i = 0$ to n there are $n+1$ terms overall, so we find that $|c_n| \geq 1$ for all n. Thus the general term does not approach zero, so the series must diverge. (Hence it also diverges if we remove any/all of the parentheses in the Cauchy product $\sum_n (a_0 b_n + \ldots + a_n b_0)$.)

6.4. **Rearrangement Theorems.** We come now to what is probably the most surprising result of the entire course, which is Riemann's theory of rearrangements of nonabsolutely convergent series. (Recall that we showed that no rearrangement of an absolutely convergent series alters either the convergence or the sum!)
Let $\sum_n a_n$ be an arbitrary real series, and introduce the following notation: for any real number r, $r^+ = r$ if $r \geq 0$ and 0 otherwise. Similarly $r^- = r$ if $r \leq 0$ and 0 otherwise. Thus, we always have $r = r^+ - r^-$ and $|r| = r^+ + r^-$ (check this!). For our series, we thus have formally

$$\sum_n a_n = \sum_n a_n^+ - \sum_n a_n^-.$$

[4]This is, in fact, Exercise 17 on page 235 of your text.

We call $\sum_n a_n^+$ and $\sum_n a_n^-$ the **positive part** and **negative part** of the original series. Now suppose that the series – i.e., the left hand side – is convergent. Thus the positive and negative parts are either both convergent or both divergent. If both $\sum_n a_n^+$ and $\sum_n a_n^-$ are convergent, then so is $\sum_n a_n^+ + a_n^- = \sum_n |a_n|$, i.e., in this case the series is absolutely convergent. Turning this argument around, we've established the following:

Proposition 4. *If a series $\sum_n a_n$ is convergent but not absolutely convergent, then both the positive and negative parts are divergent.*

Let us reflect for a moment on this state of affairs. What it means is that in any nonabsolutely convergent series we have both enough positive terms and enough negative terms to make the series diverge; it is only because of lots of **cancellation** between positive and negative terms that the series can converge. What this really means is that the order of the terms is very important: if we rearranged the terms so that, say, we took many positive terms before taking each negative terms, then we should be able to make the series diverge.

It seems plausible that this could certainly happen with certain nonabsolutely convergent series. What is shocking is that it happens for *every* nonabsolutely convergent series:

Proposition 5. *Let $\sum_n a_n$ be any series which is convergent but not absolutely convergent. Then there exists a rearrangement $a_{\sigma(n)}$ of the terms so that the series diverges to $+\infty$ and another rearrangement so that the series diverges to $-\infty$.*

Proof: Since the series is convergent, we have $a_n \to 0$, which means that the sequence a_n is bounded: choose an M such that $|a_n| \leq M$ for all n. We are not going to give an explicit function σ; rather, we are going to describe a process of when to take positive terms and when to take negative terms. For this it is convenient to imagine that the sequence (a_n) is already sifted into a disjoint union of two subsequences, one consisting of the positive terms and one consisting of the negative terms (we can of course, ignore those $a_n = 0$). If we like, we can even imagine both sequences ordered so that they are nonincreasing in absolute value. Thus we have two positive sequences $p_1 \geq p_2 \geq \ldots \geq p_n$ and $n_1 \geq n_2 \geq \ldots \geq n_n \ldots$, so that together the terms $\{p_i, n_i\}$ comprise the terms of the series. The key point is the previous proposition, which tells us that $\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} n_n = \infty$. In order to specify a rearrangement, we can specify a choice of a certain number of positive terms, then a certain number of negative terms, then a certain number of positive terms, and so on. The only thing to make sure of is that every term of the sequence p_n and of the sequence n_n gets included eventually!

To get a sequence of positive sums approaching $+\infty$, we proceed as follows: take positive terms $p_1, p_2, \ldots$ in order, until we get a partial sum that exceeds $1 + M$; then we take the first negative term n_1. Since $|n_1| \leq M$, this partial sum $p_1 + \ldots + p_{k_1} + n_1$ is still at least 1. Then we take enough positive terms to get a partial sum exceeding $2 + M$, then we take n_2, and so on. It is clear that after the nth iteration of this process, every partial sum from that point on will be at least n, so therefore the sequence of partial sums diverges to $+\infty$.

Of course a very similar argument with the roles of p_n and n_n reversed allows us to build a rearrangement for which the partial sums diverge to $-\infty$. Details are left to you.

In fact we can do much more than this; this was just a warmup to get used to the idea of specifying a rearrangement by a qualitative process rather than by an explicit pattern or formula.

Theorem 6. *(Riemann's rearrangement theorem) Let $\sum_n a_n$ be any series which converges but is not absolutely convergent. Let S be any extended real number. Then there exists a rearrangement of the series whose partial sums approach S.*

Proof: We have already taken care of the cases where $S = \pm\infty$, so now we want a rearrangement to converge to some given finite real number S. The argument is much the same: we take positive terms $p_1 + \ldots + p_{n_1}$ until we get a partial sum exceeding S (and we stop as soon as this happens; if $S < 0$ then we take zero positive terms in the first step); then we take negative terms $n_1 + \ldots + n_{n_2}$ until we get a partial sum which is less than S, again stopping as soon as this happens. We repeat this process indefinitely, using up all the positive and negative terms.

It remains to be seen that this rearrangement converges to S, but this is not difficult. We use again the fact that $a_n \to 0$. Notice after each step, the error $|S_{n_k} - S|$ is at most the size of the last term we added (think about why this is true). Thus the error approaches zero as $k \to \infty$, which was to be shown!

This theorem exposes the dark side of nonabsolutely convergent series: just by changing the order of the terms, we can make the series diverge or be any real number! Because of this, we see that the nonabsolute convergence of a series is of a more delicate and less satisfactory nature. Let us define a series to be **conditionally convergent** if it is convergent, but some rearrangement of it is divergent. Then both of our results on rearrangements can be summarized as follows:

Theorem 7. *(Main rearrangement theorem) A convergent real series is conditionally convergent if and only if it is nonabsolutely convergent.*

Comment: Many texts do not use the term "nonabsolutely convergent" and define a series to be conditionally convergent if it is convergent but not absolutely convergent. Aside from the fact that this terminology can be confusing to students (especially calculus students) to whom this rather intricate story of rearrangements has not been told, it seems correct to distinguish between the two *a priori* different notions of: $\sum_n a_n$ converges but $\sum_n |a_n|$ does not, and $\sum_n a_n$ converges but some rearrangement $\sum_n a_{\sigma(n)}$ does not. In fact there are more general contexts in which the notions of absolute convergence and conditional convergence can be pursued – namely **Banach spaces** – and in this larger domain there in fact exist series for which every rearrangement converges but for which the series of absolute values does not converge!

Exercise 30*: Even Riemann's theorem is not the strongest possible result about limiting behavior of nonabsolutely convergent series upon rearrangement. Prove the following generalization: if $m \leq M$ are any two extended real numbers, and $\sum_n a_n$ is any nonabsolutely convergent series, then there exists a rearrangement whose series of partial sums has upper limit M and lower limit m. Show that for such a rearrangement, every number r such that $m < r < M$ is also a limit point of the sequence of partial sums.

Exercise 31**: Consider, if you like, the case of series of complex numbers $\sum_n z_n$ which are convergent but for which $\sum_n |z_n|$ is divergent. What is the appropriate analogue of Riemann's theorem in this context? Is it necessarily the case that for any complex number Z there exists a rearrangement converging to z? (Hint: a real series is in particular a series of complex numbers!)

Exercise 32: There is a version of Riemann's theorem in which, instead of rearranging the terms of a series $\sum_n a_n$, we insert signs: that is, we replace $\sum_n a_n$ by $\sum_n \epsilon_n a_n$, where $\epsilon_n \in \{\pm 1\}$; we call (ϵ_n) a **sign sequence**.
a) Show that if $\sum_n a_n$ is absolutely convergent, then so is $\sum_n a_n \epsilon_n$ for any sign sequence (ϵ_n), and that there exists a number M such that $|\sum_n \epsilon_n a_n| \leq M$ for all sign sequences.[5]
b) Suppose that $a_n \to 0$ and $\sum_n |a_n| = \infty$. Show that for every extended real number $L \in [-\infty, \infty]$, there exists a sign sequence (ϵ_n) such that $\sum_n a_n \epsilon_n \to L$.

Exercise 33: Do exercise 15, from page 234 of your text, namely: if $\sum_k a_k^2$ and $\sum_k b_k^2$ converge, then $\sum_k a_k b_k$ converges absolutely. (Hint: use an inequality from Chapter 1.)

[5]Of course the sum can and will change for some sign sequence, unless $a_n = 0$ for all n.