NOTES ON INFINITE SERIES III: SERIES WITH NON-NEGATIVE TERMS

PETE L. CLARK

REVISED: Tuesday, January 18, 2005

3. Series with non-negative terms

In this section we treat the theory of convergence of infinite series $\sum_n a_n$ under the supplementary hypothesis that $a_n \geq 0$ for all n. We describe this as a **series with non-negative terms**.

This hypothesis very much simplifies matters. For instance, we can give what seems like a more direct proof of the General Term Test when $a_n \geq 0$ for all n: if the general term did not converge to zero, then there would be, for some $\epsilon > 0$, infinitely many terms which are greater than or equal to ϵ. Adding up just these terms, the sum would grow arbitrarily large, so adding more positive terms, the entire series certainly diverges to ∞.

In general, we are aided by the following key observation:

If $a_n \geq 0$ for all n, then the sequence of partial sums $S_n = a_0 + \ldots + a_n$ is an increasing sequence.

Thus from the monotone sequence lemma we get the following criterion for the convergence of a series with non-negative terms:

Proposition 1. *Let $\sum_n a_n$ be an infinite series with $a_n \geq 0$ for all n. Then $\sum_n a_n$ is convergent if and only if the partial sums are bounded, i.e., if and only if there exists a number M such that for all n, $a_0 + \ldots + a_n \leq M$. Moreover the sum of the series is precisely the least upper bound of the sequence of partial sums.*

Because of this, we may express the convergence of such a series by the notation $\sum_n a_n < \infty$ and its divergence by $\sum_n a_n = \infty$.

Example 1: Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}(\frac{1}{2})^n$, with sequence of partial sums

$$T_n = 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} + \ldots + \frac{1}{n} \cdot (\frac{1}{2})^n.$$

Compare this with the geometric series.

$$S_n = \frac{1}{2} + \frac{1}{4} + \ldots + (\frac{1}{2})^n$$

Since $\frac{1}{n} \leq 1$ for all positive integers n, we clearly have $T_n \leq S_n$ for all n. What does this tell us about the convergence of the new series? It will converge if and only if $\{T_n\}$ is bounded, but we have for all n that $T_n \leq S_n \leq 1$, so not only does the sequence T_n have to converge, its sum must be at most $\lim_{n\to\infty} S_n = \sup S_n = 1$.

Example 2: Now consider the series $\sum_{n=1}^{\infty} \sqrt{n}$. We don't need to try to put $T_n := \sqrt{1} + \ldots + \sqrt{n}$ into closed form in order to see that the series diverges; indeed we clearly have $\sqrt{n} \geq 1$ for all n, so $T_n \geq 1 + \ldots + 1 = n$. Thus the sequence of partial sums is at least as large as n, so it is unbounded.[1]

These are two instances of an amazingly powerful test for convergence:

Theorem 2. *(The Comparison Test)*
Let $\sum_n a_n$, $\sum_n b_n$ be two series with non-negative terms.
Suppose that $a_n \leq b_n$ for all n.
a) If $\sum_n b_n < \infty$, then $\sum_n a_n < \infty$, and indeed $\sum_n a_n \leq \sum_n b_n$.
b) If $\sum_n a_n = \infty$, then $\sum_n b_n = \infty$.

There is absolutely nothing more going on in the general case than in the two examples we have already examined, but because we are going to consider many variations on this theme, let's record a proof for posterity. Namely, certainly $a_n \leq b_n$ for all n implies $S_n = a_1 + \ldots + a_n \leq T_n = b_1 + \ldots + b_n$ for all n. Thus $\sum_n a_n = \lim_n S_n \leq \lim_n T_n = \sum_n b_n$, where the limit is well-defined as an element of $\mathbb{R} \cup +\infty$. The conclusion follows from this.

Example 1 revisited: we know that $\sum_{n=1}^{\infty} \frac{1}{n}(\frac{1}{2})^n$ converges to some positive number which is less than one. In general, we must begin to live with the idea that we may not always be able to find the sum of the series, but it never hurts to try. Suppose we get our favorite computer algebra system to compute S_{100}, the one hundredth partial sum of the series. (I used the system `gp/pari` and the command `sum(n=1,100,1/n*(1/2)^n) + 0.` Something very similar will work in almost any package.) My computer gives me

$$\%5 = 0.6931471805599453094172321214.^{2}$$

Well, you don't have to be an arithmetic geometer to be reminded of $\log 2$, which my computer helpfully tells me is

$$\%6 = 0.6931471805599453094172321214.$$

This cannot be a coincidence. But there is a but: all we did was compute the sum of the first hundred terms – what reason do we have to believe that the %5 is the sum of the series accurate to 28 decimal places? Well, the comparison test helps us with this too! Indeed the error in cutting the series off after the 100th term is

$$E_{100} = \sum_{n=1}^{\infty} \frac{1}{n}(\frac{1}{2})^n - S_{100} = \sum_{n=101}^{\infty} \frac{1}{n}(\frac{1}{2})^n < \sum_{n=101}^{\infty} (\frac{1}{2})^n = (\frac{1}{2})^{100} \approx 10^{-30},$$

so indeed our estimate is correct to all 28 decimal places! Thus it seems overwhelmingly likely that the sum of the series is $\log(2)$. We'll prove this later in the course via a power series manipulation.

---

[1] Of course the divergence of this series also follows from the General Term Test; a more interesting example of a divergent series will come shortly.

[2] The %5 is just its weird way of outputting things.

There are many variations on the basic comparison test. First of all, to get convergence of series $\sum_n a_n$ with non-negative terms, it is enough if there exists a convergent series $\sum_n b_n$ such that $a_n \leq b_n$ not for all n but only for $n \geq N$. Moreover, by separating out the terms of the series with indices less than N we can still get an upper bound on the sum of the series. (And exactly the same holds in the other direction, i.e., in testing for divergence.)

Example 3: We consider the series $\sum_{n=0}^{\infty} \frac{1}{n!}$. Recall that $0! = 1$ and for any positive integer n, $n! = 1 \cdot 2 \cdot \ldots n$. In less compact notation, the series is

$$1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \ldots + \frac{1}{2 \cdot 3 \cdot \ldots \cdot n} + \ldots.$$

If we ignore the first two terms of the series, we get a pattern where we start with $\frac{1}{2}$ and get from one term to the next by multiplying by successively smaller numbers: $\frac{1}{3}$, $\frac{1}{4}$ and so on. Thus the sum of the series is less than the sum of the following series:

$$1 + 1 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots + (\frac{1}{2})^n + \ldots = 2 + \frac{1}{2} \cdot (\frac{1}{1 - \frac{1}{2}}) = 3.$$

In particular, the series is convergent, and its sum is somewhere between 2.5 and 3.

Exercise 10: Using a method similar to that of Example 1 revisited, compute the sum of the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ (i.e., Example 3) to at least 10 decimal place accuracy, and make a guess as to the sum. (Later in the course, we will prove your guess is correct – at least, we will if you make the right guess.)

We want now to give a version of the *limit comparison test*, which in many calculus courses plays a more prominent role than the "basic" comparison test. The version from calculus is that if two series $\sum_n a_n$, $\sum_n b_n$ have non-negative terms and the quantity $\alpha = \lim_{n \to \infty} \frac{a_n}{b_n}$ exists and is nonzero, then the series either both converge or both diverge.

One of the main things that it is appropriate to worry about in an analysis class rather than a calculus class is: what if the limit doesn't exist?[3] It is desirable to phrase as many results as possible so as not to require the existence of the limit. This is the advantage of the $\liminf$ and the $\limsup$ which always exist (at least, as extended real numbers). so we introduce the following notation:

Let (a_n), (b_n) be two sequences of non-negative real numbers, which we assume are strictly positive for all sufficiently large n. We write $a_n \ll b_n$ if $\limsup \frac{a_n}{b_n} < \infty$, i.e., the sequence $\frac{a_n}{b_n}$ is bounded. (Our convention is that we throw away the finitely many terms for which $b_n = 0$; removing any finite number of terms does not affect the boundedness.)

Exercise 11: There are two plausible definitions for the symbol $a_n \gg b_n$:
i) $\liminf \frac{a_n}{b_n} > 0$;

---

[3] Thus the entire business of limit *points* of sequences, for instance.

ii) $b_n \ll a_n$.
Show that the two definitions are the same.

Proposition 3. *(Limit comparison test)*
Let $\sum_n a_n$, $\sum_n b_n$ be two series, with terms that are positive for all sufficiently large n.
a) Suppose that $a_n \ll b_n$ and $\sum_n b_n < \infty$. Then $\sum_n a_n < \infty$.
b) Suppose that $a_n \ll b_n$ and $\sum_n a_n = \infty$. Then $\sum_n b_n = \infty$.
c) Suppose that $a_n \gg b_n$ and $\sum_n a_n < \infty$. Then $\sum_n b_n < \infty$.
d) Suppose that $a_n \gg b_n$ and $\sum_n b_n = \infty$. Then $\sum_n a_n = \infty$.

Proof: First note that c) is equivalent to a) using the fact that $a_n \gg b_n \iff b_n \ll a_n$; similarly d) is equivalent to b). Also b) is just the contrapositive of part a). Thus it suffices to prove part a). For this, the meaning of $a_n \ll b_n$ is that there exists a number $M > 0$ such that for all sufficiently large n, $a_n \leq Mb_n$. In testing for convergence it is permissible to throw out the finitely many exceptions, and then we are in an instance of the comparison test: since $\sum_n b_n < \infty$, $\sum_n Mb_n = M\sum_n b_n < \infty$, hence the series $\sum_n a_n$ converges by comparison.

Example 4: We will show that for any $p \geq 2$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. First, it is enough to show this for $p = 2$, since if $p > 2$, we have, for all n, $\frac{1}{n^p} < \frac{1}{n^2}$. Thus the $p > 2$ case will follow by comparison. Now, it turns out that the method of telescoping sums allows us to show the convergence of a sequence which is closely related to $\sum_{n=1}^{\infty} \frac{1}{n^2}$: namely, consider the identity $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n^2+n}$. This gives us that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1.$$

Unfortunately the comparison doesn't go the way we want: $\frac{1}{n^2} > \frac{1}{n^2+n}$ for all n. On the other hand, certainly the terms of these two sequences are of the same order of magnitude, and this allows us to use the limit comparison test: explicitly, since

$$\lim_{n\to\infty} \frac{\frac{1}{n^2+n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+n} = 1,$$

the series $\sum_n \frac{1}{n^2}$ converges if and only if $\sum_n \frac{1}{n^2+n}$ converges; in other words, it does indeed converge. Note that the comparison test now steps in to tell us that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} > \sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1.$$

I have to say that I feel that the limit comparison test is often just the comparison test for people who are too lazy to work with inequalities: for instance, in the last example, we could have avoided the limit comparison test just by observing that $\frac{1}{n^2} \leq \frac{2}{n^2+n}$. On the other hand, showing that certain inequalities hold "for sufficiently large n" is really what the limiting process is all about. The real drawback of the limit comparison test is that, unlike the comparison test, it does not in itself give us an estimate on the sum.

You might think that the comparison test is of limited usefulness until we have a sizable supply of known convergent or divergent series. In a sense this is certainly true. However, in a deeper sense, it is amazing how many ways we can get information by comparing (in some clever way) a series to a geometric series: Example 3 was just the beginning.

The next example is one to remember for the rest of your days:

Example 5: Consider $\sum_{n=1}^{\infty} \frac{1}{n}$, the **harmonic series**. Note well that $a_n \to 0$. Suppose we again take the computational approach. S_{100} is approximately 5.187. Do we recognize this? Not really. $S_{150} \approx 5.591$, and $S_{200} \approx 5.878$. Maybe the partial sums never exceed 6? Let's try a much larger one: $S_{1000} \approx 7.485$, so no. Since $S_{1050} \approx 7.534$, we get the idea that whatever the series is doing, it's doing it rather slowly, so let's instead start stepping up the partial sums multiplicatively:

$$S_{100} \approx 5.878.$$
$$S_{1000} \approx 7.4854.$$
$$S_{10000} \approx 9.788.$$
$$S_{100000} \approx 12.090,$$

and now there is a pattern – the difference $S_{10^{k+1}} - S_{10^k}$ appears to be approaching $2.30\ldots = \log_e(10)$. This is consistent with $S_n \approx \log(n)$, and if so, since $\log(n) \to \infty$, the series would diverge. I hope you recognize that the relation between $\frac{1}{n}$ and $\log(n)$ is one of a function and its antiderivative. After we discuss integration, we will come back to a lovely method for comparing series to integrals – **integral test** – which will show in fact that $\sum_{k=1}^{n} \frac{1}{k} \sim \log(n)$, where $f(n) \sim g(n)$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. But we don't have integration officially available yet[4], so we had better try something else. The following clever idea is due to Cauchy:

Consider the terms arranged thusly:

$$(1) + (\frac{1}{2} + \frac{1}{3}) + (\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}) + \ldots,$$

i.e., we group the terms in blocks of length 2^k. Now the idea is that the power of $\frac{1}{2}$ which begins each block is larger than any term in the previous block, so if we replaced every term in the present block by the first term in the next block, we would only decrease the sum of the series. But this sum is much easier to work with:

$$(\frac{1}{2}) + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \ldots) + \ldots =$$
$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots = \infty.$$

Thus by shaving a bit off of most of the terms we get a series summing to infinity; by comparison, the original, harmonic, series must also sum to infinity, i.e., is divergent. Notice that the fact that we must double the number of terms to increase the partial sum by a constant amount is consistent with the earlier computations and the predicted logarithmic growth of the series. (In fact doubling the number of terms adds approximately $\log 2$) to the partial sums of the original series, so we are making an underestimate after all.)

---

[4] I must confess that the desire *not* to have integration available at precisely this point was one of my motivations for covering series before integration.

There are several lessons to be taken from this important example:

1) The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, despite the fact that its general term approaches zero. Thus, the converse of the General Term Test is **false**.

2) Computations of partial sums can often be enlightening and suggest theoretical results. However, computations cannot suggest that a series converges or diverges *per se*; they can only suggest that a series converges or diverges *at a certain rate.* If a series diverges, but very slowly, we won't see it in the partial sums unless we chance or reason upon the correct "scale," i.e., the growth rate of the partial sums. Moreover, if the growth rate is much slower than logarithmic, we may not in practice be able to compute enough partial sums (my laptop thought for a bit before giving S_{100000} in the last example).

3) Cauchy's trick is much more widely applicable than just the divergence of the harmonic series:

Theorem 4. *(Cauchy condensation test) Let* $(a_n)_{n=1}^{\infty}$ *be a sequence of real numbers whose terms decrease monotonically to zero. Then the series* $\sum_{n=1}^{a}{}_{n}$ *converges if and only if the "condensed" series* $\sum_{k=1}^{\infty} 2^k a_{2^k}$ *converges.*

Proof: First, the convergence of the condensed series is certainly equivalent to the convergence of $\frac{1}{2}$ times the condensed series, or $\sum_{k=1}^{\infty} 2^{k-1} a_{2^k}$. With this slight adjustment, the condensed series is

$$a_2 + a_4 + a_4 + a_8 + a_8 + a_8 + a_8 + \ldots,$$

which, because of the monotonicity, is less than or equal to

$$a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots = \sum_{n=2}^{\infty} a_n.$$

Thus if the original series converges, so does the condensed series. On the other hand

$$\sum_{n=1}^{\infty} a_n = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + (a_8 + \ldots + a_{15}) + \ldots \leq$$

$$a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots = a_1 + \sum_{k=1}^{\infty} 2^{k-1} a_{2^k}.$$

If the condensed series converges, evidently so does the last displayed series, hence by comparison so does the original series. This completes the proof.

Exercise 12: Show that the hypothesis that a_n decrease monotonically to zero is necessary in Theorem 4.

Example 6: Fix a real number p, and consider the **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$. This is a generalization of the harmonic series, which we recover when $p = 1$. There are several cases to consider: if $p < 0$, then $\lim_{n \to \infty} n^{-p} = \infty$, so the divergence follows from the general term test. Similarly, if $p = 0$, $\frac{1}{n^p} = 1$, and 1 does not approach zero – the series diverges. Now assume $p > 0$. Then n^{-p} is a monotonically decreasing function (because if $f(x) = x^{-p}$, $f'(x) < 0$) and by Cauchy's condensation test we can cash in our p-series in exchange for the series $\sum_{k=1}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=1}^{\infty} (2^{(1-p)})^k$. This is a geometric series with $r = 2^{1-p}$, which is less than one if and only if $p > 1$. We conclude:

Proposition 5. *The p-series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *is convergent if and only if* $p > 1$.

Digression: Well, suppose that $p > 1$, so that the p-series converges. What is its sum? Let us write for brevity $\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p}$. It turns out that there are good methods available to evaluate $\zeta(p)$ when p is an **even integer**. For instance, one has the remarkable result

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(To test it out, compute say S_{100}, multiply by 6 and take the square root. You will get what is unmistakably an approximation to π.) A proof of this appears as an exercise on pages 289-290 of your text. The suggested proof is rather elaborate; better methods are available using either Fourier analysis or basic complex analysis – you'll see someday. In fact, using either method, one can show that whenever $p = 2k$ is an even integer, $\zeta(2k)$ is of the form $\frac{a_k}{b_k} \cdot \pi^{2k}$, i.e., is a rational multiple of a power of π. One can say more about the numerator and denominator; they are obtained from **Bernouilli numbers**, which are coefficients of a certain power series.

Well, what about when $p \geq 3$ is an odd integer? Amazingly, although the results of the previous paragraph have been known for more than 200 years, absolutely nothing was known in this case until 1979, when R. Apéry proved that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational. (Since π is transcendental – that is, it is not the solution of a polynomial equation with integer coefficients – it follows that all the $\zeta(2k)$'s are transcendental, hence irrational.) Apéry's proof does not in any sense *identify* $\zeta(3)$, nor does it prove its transcendence, although most people believe it to be so. (Since the algebraic numbers form a countable set – Exercise 23 of §1.4 – and the reals are uncountable, the transcendental numbers are also uncountable: in other words there are a lot more of them. Thus, guessing that a "random" real number is transcendental is a much better bet than guessing it is algebraic.) Even more recently (2000), Tanguy Rivoal proved that the sequence $\zeta(3), \zeta(5), \zeta(7), \ldots$ contains infinitely many irrational numbers, but the proof does not even specify which they might be! As far as I know, it is completely open to show e.g. that $\zeta(5)$ is irrational.

The moral of the story: determining the sum of an infinite series can be very hard!

On the other hand, it is a much more reasonable proposition to give *approximate* values of the sums of the p-series:

Exercise 13: Use Cauchy's condensation test to determine the convergence or divergence of the following series (make sure to show that the terms are monotonically decreasing to zero):
a) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$
b) $\sum_{n=2}^{\infty} \frac{1}{n (\log n)^{\alpha}}$, where α is a real number. (Your answer may depend upon α.)

c) $\sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log\log n)}$.
d)* Attempt to experimentally verify the result of part c) by considering various partial sums. Let me know if you succeed in getting a partial sum exceeding 4. (**Note**: I actually wanted you to find a partial sum exceeding 6. It's no problem to find a partial sum exceeding 4: try it!)

Exercise 14: Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function. Choose any positive integer N which is larger than all of the roots of $Q(x)$, and consider the series $\sum_{n=N}^{\infty} \frac{p(n)}{q(n)}$.
a) (just checking) Why have we started the summation at N?
b) Suppose that all the coefficients of the polynomials $p(x)$ and $q(x)$ are non-negative. Show that the series converges if and only if the degree of the denominator is at least two more than the degree of the numerator.

Exercise 15: Let $\sum_n a_n$ be a series with non-negative terms. Show that neither the convergence nor the sum of the series is altered by any addition of parentheses.

Exercise 16: It seems that many students were confused by the notation $a_n \ll b_n$ to mean that $\sup \frac{a_n}{b_n}$ is bounded. In general, there is a certain amount of notation used for comparing rates of growth of functions. Assume that $f,\ g : [0,\infty) \rightarrow (0,\infty)$ are two functions. We write:

- $f = O(g)$, if $\sup_{x\in[0,\infty)} \frac{f(x)}{g(x)} < \infty$.
- $f = o(g)$, if $\lim_{x\rightarrow\infty} \frac{f(x)}{g(x)} = 0$.
- $f \sim g$, if $\lim_{x\rightarrow\infty} \frac{f(x)}{g(x)} = 1$.

The first symbol is pronounced, "f equals oh of g"; the second is pronounced, "f equals little oh of g"; the third is pronounced, "f is asymptotic to g."
a) Show that $2 + \sin x = O(1)$ and $\frac{1}{x+1} = o(1)$. (This is just to make sure you understand the definitions.)[5]
b) Let f and g be polynomials. Find necessary and sufficient conditions for $f = O(g)$, for $f = o(g)$ and for $f \sim g$.
c) Show that to any positive sequence $(a_n)_{n=0}^{\infty}$ we can associate a function $f_a : [0,\infty) \rightarrow (0,\infty)$ in such a way so that $(a_n) \ll (b_n) \iff f_a = O(f_b)$. (Hint: we did not say that f_a has to be continuous.)
d) Is it the case that if $f = o(g)$, then $f = O(g)$? How about if we assume that f and g are continuous?
e) Give an example of functions f and g such that we have *neither* $f = O(g)$ nor $g = O(f)$.
f) If $f = O(g)$ and $g = O(f)$ must it be the case that $f \sim g$?
g)* Find links to papers on the web where these symbols are used.

---

[5] In an earlier draft it was $\sin x$ instead of $2+\sin x$. But according to the definition both f and g must be positive, so this has been changed. In practice, it is only necessary that g be positive – since it's the one we're dividing by, and for an arbitrary function f, the symbol $f = O(g)$ means that $|\frac{f}{g}|$ is bounded above.