NOTES ON INFINITE SERIES V: ABSOLUTE CONVERGENCE

PETE L. CLARK

5. Absolute convergence

In this section we revisit the convergence theory of series with arbitrary (not necessarily non-negative) terms. It turns out that if we impose one additional hypothesis, all the techniques and results of the last section can be carried over.

A series $\sum_n a_n$ is **absolutely convergent** if the series $\sum_n |a_n|$ converges. (Note that this is a series with non-negative terms.) A series which converges but for which $\sum_n a_n$ diverges is said to be **nonabsolutely convergent**.

Proposition 1. *Absolutely convergent series are convergent.*

We give two proofs.

First proof: Consider the three series $\sum_n a_n$, $\sum_n |a_n|$ and $\sum_n a_n + |a_n|$. The second series converges by hypothesis. If the third series converges, then the first series, being the difference of two convergent series, must also converge. But $a_n + |a_n|$ is equal to $2a_n$ when a_n is non-negative and zero otherwise. Thus it is a series of non-negative terms which is term-by-term less than or equal to $\sum_n 2|a_n|$; hence the third series converges by the comparison test.

Second proof: Because $\sum_n |a_n|$ converges, for every $\epsilon > 0$ there exists N such that $n \geq N$ implies $\sum_{n=N}^{\infty} |a_n| < \epsilon$. Therefore

$$|\sum_{n=N}^{\infty} a_n| \leq \sum_{n=N}^{\infty} |a_n| < \epsilon,$$

so the series converges by the tail (Cauchy) criterion for convergence.

Example: Consider the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$. Since $|\sin n| \leq 1$ for all n, we have $\sum_n \frac{|\sin n|}{n^2} \leq \sum_n \frac{1}{n^2}$, so the series is absolutely convergent by comparison. In particular it is convergent.

There are "absolute" versions of all the tests of the previous section. In particular, we get absolute versions of the ratio and root tests:

For an arbitrary series $\sum_n a_n$, define

$$\underline{\rho} := \liminf_n \frac{|a_{n+1}|}{|a_n|}.$$

$$\overline{\rho} := \limsup_n \frac{|a_{n+1}|}{|a_n|}.$$

$$\underline{\theta} := \liminf_n |a_n|^{\frac{1}{n}}.$$

$$\overline{\theta} := \limsup_n |a_n|^{\frac{1}{n}}.$$

Proposition 2. *(Ratio & Root Tests for Absolute Convergence) Let $\sum_n a_n$ be a series. If $\underline{\rho} < 1$ or $\underline{\theta} < 1$, then the series is absolutely convergent. If $\overline{\rho} > 1$ or $\overline{\theta} > 1$, then the series diverges.*

Exercise 23: Prove Proposition 2.

Rearrangements: To investigate the commutative law for infinite sums we need the notion of a **rearrangement** of a series, in which we take the terms in a different order. Formally, let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection.[1] Then the rearrangement of $\sum_{n=0}^{\infty} a_n$ by σ is $\sum_{n=0}^{\infty} a_{\sigma(n)}$. For instance, if σ is the function which for all $n \geq 0$ interchanges $2n$ and $2n+1$, then the rearrangement corresponding to σ is

$$a_1 + a_0 + a_3 + a_2 + a_5 + a_4 + \ldots.$$

Theorem 3. *Let $\sum_n a_n$ be an absolutely convergent series. Then every rearrangement $\sum_n a_{\sigma(n)}$ converges absolutely to the same sum.*

Proof: For clarity, let us write $b_n = a_{\sigma(n)}$ for the terms of the rearranged series. To show that the second series converges to the same sum as the first is equivalent to showing that the difference of the two sequences of partial sums $D_n = \sum_{k=0}^{n} a_n - b_n$ converges to zero. Now, since $\sum_n |a_n|$ converges, for any $\epsilon > 0$ there is an N_1 such that $\sum_{n=N_1+1}^{\infty} |a_n| < \frac{\epsilon}{2}$.

Now comes the key point: all the terms in the rearranged series will occur eventually: thus, there exists an $N_2 \geq N_1$ such that among the terms $a_{\sigma(1)}, \ldots, a_{\sigma(N_2)}$ we find all the terms $a_1, \ldots, a_{N_1}$. (For example, in the simple rearrangement considered above, we can take $N_2 = N_1 + 1$.) Now suppose $n \geq N_2$, and consider the difference $|D_n| = |\sum_{k=0}^{n} a_k - b_k|$. Every term a_k with $k \leq N_1$ in $\sum_k a_k$ will be cancelled out by a corresponding term of b_k; thus in either sum the only uncancelled terms can have index at least $N_1 + 1$, so

$$|\sum_{k=0}^{n} a_k - b_k| \leq \sum_{k=N_1+1}^{\infty} |a_k| + \sum_{k=N_1+1}^{\infty} |a_k| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $D_n \to 0$, which is what we wanted to show.

Exercise 24: For an infinite series $\sum_{n=0}^{\infty} a_n$, if we sum only over the terms of a subsequence $(n_k)_{k=0}^{\infty}$ we get a **subseries** $\sum_{n=0}^{\infty} a_{n_k}$. Informally speaking, we take infinitely many, but not all, of the terms of the original series. (Warning: do not confuse a subseries with passing to a subsequence of the sequence of partial sums. The latter operation corresponds to inserting parentheses in the infinite sum, whereas this operation corresponds to the more drastic process of omitting some of the a_i's entirely.) Show that if $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then every subseries is absolutely convergent. Must the sum be the same?

[1]We denote by $\mathbb{N}$ the set $\{0, 1, 2, \ldots\}$. Some others would use this notation to mean that zero is not included.

5.1. **Products.** Let us now revisit the issue of the product of two series $(\sum_{n=0}^{\infty} a_n)\cdot(\sum_{n=0}^{\infty} b_n)$. Recall that the challenge is how to order/parenthesize the terms which should appear in the product, namely $a_i b_j$ as i and j range over all non-negative integers. The first thing to say is that if *both* series are absolutely convergent, then it does not matter how we arrange the terms.

Theorem 4. *(Product of two absolutely convergent series) Let $\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$ be two absolutely convergent series. Let $(p_n)_{n=0}^{\infty}$ be any ordering of the countable set $(a_i b_j)_{0 \leq i,j \leq \infty}$. Then the series $\sum_{n=0}^{\infty} p_n$ is absolutely convergent, with sum equal to AB.*

Proof: First consider the case when both a_n and b_n are non-negative for all n. I claim that it is then rather obvious that the series $\sum_n p_n$ converges to AB. Indeed, every partial sum of the series is less than or equal to a partial sum which includes all terms $0 \leq i \leq N$ and $0 \leq j \leq N$, in which case the partial sum is at most $(a_0 + \ldots + a_N)\cdot(b_0 + \ldots + b_N) \leq AB$. Thus the sequence of partial sums is bounded by AB, and being a series of non-negative terms, is therefore convergent. Moreover, for any $\epsilon > 0$, we can choose N so that $\sum_{i=0}^{N} a_i > A - \epsilon$ and $\sum_{j=0}^{N} b_j > B - \epsilon$, so then any partial sum which includes all $i, j \leq N$ is at least

$$(A - \epsilon)(B - \epsilon) = AB - \epsilon(A + B) + \epsilon^2,$$

which, as $\epsilon \to 0$, approaches AB.

Now the case where both series are absolutely convergent follows: indeed, since the series $\sum_n |a_n|$ and $\sum_n |b_n|$ are both assumed convergent, by the above paragraph we have that $\sum_n |p_n|$ is convergent, so the series $\sum_n p_n$ is absolutely convergent. To show that the sum is indeed equal to AB, we need a final little trick: namely, the sequence

$$\Box_n := (a_0 + \ldots + a_n)\cdot(b_0 + \ldots + b_n)$$

is a subsequence of a certain ordering of the products $a_i b_j$ – e.g., the one starting $a_0b_0,\ a_0b_1,\ a_1b_0,\ a_0b_2,\ a_1b_1,\ a_2b_0, \ldots$. Since $\Box_n \to AB$, AB is a limit point of the convergent series $\sum_n p_n$; thus, $\sum_n p_n$ itself converges to AB.

Suppose now that the series $\sum_n b_n$ is *non*absolutely convergent. Then, except in the trivial case where $a_n = 0$ for all n, no series p_n enumerating the $a_i b_j$'s can be absolutely convergent. Indeed, by Exercise 24, if the series were absolutely convergent, then so would every subseries be absolutely convergent, but taking any a_N which is not zero, the series $\sum_j a_N b_j = a_N \sum_j b_j$ is not absolutely convergent. We will see in the next section that this means we must be very careful in what order we take the terms of the product. The next result provides evidence that the Cauchy product $c_n := \sum_{k=0}^{n} a_k b_{n-k}$is "the right way" to order and group the terms:

Theorem 5. *(Mertens) Let $\sum_{n=0}^{\infty} a_n = A$ be an* ***absolutely convergent*** *series and $\sum_{n=0}^{\infty} b_n = B$ be a convergent series. Then the Cauchy product series $\sum_{n=0}^{\infty} c_n$ converges to AB.*

Proof:[2] Put

$$A_N = \sum_{k=0}^{n} a_k, B_n = \sum_{k=0}^{n} b_k, C_n = \sum_{k=0}^{n} c_k, \beta_n = B_n - B.$$

Then

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \ldots + (a_0 b_n + \ldots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \ldots + a_n b_0 \\ &= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \ldots + a_n (B + \beta_0) \\ &= A_n B + A_0 \beta_n + a_1 \beta_{n-1} + \ldots + a_n \beta_0. \end{aligned}$$

Put $\gamma_n = a_0\beta_n + a_1\beta_{n-1} + \ldots + a_n\beta_0$. Since our goal is to show that $C_n \to AB$, and $A_n B \to AB$, it suffices to show that $\gamma_n \to 0$. Note that we have not yet used the absolute convergence of the first series, but we use it now: put $\alpha = \sum_{n=0}^{\infty} |a_n|$. Since $\beta_n \to 0$, for any $\epsilon > 0$, we can choose N such that $|\beta_n| \leq \epsilon$ for all $n \geq N$, so

$$\begin{aligned} |\gamma_n| &\leq |\beta_0 a_n + \ldots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \ldots + \beta_n a_0| \\ &\leq |\beta_0 a_n + \ldots + \beta_N a_{n-N}| + \epsilon\alpha. \end{aligned}$$

Keeping N fixed and letting $n \to \infty$, we get

$$\limsup_n |\gamma_n| \leq \epsilon\alpha,$$

which since ϵ was arbitrary, implies that $\gamma_n \to 0$, completing the proof.

As you might expect from the increased difficulty of the proof when only one of the two series is absolutely convergent, when both series are nonabsolutely convergent, the Cauchy product need not be convergent. However, we do not yet have any examples of nonabsolutely convergent series, so we will revisit this point shortly.

Another important property of the Cauchy product is that it never converges to "the wrong value." This is formalized in the following theorem (due to Cauchy):

Theorem 6. *Let $\sum_n a_n = A$ and $\sum_n b_n = B$ be two convergent series, with Cauchy product $\sum_n c_n$. If $\sum_n c_n$ converges, then its sum is AB.*

We do not have the tools to prove this theorem at the moment; we will come back to it after a discussion of power series.

Exercise 25: Show that a nonabsolutely convergent series must have infinitely many positive terms and infinitely many negative terms.

[2]We follow Rudin's *Principles of Mathematical Analysis*, pp. 74-75.