THE RIEMANN-DARBOUX INTEGRAL II

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1. Addendum: Examples of the Fundamental Theorem

(This material should have occurred after the statement and proof of the fundamental theorem of calculus for the Riemann integral.)

Let a and b be positive integers and consider the function $f = f_{a,b} : [0,1] \to \mathbb{R}$ given by

$$f(x) = x^a \sin(1/x^b), x \neq 0.$$
$$f(0) = 0.$$

Evidently f is perfectly well-behaved except possibly at $x = 0$. At $x = 0$ it is at least continuous, since $a > 0$: we have $|x^a \sin(1/x^b)| \leq x^a \to 0$ as $x \to 0$, therefore $\lim_{x\to 0} f = 0 = f(0)$. To check for differentiability at $x = 0$, we resort to the definition of the derivative:

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} h^{a-1} \sin(1/h^b).$$

If $a = 1$ then the function is just $\sin(1/h^b)$, which oscillates more and more rapidly as $h \to 0$, so the limit does not exist.[1] If $a \geq 2$ then the limit exists as before. Therefore we get that $f_{a,b}$ is differentiable at 0, with $f'(0) = 0$ if and only if $a \geq 2$.

Now I can't resist a digression. As a graduate student I taught a section of a first semester calculus class in which the course head was simultaneously one of the greatest living analysts and an example of someone who, not having been educated in North America, had completely unrealistic ideas about what first-year American students (not future math majors!) were capable of. One of his proposed problems for the first midterm exam had been voted down as too difficult, so at the last minute he substituted the following problem: consider $f_{a,2}$ (he wrote it out). Find the smallest integer a for which the function is a) continuous, b) differentiable, c) twice-differentiable. We have just seen that the answers to part a) and b) – which are independent of the value of b – are $a = 1$ and $a = 2$ respectively. I leave you to think about the answer to part c). If you don't get it, you're in good company: not one of the more than 200 students writing the exam got it right[2]; most mathematics graduate students are not able to solve it on demand; and I have sprung this problem on at least one leading mathematician (a winner of the Fields Medal, the "mathematical equivalent of the Nobel Prize"), who became quite confused.

[1] One can make sense of limit points of functions as $x \to c$ as for sequences, and in this case every real number x with $-1 \leq x \leq 1$ is a limit point as $h \to 0$: i.e., the function assumes all these values on every open interval containing zero.

[2] Actually there was exactly one student who wrote down the correct numerical value but with no justification.

Anyway, assume $a \geq 2$, so that f' exists. What concerns us here is whether the fundamental theorem of calculus holds for f': i.e., is the Riemann integral $\int_a^b f'$ equal to $f(b) - f(a)$? By the second part of FTC we know that the answer is "yes" as long as f' is Riemann-integrable (and of course the answer is "no" otherwise, since we cannot take the integral at all). For this let's look at the formula for $f'(x)$ at a nonzero point: it is

$$f'(x) = ax^{a-1}\sin(x^{-b}) + x^a \cos(x^{-b}) \cdot (-bx^{-b-1}) = ax^{a-1}\sin(x^{-b}) - bx^{a-b-1}\cos(x^{-b}).$$

So f' is again continuous except possibly at $x = 0$. Moreover, if $a - b - 1 > 0$ – i.e., if $a > b + 1$, then $\lim_{x \to 0} f' = 0$, so the derivative is even continuous at $x = 0$. If $a = b + 1$ then the derivative is not continuous at zero but at least is bounded as a function on $[0, 1]$. If $a < b + 1$, then x^{a-b-1} blows up as $x \to 0$, so the derivative is unbounded as $x \to 0$.

Since unbounded functions cannot be Riemann-integrable, we conclude that f' is not Riemann-integrable when $a < b + 1$; in particular, $f_{2,2}$ gives an example of a differentiable function whose derivative is not Riemann integrable. Of course continuous functions are Riemann-integrable, so e.g. the derivative of $f_{4,2}$ is Riemann-integrable. On the other hand, $f'_{a,b+1}$ is a bounded function which is continuous except at $x = 0$, so by Corollary 13 from the first integration handout is Riemann-integrable. Thus e.g. $f_{2,1}$ gives an example of a function which is the Riemann integral of its derivative despite having a discontinuous derivative. Thus the Riemann-integrability of discontinuous derivatives is a very subtle issue.

2. The Darboux Integral

Our goal is to enlarge our collection of known integrable functions in at least two ways: we want to show that all monotone functions are integrable and to show that products of integrable functions are integrable. This turns out to be quite difficult from the definition of the Riemann integral. Instead, we start again with a simpler abstract integral, the Darboux Integral, which we can show shares all the nice propeties of the Riemann integral and for which it is easy to prove that monotone functions are integrable. (The proof that products of Darboux-integrable functions are integrable will still require some work, but it would be hopeless to show this directly for the Riemann integral.) After developing fully this "rival" integration theory, we will show that the Riemann and Darboux integrals are "the same" in the sense that the class of Riemann-integrable functions is exactly the same as the class of Darboux-integrable functions and for such a function f, $R\int_a^b f = D\int_a^b f$.

In fact the definition of the Darboux integral is familiar in the sense that we all but introduced it in order to show that continuous functions are Riemann-integrable. Namely, it is defined in terms of upper and lower sums. Let $f : [a, b] \to \mathbb{R}$ be any function and P be any partition of $[a, b]$. For $1 \leq i \leq n$, let $f_i^{\sup}$ denote the supremum of f restricted to the subinterval $[x_{i-1}, x_i]$: note that this could be $+\infty$. Similarly, let $f_i^{\inf}$ denote the infiumum of f restricted to the subinterval $[x_{i-1}, x_i]$;

this could be $-\infty$. With these provisos, we define

$$U(f,P)=\sum_{i=1}^{n} f_i^{\sup}(x_i-x_{i-1})\in(-\infty,\infty]$$

$$L(f,P)=\sum_{i=1}^{n} f_i^{\inf}(x_i-x_{i-1})\in[-\infty,\infty).$$

Note that we always have $L(f,P)\le U(f,P)$. As we will very often be interested in the difference between these two quantities, we introduce the notation

$$\omega(f,P)=U(f,P)-L(f,P)\in[0,\infty].$$

Now we define the **lower integral** $\underline{\int}_a^b f$ as the supremum of $L(f,P)$ as P ranges over all partitions of $[a,b]$ and the **upper integral** $\overline{\int}_a^b f$ as the infimum of $U(f,P)$ as P ranges over all partitions of $[a,b]$. It follows that

$$\underline{\int}_a^b f\le\overline{\int}_a^b f;$$

we leave it to you to convince yourself of this. Note that the upper integral could be $+\infty$ and the lower integral could be $-\infty$.

Exercise 50: Let $f:[a,b]\to\mathbb{R}$ be any function. Show that the following conditions are equivalent:
(i) f is bounded above.
(ii) For all partitions P of $[a,b]$, $U(f,P)<+\infty$.
(iii) $\overline{\int}_a^b f<+\infty$.
What is the analogous statement for lower sums, lower integrals, ...?

Definition: We say that f is **Darboux-integrable** if $\underline{\int}_a^b f=\overline{\int}_a^b f$, and we denote this common value by $D\int_a^b f$.

Exercise 51: Show that any Darboux-integrable function is bounded.

We now derive an equivalent condition on Darboux-integrability which is the point of the usefulness of the Darboux-integral: it can be signficantly easier to check than the condition for Riemann-integrability. The proof will require the following result, which in fact we used already in the proof that continuous functions are Riemann-integrable, so we owe it to ourselves to write out a careful proof.

Exercise 52: Let $f:[a,b]\to\mathbb{R}$ and $P\subset Q$ be two partitions of $[a,b]$ such that Q refines P. Show that $U(f,Q)\le U(f,P)$ and $L(f,Q)\ge L(f,P)$.

Proposition 1. *$f:[a,b]\to\mathbb{R}$ is Darboux-integrable if and only if: for all $\epsilon>0$, there exists a partition P of $[a,b]$ such that $\omega(f,P)=U(f,P)-L(f,P)<\epsilon$.*

Proof: We will first show the "if" implication: by assumption, for any $\epsilon>0$ there exists a partition P such that $U(f,P)-L(f,P)<\epsilon$. Since $\overline{\int}_a^b f\le U(f,P)$ and

$\underline{\int}_a^b \geq L(f,P)$ (so $-\underline{\int}_a^b \leq -L(f,P)$), we have

$$\overline{\int}_a^b f - \underline{\int}_a^b f \leq U(f,P) - L(f,P) < \epsilon.$$

Since this holds for every $\epsilon > 0$, we must have $\overline{\int}_a^b f = \underline{\int}_a^b f$.
Now assume that f is Darboux-integrable. For any $\epsilon > 0$, by the definition of the upper integral as the infimum over upper sums, we can chooose one partition P_1 such that $U(f,P) < \overline{\int}_a^b f + \frac{\epsilon}{2}$; similarly, we can choose another partition P_2 such that $L(f,P) > \underline{\int}_a^b f - \frac{\epsilon}{2}$. If P_1 and P_2 were the same partition, we'd be done. The slight annoyance that they are not necessarily the same is easily fixed by taking a common refinement: let $P = P_1 \cup P_2$, so that $P_1 \subset P$ and $P_2 \subset P$. By Exercise 52 we get that

$$\omega(f,P) = U(f,P) - L(f,P) \leq U(f,P_1) - L(f,P_2) < \overline{\int}_a^b f + \frac{\epsilon}{2} - \left(\underline{\int}_a^b f - \frac{\epsilon}{2}\right).$$

But the definition of Darboux integrability is that the lower and upper integrals are equal, so the last term in the above expression is just ϵ, so we found a partition P with $\omega(f,P) < \epsilon$, which was to be shown.

Exercise 53: Show that the Darboux-integral on the collection of Darboux-integrable functions is an abstract intgral in the sense of the first integration handout. Note that we have already seen that Darboux-integrable functions are bounded. Thus it remains to show:
a) $f(x) \equiv C$ on $[a,b]$ is Darboux-integrable with $D\int_a^b = (b-a)C$.
b) If $f_1 \leq f_2$ are two Darboux-integrable functions, then $D\int_a^b f_1 \leq D\int_a^b f_2$.
c) If $a < c < b$, then $f : [a,b] \to \mathbb{R}$ is Darboux-integrable if and only if $f|_{[a,b]}$ and $f|_{[c,b]}$ are both Darboux-integrable, and if so $D\int_a^b f = D\int_a^c f + D\int_c^b f$.
d) All continuous functions are Darboux-integrable.

Comment: The point of this exercise is that it is much easier to show this for the Darboux integral then the Riemann integral. Especially, the proof of part d) can (and should) be obtained by adapting the first few lines of the proof of the corresponding fact for the Riemann integral.

Thus the abstract fundamental theorem of calculus holds for the Darboux integral as well. As a consequence, we must have $D\int_a^b f = R\int_a^b f$ at least for all continuous functions f, since the abstract integral of any continuous function must be $F(b) - F(a)$, where F is any function with $F' = f$. (In other words, if all we wanted to do was to show that continuous functions have antiderivatives, we could have done it with a lot less trouble!)

3. Darboux integrability of monotone functions

A triumph of the Darboux integral is the easy proof of the following result.

Theorem 2. *Any monotone function is Darboux-integrable.*

Proof: We will assume that $f : [a, b] \to \mathbb{R}$ is increasing. To deal with the decreasing case, the reader may either modify the proof, or use the observation that $-f$ is increasing if and only if f is decreasing. Now, consider the uniform partition P_n which divides the interval $[a, b]$ into n equally spaced subintervals, so each $x_i - x_{i-1} = \frac{b-a}{n} = \Delta x$, say. Observe that if f is increasing, its supremum on any subinterval is its right endpoint and its infimum on any subinterval is its left endpoint. Thus:

$$\omega(f, P_n) = U(f, P_n) - L(f, P_n) = \sum_{i=1}^{n} f(x_i)\Delta x - \sum_{i=1}^{n} f(x_{i-1})\Delta x =$$

$$\frac{b-a}{n}\sum_{i=1}^{n} f(x_i) - f(x_{i-1}) = \frac{b-a}{n}(f(x_n) - f(x_0)) = \frac{b-a}{n}(f(b) - f(a)).$$

So as $n \to \infty$, $\omega(f, P_n) \to 0$, so that f is Darboux-integrable.

Recall from Exercise 49 that for any countable subset S of an interval $[a, b]$ there exists a monotone function f defined on $[a, b]$ whose set of discontinuities is precisely S. Thus this result takes us far beyond the realm of bounded functions with finitely many discontinuities.

4. Linearity of the integrals

In our previous discussion of the Riemann integral, we forgot to mention that the operations of addition of functions and multiplication of functions by constants commute with Riemann integration (and preserve the class of Riemann-integrable functions). For completeness, we give the proofs now both for Riemann and Darboux.

Proposition 3. *Let $f, g : [a, b] \to \mathbb{R}$ be two Riemann-integrable functions. Then $f + g$ is Riemann-integrable and $R\int_a^b f + g = R\int_a^b f + R\int_a^b g$.*

Proof: The first thing to note is that Riemann sums are additive: for any tagged partition (P, τ) of $[a, b]$, we have

$$R(f + g, P, \tau) = \sum_{i=1}^{n}(f(x_i^*) + g(x_i^*))(x_i - x_{i-1}) =$$

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) + \sum_{i=1}^{n} g(x_i^*)(x_i - x_{i-1}) = R(f, P, \tau) + R(g, P, \tau).$$

Thus if we can show that $f + g$ is Riemann-integrable, the equality of the integral of the sum and the sum of the integrals follows immediately. For this, we use the Cauchy criterion: let δ be sufficiently small so that whenever (P_1, τ_1) and (P_2, τ_2) are two tagged partitions of mesh at most δ, we have both that $|R(f, P_1, \tau_1) - R(f, P_2, \tau_2)| < \frac{\epsilon}{2}$ and $|R(g, P_1, \tau_1) - R(g, P_2, \tau)2)| < \frac{\epsilon}{2}$. (A different choice of δ works for each of f and g; we need only take the smaller of the two.) But then using the additivity of the Riemann sums and the triangle inequality we get

$$|R(f + g, P_1, \tau_1) - R(f + g, P_2, \tau_2)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

On the other hand, it is not true that $U(f + g, P) = U(f, P) + U(g, P)$ necesarily: in general, we can only say that $\sup_{x\in A} f(x) + g(x) \leq \sup_{x\in A} f(x) + \sup_{x\in A} g(x)$.

This shows that $U(f+g,P) \leq U(f,P)+U(g,P)$; similarly $L(f+g,P) \geq L(f,P)+L(g,P)$. This is enough to show that if f and g are Darboux-integrable, then so is their sum: let P_1 be a partition such that $\omega(f,P_1) <\epsilon_{\overline{2}}$ and P_2 be a partition such that $\omega(g,P_2) < \frac{\epsilon}{2}$. Then taking $P = P_1 \cup P_2$ we get $\omega(f,P) \leq \omega(f,P_1)$ and $\omega(g,P) \leq \omega(g,P_2)$, so $\omega(f+g,P) =$

$$U(f+g,P)-L(f+g,P) \leq U(f,P)+U(g,P)-L(f,P)-L(g,P) = \omega(f,P)+\omega(g,P) < \epsilon.$$

Moreover, we can at least say that

$$U(f+g,P) \geq L(f+g,P) \geq L(f,P)+L(g,P) > (\int_a^b f-\frac{\epsilon}{2})+(\int_a^b g-\frac{\epsilon}{2}) = \int_a^b f+\int_a^b g-\epsilon$$

and

$$L(f+g,P) \leq U(f+g,P) \leq U(f,P)+U(g,P) < (\int_a^b f+\frac{\epsilon}{2})+(\int_a^b g+\frac{\epsilon}{2}) = \int_a^b f+\int_a^b g+\epsilon,$$

which means that $\int_a^b f+g$ lies in the interval

$$(\int_a^b f + D\int_a^b g - \epsilon, \int_a^b f + \int_a^b g + \epsilon).$$

Since ϵ was arbitrary, this means that $D\int_a^b f+g = D\int_a^b f + D\int_a^b g$.[3]

Thus we have also:

Proposition 4. *Let $f, g : [a,b] \to \mathbb{R}$ be two Darboux-integrable functions. Then $f+g$ is Darboux-integrable and $D\int_a^b f+g = D\int_a^b f + D\int_a^b g$.*

The following result is easier:

Proposition 5. *a) Let $f : [a,b] \to \mathbb{R}$ be a Riemann-integrable function and C a constant. Show that Cf is Riemann-integrable and $R\int_a^b Cf = CR\int_a^b f$.*
b) The same holds with "Riemann" replaced everywhere by "Darboux."

Exercise 54: Prove Proposition 5.

We may summarize the results of this section by saying that the Riemann integral, viewed as a map $I : \mathcal{R}[a,b] \to \mathcal{C}[a,b]$ from the $\mathbb{R}$-vector space of Riemann-integrable functions to the $\mathbb{R}$-vector space of continuous functions, is in fact a *linear* map, and a similar statement holds for the Darboux integral. (We will, remember, eventually be showing that these two linear maps are the same.)

5. Further integrability theorems

Having shown without terrible trouble that sums of integrable functions are integrable and that the product of a constant function and an integrable function is integrable (again, in either sense), it is natural to wonder whether the *product* of two integrable functions is integrable. Of course, this is true when f and g are continuous, or even both bounded with finitely many discontinuities (because the product of two such functions is again bounded with finitely many discontinuities). In asking the question in this full generality, we are venturing well outside the safe

[3]Note that this was an exception to the rule that the Darboux integral is easier to work with than the Riemann integral.

territory of "nice functions" that one encounters in calculus class.

Let us also recall that it is not true in general that the (Cauchy) product of two convergent series is convergent, so it might be fair to be skeptical as to whether it is even true, in general, that the product of integrable functions is integrable. However, it will turn out that every integrable function is "absolutely integrable" in the sense that the integrability of f implies that of $|f|$. In other words, integrals are analogous to absolutely convergent series, for which the "product theorem" does indeed hold.

It turns out that the key statement is the following result.

Theorem 6. *Let $f : [a, b] \to [c, d]$ be Darboux-integrable and $\varphi : [c, d] \to \mathbb{R}$ be continuous. Then the composition $\varphi \circ f$ is Darboux-integrable.*

Remarks: The result is stated for the Darboux integral because it is too hard to prove directly for the Riemann integral. (Later, after we show that Riemann integrability is equivalent to Darboux integrability, we will of course know the result for the Riemann integral as well).

One might ask why the theorem does not simply assert that the composition of integrable functions is integrable. In fact this is false: take $f : [1, 2] \to [0, 1]$ to be the function which takes the value $\frac{1}{q}$ at every rational $\frac{p}{q}$ number and 0 at every irrational number, and $\varphi : [0, 1] \to \mathbb{R}$ to be the function which takes 0 to zero and every $x \in (0, 1]$ to 1. Both of these functions are (Riemann and Darboux)-integrable, but $\varphi \circ f : [1, 2] \to \mathbb{R}$ takes every rational number to 0 and every irrational number to 1, so is not (Riemann or Darboux)-integrable.

Neither is it true that an integrable function composed with a continuous function (i.e., the other order) must be integrable, although the only counterexamples that I was able to construct require concepts beyond what will be discussed in this course.

Proof:[4] Say $|\varphi| \leq M$. Fix $\epsilon > 0$, and let $\epsilon_1 = \epsilon/(b - a + 2M)$. Since φ is uniformly continuous on $[c, d]$, there exists a positive number η – which may take to be less than ϵ_1 – such that $|f(x) - f(y)| < \epsilon_1$ whenever $|x - y| \leq \eta$. Apply Proposition 1 to the Darboux-integrable function with η^2 in place of ϵ: there exists a partition $P = \{x_0 = a < x_1 < \ldots < x_n = b\}$ of $[a, b]$ such that $\omega(f, P) < \eta^2$.

We divide the index set $\{i \mid 1 \leq i \leq n\}$ into two subsets:[5] S_1 is to be the set of all i such that $\omega(f, [x_{i-1}, x_i]) \leq \eta$ and S_2 is the complementary set of all i such that $\omega(f, [x_{i-1}, x_i]) > \eta$. Note that things are set up so that $\omega(\varphi \circ f, [x_{i-1}, x_i]) \leq \epsilon_1$ for $i \in S_1$; on the other hand, the oscillation of $\varphi \circ f$ on any subinterval is certainly at most the supremum of $\varphi \circ f$ minus the infimum of $\varphi \circ f$, so at most $2M$. Now we calculate

$$\sum_{i \in S_2} (x_i - x_{i-1}) = \frac{1}{\eta} \sum_{i \in S_2} \eta(x_i - x_{i-1}) < \frac{1}{\eta} \sum_{i \in S_2} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1})$$

$$\leq \frac{1}{\eta} \sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) = \frac{1}{\eta}\omega(f, P) < \frac{1}{\eta} \cdot \eta^2 = \eta.$$

[4]The proof is taken directly from p. 185 of your text.

[5]This is the tricky part: we offer no motivation as to why to do this, other than that it works!

The first inequality is just from the definition of S_2, and the last is because $\omega(f, P) < \eta^2$. It follows that

$$\omega(\varphi \circ f, P) = \sum_{i=1}^{n} \omega(\varphi \circ f, [x_{i-1}, x_i])(x_i - x_{i-1}) =$$

$$\sum_{i \in S_1} \omega(\varphi \circ f, [x_{i-1}, x_i])(x_i - x_{i-1}) + \sum_{i \in S_2} \omega(\varphi \circ f, [x_{i-1}, x_i])(x_i - x_{i-1})$$

$$\leq \sum_{i \in S_1} \epsilon_1 (x_i - x_{i-1}) + \sum_{i \in S_2} 2M(x_i - x_{i-1}) < \epsilon_1(b-a) + 2M\eta)$$

$$< \epsilon_1(b - a + 2M) = \epsilon.$$

Therefore $\varphi \circ f$ is Darboux-integrable on $[a, b]$.

Taking $\varphi(x) = |x|$, we get as advertised the following result:

Corollary 7. *If $f : [a, b] \to \mathbb{R}$ is Darboux-integrable, so is $|f|$.*

Corollary 8. *If f and g are Darboux-integrable, so is fg.*

Remark: Of course it is **not** true that $\int_a^b fg = \int_a^b f \int_a^b g$: integration is more complicated than that!

Proof: It is nothing but a trick:

$$f \cdot g = \frac{(f+g)^2 - (f-g)^2}{4}.$$

Since f and g are Darboux-integrable, so is $f + g$ and $f - g$. Moreover, taking $\varphi(x) = x^2$ in Theorem 6, we get that $(f+g)^2$ and $(f-g)^2$ are Darboux-integrable. By Proposition 5, multiplying by $\frac{1}{4}$ preserves the integrability, so we're done.

6. The equivalence of the Riemann and Darboux Integrals

In this section we tie everything together with the following result.

Theorem 9. *The Riemann and Darboux integrals give* the same *abstract integral. More precisely:*
a) A function $f : [a, b] \to \mathbb{R}$ is Riemann-integrable if and only if it is Darboux-integrable.
b) If f is integrable, $R \int_a^b f = D \int_a^b f$.

Proof: **Omitted as of 1:15 am 3/04/05. I will upload a revised draft soon!**

As immediate consequences of this theorem, the results that we were only able to prove for the Darboux integral now hold true for the Riemann integral. In particular, monotone functions are Riemann-integrable; compositions of continuous functions with Rimeann-integrable functions are Riemann-integrable; and products of Riemann-integrable functions are Riemann-integrable. Conversely, we now know that part c) of the Fundamental Theorem of Calculus applies to the Darboux integral (although in fact it would not have been difficult to couch the proof in the language of Darboux integrability).

It should be clear at this point that we have not taken the path of least resistance: in particular, we proved the most important result – that all continuous functions have antiderivatives – twice, and the second proof was much easier than the first. One might well ask: why didn't we work exclusively with the Darboux integral? The answer is that is often very convenient to be able to compute the integral via a sequence of Riemann sums. First, this allows us to evaluate certain limits by recognizing them as Riemann sums. (Exercise 57 gives an example of this.) Second, in all the applications of the Riemann integral that one meets in calculus (pages 202-203 of your text recall some of these), it is arguably more natural, and more intuitive, to be able to take any sample point x_i^* as the approximating value; and numerically, taking a "random" Riemann sum is guaranteed to be at least as good an approximation to $\int_a^b f$ than $U(f,P)$ and $L(f,P)$, which are the worst-case scenarios. Indeed, $U(f,P)$ and $L(f,P)$ are better in theory than in practice: if $P = \{x_0 < \ldots < x_n\}$, then even for a continuous function, computing $U(f,P)$ and $L(f,P)$ involves solving n minimization problems and n maximization problems.

Having come to the end of the theory of integration, we must reveal a secret: most people use "Riemann integral" to mean *either* the Riemann integral as we have defined it or the Darboux integral. (In fact one often finds a third definition of the Riemann integral, intermediate between the two we have considered. This is explored in Exercise 58.) Thus in the rest of the course we will follow this practice: we will write $\int_a^b f$, say "Riemann integral" and feel free to use whichever limiting process seems most appropriate to the problem at hand.

You should be aware that there are many other integrals out there which are *essentially* different from either the Riemann/Darboux integral: in other words, there are integrals for which the supply of integrable functions is strictly larger than the supply of Riemann-integrable functions. The most famous such integral is called the **Lebesgue integral**, the development of which is the centerpiece of a more advanced course (here, Math 355). Just to tantalize you, we mention that the function $f : [0,1] \rightarrow \mathbb{R}$ which is zero on the irrationals and one on the rationals is Lebesgue integral: can you guess what its value is? (Hint: It is *not* equal to $\frac{1}{2}$!) In the late 1950's a new integral was discovered independently by Kurzweil and Henstock, which is significantly simpler than the Lebesgue integral (it would have been possible to cover it in this course, had we the time) yet yields an even larger set of integrable functions: for instance the derivative $f'_{2,2}$ of Section 1 fails to be Lebesgue integrable but *is* nevertheless Kurzweil-Henstock integrable[6]. In fact, every derivative is Kurzweil-Henstock integrable, so that one gets a much more satisfying version of the fundamental theorem of calculus.[7]

Exercise 55: Give an example of a function $f : [a,b] \rightarrow \mathbb{R}$ such that $|f|$ is Darboux-integrable but f is not.

Exercise 56: a) Show that if f is Darboux-integrable, f^3 is Darboux-integrable.

[6] In these more general theories certain unbounded functions can be integrable, so they do not satisfy our first axiom for an abstract integral.

[7] On the other hand, the Lebesgue integral can be defined on a vastly more general class of spaces than the Kurzweil-Henstock integral; it is not in danger of becoming obsolete.

(Hint: use Theorem 6.)
b) For which positive integers n is it the case that if f is Darboux-integrable, f^n is Darboux-integrable?

Exercise 57: Compute $\lim_{n\to\infty}\sum_{k=1}^n \frac{n}{k^2+n^2}$.
Hint: Show that if $f:[0,1]\to\mathbb{R}$ is any integrable function, $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n f(\frac{i}{n}) = \int_a^b f$.

Exercise 58*: We might say that a function $f:[a,b]\to\mathbb{R}$ is **weakly Riemann integrable** to a number L if: for any sequence (P_n,τ_n) of tagged partitions with $||P_n||\to 0$, then the sequence of Riemann sums $R(f,P_n,\tau_n)$ converges to L. Note that we have used many times now that Riemann integrable functions are weakly Riemann integrable (see the solutions to Homework 6). Show that, conversely, a weakly Riemann integrable function is Riemann integrable. (Hint: show instead that it is Darboux integrable!)

Exercise 59: Let $f:[a,b]\to\mathbb{R}$ be an integrable function. The **average value** of f is defined to be $\frac{\int_a^b f}{b-a}$.
a*) The average of a finite set $\{y_1,\ldots,y_n\}$ of numbers is of course $\frac{y_1+\ldots+y_n}{n}$. In a sense, the average value f_0 is a limit of the averages $\frac{f(x_1)+\ldots+f(x_n)}{n}$ of the values of f over various finite set of points $\{x_1,\ldots,x_n\}$ of $[a,b]$. Explain the precise sense in which this limit exists, using Riemann sums.
b) Prove the Mean Value Theorem for integrals: if f is continuous, then there exists a point $c\in[a,b]$ such that $f(c)=\frac{\int_a^b f}{b-a}$. (Hint: show that the average value is somewhere between the minimum and the maximum values of f!)
c) Does the result of part b) continue to hold – i.e., is the average value always attained at some point – if f is not assumed to be continuous?