NOTES ON INFINITE SERIES: I

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1. Introducing infinite series

Mankind has had a fascination with, but also a suspicion of, infinite processes for well over two thousand years. Historically, the first kind of infinite process that received detailed consideration was the idea of adding infinitely many things together; or, to put a slightly different emphasis on the same idea, to divide the whole into infinitely many parts.

The idea that any sort of infinite process can lead to a finite answer has been deeply unsettling to philosophers going back at least to Zeno of Elea. Zeno believed that such a "convergent infinite process" was absurd; since he had a penetrating enough eye to see convergent infinite processes in a variety of everyday phenomena, he ended up at the lively conclusion that many everyday phenomena are in fact absurd.

We will get the flavor of his ideas by considering just one paradox, the arrow paradox. Suppose an arrow is launched at a target one stadium away. Can the arrow hit the target? Call this event E. Before it does, it must arrive halfway at the target: call this event E_1. But before it does *that* it must arrive halfway to the halfway point; call this event E_2. Clearly we can continue in this way, getting infinitely many events $E_1, E_2, \ldots$ all of which must occur before the event E. That infinitely many things can happen before some predetermined thing Zeno regarded as absurd, and he concluded that the arrow never hits its target. Similarly, it is easy to deduce that all motion was impossible.[1]

Nowadays we find no mathematical contradiction in the infinitely many events occurring before E. Indeed, assuming for simplicity that the arrow takes one second to hit its target and (rather unrealistically) travels at uniform velocity, we know exactly when these events E_i take place: E_1 takes place after $\frac{1}{2}$ seconds, E_2 takes place after $\frac{1}{4}$ second, and in general E_n takes places after $\frac{1}{2^n}$ seconds. Nevertheless there is something interesting going on here: we have divided the total time of the trip up into infinitely any parts, and the inescapable conclusion seems to be

$$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} + \ldots = 1.$$

[1]This is such a bizarre conclusion that one must ask whether Zeno actually believed it, or whether it was posed as a puzzle to be solved. In fact this ambiguity is inherent in the meaning of term "paradox": it can mean either a correct piece of reasoning that nevertheless arrives at a contradiction, or (and this is its more modern meaning) reasoning leading to an *apparent* contradiction, which needs to be resolved through closer study. My understanding is that Zeno intended the former meaning. One wonders whether he got around much.

But now we have, if not a paradox, then at least a problem: what meaning can be given to the left hand side of the equation? Obviously we are not completely confident in our ability to add infinitely many things together and get a finite number: the expression

$$1 + 1 + \ldots + 1 + \ldots$$

represents an infinite sequence of events, each lasting one second. Surely the aggregate of these events takes forever.

This leads us to the mathematical definition of a series of numbers, and its sum. Namely, if $a_1, \ldots, a_2, \ldots$ are real numbers and S is a real number, we will give meaning to the equation

$$a_1 + a_2 + \ldots + a_n + \ldots = S.$$

Namely, we don't try to add everything together at once. Instead, we form a sequence S_n whose terms represent adding up the first n numbers: namely $S_n := a_1 + \ldots + a_n = \sum_{k=1}^n a_k$. This special kind of sequence (S_n) which is itself associated to a sequence (a_k) by adding more and more terms is called a **sequence of partial sums**. We now say that the infinite series $a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^\infty a_n$ **converges** to S – or has **sum** S, if $\lim_{n\to\infty} S_n = S$ in the usual sense of limits of sequences. A series which does not converge to any real number S is said to **diverge**. Just as with sequences, we have symbols describing certain types of divergence: $\sum_{n=1}^\infty a_n = +\infty$ means: for any real number M, there exists a natural number n such that $n \geq N$ implies $a_1 + \ldots + a_n \geq M$. A similar definition applies to $\sum_{i=1}^\infty a_n = -\infty$.

We refer to the sequence of partial sums ($S_n = a_1 + \ldots a_n$) as the **infinite series** whose **terms** are the sequence a_n. (Thus there are two different sequences being considered, and one must be careful to distinguish between them.) Actually, one rarely writes the series as (S_n); instead it is common to write $\sum_{n=1}^\infty a_n$ for both the series itself and for its limit, or sum. This is, strictly speaking, wrong: e.g. for the example above we can say either "The first term in $\sum_{i=1}^\infty \frac{1}{2^i}$ is $\frac{1}{2}$" or" $\sum_{i=1}^\infty \frac{1}{2^i} = 1$", but we cannot say "The first term of 1 is $\frac{1}{2}$"! But this sort of conflation is consecrated by time (because it turns out to be much more convenient than the precise language), and in context there is seldom any doubt whether one is referring to the series itself or to its sum.

Let us come back to our basic examples.

Example 1: If $a_n = 1$ for all n, we are getting the series $1 + 1 + \ldots$. Here the partial sums are $S_n = n$, so $\sum_{n=1}^\infty 1 = \lim_{n\to\infty} n = \infty$. Thus indeed this series diverges.

Example 2: If $a_n = \frac{1}{2^n}$ for all n, then

$$S_n = \frac{1}{2} + (\frac{1}{2})^2 + \ldots + \frac{1}{2^n}.$$

There is a standard trick for evaluating this sum in closed form. Namely, multiplying the previous equation by $\frac{1}{2}$ and subtracting, we get $\frac{1}{2}S_n - S_n = \frac{1}{2^{n+1}} - \frac{1}{2}$, or

$$S_n = \frac{\frac{1}{2^{n+1}} - \frac{1}{2}}{\frac{1}{2} - 1}.$$

But now we can take $\lim_{n\to\infty} S_n$, using the fact that $(\frac{1}{2})^n \to 0$ since $|\frac{1}{2}| < 1$, we find that $S_n \to 1$. Thus, indeed our above geometric reasoning was correct: $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

It is not necessary for the indexing of the terms of a series to start with 1. Almost as common is to start with zero, i.e., $\sum_{n=0}^{\infty} a_n$ is the sequence of partial sums $a_0,\ a_0 + a_1, \ldots$. And indeed if N is any integer, the series $\sum_{n=N}^{\infty} a_n$ has the obvious meaning: $a_N,\ a_N + a_{N+1}, \ldots$.[2]

Sometimes it is convenient to use the vaguer notation $\sum_n a_n$ for a series, where the index set is not specified. (This is especially useful for theoretical results where the specific indexing is immaterial.)

There is one class of series which is more important than all others put together, namely the **geometric series**. Here, we fix any real number r, and consider the series starting at zero whose general term is $a_n = r^n$, thus $\sum_{n=0}^{\infty} r^n$. Since you must be intimately familiar with geometric series, I leave the derivations of the important properties as an exercise:

Exercise 1:
a) Identify the geometric series when $r = 1$. Is it convergent?
b) Take $r = -1$, and find an explicit expression for the partial sums S_n. (Hint: the answer will depend upon the parity of n.) Is the geometric series with $r = -1$ convergent?
c) For any r different from 1, use the above "multiplication trick" (or any other method you devise) to show that

$$S_n = \frac{1 - r^{n+1}}{1 - r}.$$

Check in particular that this formula agrees with your formula in part b) when $r = -1$.
d) By taking limits in part c), conclude that the geometric series is convergent if and only if $|r| < 1$, in which case its sum is $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.
e) Note that "Zeno's" geometric series is a little different from the series we have considered in this exercise, since it starts with $\frac{1}{2}^1$ rather than $\frac{1}{2}^0 = 1$. Deduce that $\sum_{n=1}^{\infty} (\frac{1}{2})^n = 1$ as a consequence of the formula of part d).
f) More generally, if N is any positive integer and $|r| < 1$, derive a formula for the sum of the series $\sum_{n=N}^{\infty} r^n$. This more general series is still called a geometric series.

[2]Actually it is sometimes useful to consider series of the form $\sum_{n=-\infty}^{\infty} a_n$, but the definition here requires a little more care. We will come back to this point later.