NOTES ON INFINITE SERIES IV: RATIOS AND ROOTS

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4. Ratios and roots

We are still considering series $\sum_n a_n$ with $a_n \geq 0$ for all n. In fact, in this section we will assume $a_n > 0$ for all n. (Of course in theory we may omit as many instances of "$a_n = 0$" in the terms of a series as we care to: adding or subtracting any number of zeros does not change anything!)

Both the ratio and root tests are predicated on the idea of comparing our series to a geometric series, so it may clarify matters to try to consider, even before stating and proving the two tests, under what circumstances such a situation could succeed.

First, the series $\sum_n r^n$ has the property that the ratio between consecutive terms is always $\frac{r^{n+1}}{r^n} = r$. Suppose that $r < 1$ and we want to show that a series $\sum_n a_n$ is convergent. Suppose first that $\frac{a_{n+1}}{a_n} \leq r < 1$ for all n. Thus $a_1 \leq ra_0$, $a_2 \leq ra_1 \leq r^2 a_0$, and in general $a_n \leq r^n a_0$. Thus, under this hypothesis we have $\sum_{n=0}^{\infty} a_n \leq a_0 \sum_{n=0}^{\infty} r^n = \frac{a_0}{1-r} < \infty$, and in particular the series converges.

But remember that we are always free to neglect any finite number of terms of a series as far as its convergence is concerned. Thus, if there exists an r, $0 < r < 1$, with the property that for all sufficiently large r, $\frac{a_{n+1}}{a_n} \leq r$, just by starting the sum at some larger N, we may still deduce the convergence of the series (but not an explicit upper bound for the sum, unless we know how many terms to neglect). But this condition can be phrased much more succinctly:

Exercise 17a): Let $\sum_n a_n$ be a series with $a_n > 0$ for all n. Show that the following are equivalent:
i) There is a positive number $r < 1$ such that $\frac{a_{n+1}}{a_n} \leq r$ for all sufficiently large n.
ii) $\limsup \frac{a_{n+1}}{a_n} \leq r < 1$.

There is a very similar argument for divergence: if there exists $r > 1$ such that $\frac{a_{n+1}}{a_n} \geq r > 1$ for all n, then we have $a_1 > ra_0$, $a_2 > ra_1 > r^2 a_0$, and in general $a_n > r^n a_0$, so that the series diverges by comparison to $\sum_n a_0 r^n$. And again, if these inequalities are true only for sufficiently large n, then we get the same conclusion just by starting the summation at some large value N. And again, this condition can be rephrased:

Exercise 17b): Let $\sum_n a_n$ be a series with $a_n > 0$ for all n. Show that the following are equivalent:
i) There exists a number $r > 1$ such that $\frac{a_{n+1}}{a_n} \geq r$ for all sufficiently large n.

ii) $\liminf \frac{a_{n+1}}{a_n} \geq r > 1$.

In summary, we have proved the following important result, which is a generalization of the usual ratio test.

Theorem 1. *(d'Alembert's Ratio Test) Let* $\sum_n a_n$ *be a series with non-negative terms, and let* $\underline{\rho}, \overline{\rho}$ *be, respectively, the limits inferior and superior of the sequence* $\frac{a_{n+1}}{a_n}$.
a) If $\overline{\rho} < 1$, *then the series* $\sum_n a_n$ *is convergent.*
b) If $\underline{\rho} > 1$, *then the series* $\sum_n a_n$ *is divergent.*

At the risk of stating the obvious, we point out that the most favorable situation is when the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$ exists, and then the ratio test says that if $\rho < 1$ the series converges, while if $\rho > 1$ the series diverges. It does not say anything at all when $\rho = 1$.

Example: Let x be any positive number. We will show that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges. (Recall that we showed this earlier when $x = 1$.) We consider the quantity

$$\frac{a_{n+1}}{a_n} = \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x}{n+1}.$$

So evidently we have $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$ for any x; thus the series converges.

There is variant of the ratio test which is also sometimes useful. Namely, instead of focusing on the property that the geometric series $\sum_n r^n$ has constant ratios of consecutive terms, we remark that the sequence has the property that the nth root of the nth term is equal to r. Suppose now that $\sum_n a_n$ is a series with positive terms with the property that $a_n^{\frac{1}{n}} \leq r$ for some $r < 1$. Then raising both sides to the nth power, we get $a_n \leq r^n$, and again we find that the series converges by comparison to a geometric series. As above, by omitting finitely many terms we get a test similar to the ratio test but involving the limsup of the sequence $a_n^{\frac{1}{n}}$. This gives part a) of the following theorem – this time we ask you to write down the details.

Theorem 2. *(Cauchy's Root Test) Let* $\sum_n a_n$ *be a series with non-negative terms, and let* $\overline{\theta}$ *be the upper limit of the sequence* $a_n^{\frac{1}{n}}$.
a) If $\overline{\theta} < 1$, *then the series* $\sum_n a_n$ *is convergent.*
b) If $\overline{\theta} > 1$, *then the series* $\sum_n a_n$ *is divergent.*

Exercise 18a): Prove part a) of Cauchy's Root Test by fleshing out the above argument.

It remains to prove part b). Before discussing the proof, note that something is different from the case of the Ratio Test: we are dealing with the limsup again (rather than the liminf, as in the Ratio Test).

Exercise 18b): Prove part b) of Cauchy's root test. In fact, prove the following stronger result: if, for infinitely many n we have that $a_n^{\frac{1}{n}} \geq 1$, then the series diverges. (Hint: can the general term approach zero?) Make sure to explain why the hypothesis in part b) implies this condition is satisfied!

It is a fact that the root test is "strictly stronger" than the ratio test: this means that whenever the ratio test succeeds in determining the convergence or divergence of an infinite series, the root test would also succeed, and there are some examples where the root test succeeds but the ratio test fails. The following lemma allows us to explain this statement:

Lemma 3. *For any series with non-negative terms, we have*

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf a_n^{\frac{1}{n}} \leq \limsup a_n^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}.$$

Proof: Of course the middle inequality holds for any sequence. With notation as in the statement of the theorems, let us show the inequality $\overline{\rho} \leq \overline{\theta}$. Suppose $r > \overline{\rho}$, so that for all sufficiently large n, $\frac{a_{n+1}}{a_n} < r$. As in the proof of the Ratio Test, we have $a_{n+k} < r^k a_n$. We can rewrite this as

$$a_{n+k} < r^{n+k} \cdot (\frac{a_n}{r^n}),$$

or

$$a_{n+k}^{\frac{1}{n+k}} < r \cdot \left(\frac{a_n}{r^n}\right)^{\frac{1}{n+k}}.$$

Holding n fixed and letting $k \to \infty$, the last parenthesized quantity approaches 1, so that the limit superior of the sequence $a_n^{\frac{1}{n}}$ is at most r, so $\overline{\theta} \leq r$. That is, $\overline{\rho} \leq \overline{\theta}$. The proof of the inequality $\underline{\rho} \leq \underline{\theta}$ is similar, and is left as an exercise.

Exercise 19: Complete the proof of the lemma by showing that $\underline{\theta} \leq \underline{\rho}$.

Exercise 20: Do exercise 16 on page 227 of your text, which asks you to use the preceding lemma to explain why the root test is stronger than the ratio test. In order to explain the "strictly" stronger part, you may want to do Exercise 14 on the same page.

If the root test is strictly stronger than the ratio test, one might well ask why we bother with the ratio test at all. The answer is that in many cases the ratio test is simpler to apply. Moreover, sometimes knowing the compatibility of the ratio test and the root test is itself useful for evaluating limits:

Exercise 21: Use Lemma 3 to show that that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Exercise 22: What do the ratio and root tests have to say about the convergence of p-series?