NOTES ON INFINITE SERIES: II

PETE L. CLARK

2. Basic observations about series

In the next few lectures we will see the beginning and a bit of the middle (there is no end!) of the theory of infinite series. It is a strange fact that the topic of infinite series is at the same time covered in many calculus classes and one that has profoundly perplexed some of the greatest minds in history, up to and including the present day.

Let us begin by taking the concept of series literally as a special case of infinite sequences. In fact, it is not a special case at all:

Exercise 2: Show that any infinite sequence $(S_n)_{n=1}^\infty$ is the sequence of partial sums of some other sequence $(a_n)_{n=1}^\infty$. (Hint: take $a_1 = S_1$, and then using the equations $S_n = a_1 + \ldots + a_n$, solve for the a_i's. You are looking for a very simple formula in the end.)

It must be admitted that this observation is of basically no use, although it gives us plenty of (doctored) examples of series whose sum we can compute explicitly. As we will see later, this is not the typical state of affairs.

However, just by interpreting a series as a sequence we get immediate proofs of certain desired algebraic properties.

Proposition 1. *Let $\sum_n a_n$, $\sum_n b_n$ be two infinite series, and α any real number.*
a) If $\sum_n a_n = S_a$ and $\sum_n b_n = S_b$ are both convergent series, then the series with general term $a_n + b_n$ is also convergent, and indeed we have $\sum_n a_n + b_n = S_a + S_b$.
b) If $\sum_n a_n = S$ is convergent, then so is the series with general term αa_n, and indeed $\sum_n \alpha a_n = \alpha S$.
c) If $\sum_n a_n$ is convergent and $\sum_n b_n$ is divergent, then $\sum_n a_n + b_n$ is divergent.

Proof: We prove part a) only, leaving the others as exercises. From the definition of convergence, we are looking at the quantity $|S_a + S_b - (a_1 + \ldots + a_n + b_1 + \ldots + b_n)|$. By the triangle inequality this is at most

$$|(S_a - (a_1 + \ldots + a_n))| + |(S_b - (b_1 + \ldots + b_n))|.$$

Fix any $\epsilon > 0$. From the convergence of the two original series there exists N such that when $n \geq N$, both terms in the above expression are at most $\frac{\epsilon}{2}$. Thus it follows that for all sufficiently large n,

$$|S_a + S_b - (a_1 + \ldots + a_n + b_1 + \ldots + b_n)| < \epsilon,$$

completing the proof of part a).

(We also discussed parts b) and c) in class. You should make sure you know their proofs, but we will not formally assign them as homework.)

In order to understand the convergence or divergence of an infinite series, it is permissible to neglect any finite number of terms. More formally:

Exercise 3: Let $\sum_{n=0}^{\infty}$ be an infinite series and N any positive integer.
a) $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges.
b) Assuming both series are convergent, what must occur for the truncated series to have the same sum as the original series?

Exercise 4: Many ordinary citizens are uncomfortable with the equality

$$0.999\ldots = 1.$$

Interpret this statement in terms of infinite series, and prove that it is correct.

2.1. **Algebraic properties of infinite series.** Since we write infinite series as $a_1 + a_2 + \ldots + a_n + \ldots$, it is natural to wonder whether the familiar algebraic properties of addition carry over to the context of infinite sums. For instance, one can ask whether the commutative law holds: if we were to give the terms of the series in a different order, could the sum (or even the convergence/divergence) change? This question has a surprisingly complicated answer: yes, in general, but no if we place an additional requirement on the convergence of the series. These are rather subtle matters (giving an example of the failure of the commutative law is not so easy), and we will come back to them.

Then again there is the associative law: can we at least insert and delete parentheses as we like? For instance, is it true that the series

$$a_1 + a_2 + a_3 + a_4 + \ldots + a_{2n-1} + a_{2n} + \ldots$$

is convergent if and only if the series

$$(a_1 + a_2) + (a_3 + a_4) + \ldots + (a_{2n-1} + a_{2n}) + \ldots$$

is convergent? Here it is easy to see that the answer is no: take the geometric series with $r = -1$ and starting with $n = 0$, so

$$1 - 1 + 1 - 1 + \ldots.$$

The series of partial sums is $1,\ 0,\ 1,\ 0, \ldots$; this is not convergent. However, if we group the terms together as above, we get

$$(1 - 1) + (1 - 1) + \ldots + (1 - 1) + \ldots = 0,$$

so this regrouped series is convergent. Maybe we're just being too fastidious in calling this series divergent? Indeed we are not, because we could also group the terms as

$$1 + (-1 + 1) + (-1 + 1) + \ldots + (-1 + 1) + \ldots = 1.$$

In fact Leonhard Euler (one of the great analysts of all time) believed that "in some sense" the sum of the series is $\frac{1}{2}$.[1]

It is not difficult to see that adding parentheses to a series can only help it to converge. Indeed, suppose we add parentheses in blocks of length $n_1, n_2, \ldots$. E.g., the case of no parentheses at all is $n_1 = n_2 = \ldots = 1$, and the cases we considered above were, first, $n_1 = n_2 = \ldots = 2$ and second, $n_1 = 1$, $n_2 = n_3 = \ldots = 2$. Then what we are really doing is considering not every partial sum, but only some of them, namely the sequence of partial sums we get is

$$S_{n_1}, S_{n_1+n_2}, S_{n_1+n_2+n_3}, \ldots.$$

What this means is that in adding parentheses we are passing from to a *subsequence* of the sequence of partial sums. But recall the theory of this: if a sequence converges, every subsequence converges.

Following our experience in Wednesday's class, we include a short dicussion on limit points of sequences.

Let $(a_n)_{n=1}^{\infty}$ be a real sequence and L a real number. We say that L is a **limit point**[2] of the sequence if, for every $\epsilon > 0$, the set of n such that $|a_n - L| < \epsilon$ is infinite. We say that $+\infty$ is a limit point if the sequence is unbounded above, i.e., if for every real number M there exists n such that $a_n > M$. (It is equivalent to require infinitely many such n; why?) We say that $-\infty$ is a limit point if the sequence is unbounded below. Occasionally we may refer to the set of all limit points of a sequence as the **limit set**. Note that this is a set of *extended* real numbers, i.e., a subset of $\mathbb{R} \cup \{\pm\infty\}$.

The terminology of limit points is convenient and useful. The following exercise should give you the hang of it:

Exercise 5: Let (a_n) be any real sequence.
a) Show that (a_n) is convergent if and only if it has a unique, finite limit point.
b) Show that the lim inf and lim sup of a sequence are limit points of the sequence. Indeed, the lim inf is the smallest element of the limit set and the lim sup is the largest element of the limit set.
c) Show that L is a limit point if and only if there exists a subsequence (a_{n_k}) such that $a_{n_k} \to L$.
d) Show that every sequence has at least one limit point. (Hint: this boils down to an important result from last semester.)
e) For any finite set $S \subset \mathbb{R}$, construct a sequence whose limit set is S. Same question if $S \subset \mathbb{R} \cup \{\pm\infty\}$.
*f) There are sequences whose limit set is much more complicated in structure.

[1] This is a bit unfair. As we may see later there are other "summation processes" which can, in some cases, assign a well-defined number to series that our basic definition regards as divergent, and in (any of) these alternate theories the sum of this series really is $\frac{1}{2}$! The definition we have given for the sum of a series is due to Cauchy and Weierstrass, who lived almost a century later. Of course Euler would have agreed that the series diverges in our sense had he been presented with the definition.

[2] Other terms include **accumulation point** and **cluster point**. My officemate Gil Alon tells me that in Israel, they say (the translation into Hebrew of) **partial limit**.

Prove that if a set $S \subset \mathbb{R} \cup \{\pm\infty\}$ is the limit set of a sequence, it must satisfy the following restrictions:
i) If a real number α is *not* in S, then there exists some open interval I containing α such that $I \cap S = \emptyset$.
ii) If S is bounded above, it does not contain $+\infty$; if it is bounded below, it does not contain $-\infty$.
In fact it can be shown that any set S of extended real numbers satisfying these two properties is the limit set of a real sequence.
g) Let (a_n) be any sequence in which each rational number appears exactly once. Find its limit set.

Exercise 6: Let $\sum_n a_n$ be an infinite series.
a) Show that there exists a way of adding parentheses so that the series converges to S if and only if S is a limit point of the sequence of partial sums. In particular, show that the only numbers obtainable by adding parentheses to the series $\sum_{n=0}^{\infty}(-1)^n$ are 0 and 1.
b) Let $\sum_n a_n$ be any series whose sequence of partial sums is bounded. Show that, by adding parentheses suitably, the series becomes convergent. (Hint: This question is really testing whether you remember a key theorem from the first semester.)

Nevertheless we can hope to remove parentheses, not in general, but under suitable additional hypotheses. (Many aspects of the theory of infinite series are like this.) A case where the parentheses can be removed is given in Exercise 8.

Finally, let us consider the question of products of series: if $\sum_n a_n$ and $\sum_n b_n$ are two infinite series, is there are series $\sum_n c_n$ giving meaning to the equation

$$(a_0 + a_1 + \ldots + a_i + \ldots)(b_0 + b_1 + \ldots + b_j + \ldots) = (c_0 + c_1 + \ldots + c_n + \ldots)?$$

Expanding out the product formally, we see that the product should contain terms of the form $a_i b_j$ for all $i, j \geq 0$. What is not clear is what order the terms should appear in – in other words, we need to resolve our problems with commutativity in order for the product to become well-defined. In fact we will see later that there is an ordering and grouping of the terms which has especially nice properties. Namely, we collect all terms $a_i b_j$ such that $i+j = n$, and define $c_n := a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$. This is the so-called **Cauchy product** of the two series. Later we will show that, under an additional hypothesis, if $\sum_n a_n = A$ and $\sum_n b_n = B$ are both convergent, then $\sum_n c_n$ is convergent and especially $C = AB$.

At the moment we content ourselves with the observation that the process which associates to two series $\sum_n a_n$ and $\sum_n b_n$ the series $\sum_n a_n b_n$ is *not* the desired product of the two series: even when all three series converge, there is no reason to have $\sum_n a_n b_n = (\sum_n a_n)(\sum_n b_n)$.

Exercise 7: Find an example of convergent series $\sum_n a_n = A$, $\sum_n b_n = B$ and $\sum_n a_n b_n = C$ such that $C \neq AB$.

Exercise 8*: Let $\sum_n a_n$ be a series such that $a_n \to 0$. Suppose that we insert parentheses in blocks of lengths $n_1, n_2, \ldots$ with the property that the lengths of the blocks is bounded: $n_i \leq N$ for some n. Show that the parentheses can be removed, in the sense that the convergence of the parenthesized series implies the convergence of the original series, and the sums are the same.

2.2. **Cauchy sequences and tails.** Recall that a sequence (x_n) of real numbers converges if and only if it is Cauchy, i.e., if for every $\epsilon > 0$, there exists a positive integer N such that $m, n \geq N$ implies $|x_n - x_m| < \epsilon$. One of the merits of the Cauchy condition is that this allows us to express the convergence of a sequence without explicitly mentioning what it converges to.

Applying the Cauchy condition to a sequence of partial sums $S_n = a_1 + \ldots + a_n$, we get convergence is equivalent to: for every $\epsilon > 0$, there exists a number N such that for all natural numbers k, $|\sum_{n=N}^{N+k} a_n| < \epsilon$. Since this must hold for all k, we have $\limsup_k |\sum_{n=N}^{N+k} a_n| \leq \epsilon$. Note that if this sequence converges, it is equal to $|\sum_{n=N}^{\infty} a_n|$.

Thus, we will allow ourselves the slight imprecision of the following notation: $|\sum_{n=N}^{\infty} a_n| \leq \epsilon$ to mean that the $\limsup$ is at most ϵ. Informally, speaking, one refers to a series of the form $\sum_{n=N}^{\infty} a_n$ for $N \gg 0$ a **tail** of the original series. We get then the following criterion for convergence:

Proposition 2. *A series $\sum_{n=0}^{\infty}$ converges if and only if for any $\epsilon > 0$, for all sufficiently large N, $|\sum_{n=N}^{\infty} a_n| < \epsilon$.*

Roughly speaking, the result says that a series converges if and only if its sequence of tails approaches zero. We have the following immediate consequence:

Corollary 3. *(The General Term Test) Let $\sum_n a_n$ be a convergent infinite series. Then $\lim_n a_n \to 0$.*

Applying the contrapositive, we get the more commonly used form: if the general term of the series *does not* tend to zero, then the series must diverge.

Because we passed from the entire tail of a series to a single term, one should not expect the condition $a_n \to 0$ to be *sufficient* for convergence of a series. Very soon we will see examples of divergent series for which $a_n \to 0$.

Exercise 9: Let $\frac{P(x)}{Q(x)}$ be a rational function, i.e., a quotient of polynomials with real coefficients. Assume that $Q(x) \neq 0$ for any positive integer x. Suppose that the degree of P is at least as large as the degree of Q. Show that the series $\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$ is divergent.