NOTES ON THE ELEMENTARY FUNCTIONS

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1. Complex numbers and complex series

It turns out that the sine, cosine and exponential functions are all very closely related to each other, provided we are willing to work with complex values. Thus in this section we provide a review of complex numbers and the rudiments of complex power series. This theory can be developed to an amazing extent (in some ways, much further than the theory of real power series), but such qualitatively different developments are the subject of another course. Here we just want to develop the theory enough so that we can make sense of plugging complex numbers into power series.

Recall that a complex number is an expression of the form $z = a + bi$. Here a and b are real numbers and i is a formal symbol having the property that $i^2 = -1$.

For many years people had "philosophical difficulties" with complex numbers; indeed, numbers of the form ib were called "imaginary," and the prevailing view was that although they did not exist, they were nevertheless very useful.

From a modern point of view this is neither acceptable (we cannot work with things that don't exist, no matter how useful they may be!) nor necessary: we can define the complex numbers entirely in terms of the real numbers. Namely, we may identify a complex number $a + bi$ with the ordered pair (a, b) of real numbers, and we will define addition and multiplication. Since we would want $(a+bi)+(c+di) = (a+c)+(b+d)i$, in terms of ordered pairs this is just $(a, b)+(c, d) = (a+c, b+d)$. In other words, this is the usual addition of vectors in the plane. The multiplication operation is more interesting but still easy enough to write down in terms of only real numbers: to compute $(a + bi)(c + di)$, we would want to use the distributive law of multiplication over addition and the relation $i^2 = -1$. In other words, we would like $(a + bi) \cdot (c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (ad + bc)i$. Thus in terms of ordered pairs we *define* a multiplication operation

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

Note that with this convention, we may identify real numbers a (i.e., those with $b = 0$) with pairs of the form $(a, 0)$; moreover, what we were formally calling i corresponds to $(0, 1)$, and now any ordered pair $a + bi$ can be expressed as $(a, 0) + (b, 0) \cdot (0, 1)$.

Moreover one can show that the collection of ordered pairs of real numbers endowed with componentwise addition and the slightly unusual multiplication satisfies all the nice algebraic properties that we would expect from a "number system." Namely:

• The addition operation is commutative, associative, has an additive identity $0 := (0, 0)$, and every element $(a, 0)$ has an additive inverse: $(-a, 0)$.
• The multiplication operation is commutative, associative, has a multiplicative identity $1 = (1, 0)$, and every nonzero element has a multiplicative inverse.
• The multiplication operation distributes over addition: if z_1, z_2, z_3 are any complex numbers, we have $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$, as can be checked by a straightforward (if slightly tedious) calculation.

In general, a set endowed with two operations $+$ and $\times$ satisfying the above axioms is called a **field**. Thus the real numbers form a field, and so do the complex numbers.

Two other important operations on the complex numbers are conjugation and taking the modulus. For any complex number $z = a + bi$, we define its complex conjugate to be $\overline{z} = a - bi$. Conjugation fits in nicely with the rest of the algebraic structure: one has $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{(z_1 z_2)} = \overline{z_1}\ \overline{z_2}$.

For any complex number $z = a + bi$, we define its **modulus** (or norm, or absolute value) to be $||z|| = \sqrt{a^2 + b^2}$. Note that, viewing z as the point (x, y) in the Euclidean plane, its modulus is just its distance from the origin. The following are useful properties of the modulus: here z_1 and z_2 are any complex numbers:

(i) $||z_1 + z_2|| \leq ||z_1|| + ||z_2||$.
(ii) $||z_1 z_2|| = ||z_1|| ||z_2||$.
(iii) $z_1 \overline{z_1} = ||z_1||^2$.

Finally, we mention that the theory of complex series, and especially, of complex power series, works (at least) as well as the theory of real series. Namely, if $\sum_{n=0}^{\infty} a_n z^n$ is a power series with complex coefficients, then defining $\rho = \limsup ||a_n||^{\frac{1}{n}}$, we find that ρ is the radius of convergence of the complex power series in the sense that the series converges for all z with $||z|| < \rho$ and diverges for all z with $||z|| > \rho$. (What happens on the circle $||z|| = \rho$ can be much more complicated than in the real case.) Especially, if $\sum_n a_n x^n$ is a power series with real coefficients and infinite radius of convergence, then because for a real number x, its absolute value $|x|$ is the same as the modulus of the complex number $x + 0i$, then the power seres $\sum_n a_n z^n$ must converge for all complex numbers z.

2. The exponential function

Consider the following complex power series:

$$E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Because the ratio test limit is $\lim_{n \to \infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n \to} \frac{1}{n+1} = 0$, the radius of convergence is infinite: the series converges for all complex numbers z.

Proposition 1. *For all complex numbers z and w, we have $E(z+w) = E(z)E(w)$.*

Proof: Since the series representations of $E(z)$ and $E(w)$ are absolutely convergent, we know that $E(z)E(w)$ is given by the Cauchy product, namely

$$E(z)E(w) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = E(z+w).$$

This completes the proof.

Since $E(0) = 1$, we have for all z that

$$E(z)E(-z) = E(z-z) = E(0) = 1,$$

or $E(-z) = \frac{1}{E(z)}$. Note in particular that $E(z)$ is never zero. Restricting attention to real values, since $E : x \mapsto E(x)$ is a continuous function which is never zero and such that $E(0) = 1$, we conclude $E(x) > 0$ for all real x.

We have seen that the derivative of a convergent power series may be computed term-by-term, so in this case we get:

$$E'(z) = \sum_{n=0}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = E(z).$$

In fact the derivative can be computed without this theorem about power series: from the definition, we have

$$E'(z) = \lim_{h \to 0} \frac{E(z+h) - E(z)}{h} = E(z) \lim_{h \to 0} \frac{E(h) - 1}{h} = E(z) \cdot \left(\lim_{h \to 0} 1 + \frac{h^2}{2} + \frac{h^3}{3} + \dots \right)$$
$$= E(z) \cdot 1 = E(z).$$

It follows from this and the positivity of $E(x)$ that $x \mapsto E(x)$ is a positive, increasing, convex function.

We define $e = E(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$. From a previous homework assignment we can easily compute e accurate to as many decimal places as we wish: $e \approx 2.71828\ldots$.

An immediate induction argument allows us to extend the relation $E(z+w) = E(z)E(w)$ to $E(z_1 + \ldots + z_n) = E(z_1) \cdots E(z_n)$ for all n. In particular $E(nz) = E(z)^n$ and $E(n) = E(1)^n = e^n > 2^n$. Thus $\lim_{x \to \infty} E(x) = +\infty$. A similar argument shows that $\lim_{x \to -\infty} E(x) = 0$.

Comparison with the definition of e^x given in calculus: as will be the case throughout this section, it was rather easy to define a power series $E(x)$ and show that it has all the properties that are usually claimed (sometimes without convincing proof) in a basic calculus class. What is more challenging is to show that the power series is *actually equal* to the function as it is defined in calculus classes.

How is $f(x) = e^x$ defined in calculus class? It happens that I am also teaching a calculus class, so I can tell you how the text we are using does it: it is first claimed that for any real number $a > 1$ there is a function $f_a(x) := a^x$, which, for any real number x, "raises a to the xth power." Unfortunately it is difficult to make sense of this for all values of x. Certainly it is okay when x is a positive integer:

$a^n = a \cdot a$ (n times). Moreover, our desire to have the identity $a^{x+y} = a^x \cdot a^y$ hold as widely as possible helps us define $a^0 = 1$, $a^{-n} = \frac{1}{a^n}$, and finally $a^{\frac{1}{n}}$ to be the positive number whose nth power is a, so in all we can define a^x whenever x is a *rational number.* Of course we want the function defined for all *real numbers.* My calculus text tells me (and so I told my students) that for irrational x_0 we should define a^{x_0} to be the supremum of $a^{\frac{p}{q}}$ as $\frac{p}{q}$ ranges over rational numbers which are strictly smaller than x.[1] At least we can see that the function $f_a : \mathbb{Q} \to \mathbb{Q}$ given by $x \mapsto a^x$ is increasing: if $h > 0$, $f_a(x+h) - f_a(x) = a^{x+h} - a^x = a^x(a^h - 1) > 0$, since $a^h > 1$. Thus any rational number *larger* than x_0 gives an upper bound for this set, so indeed the supremum is well-defined (not that this is explained anywhere to the calculus students).

So we get *some* extension of the function $f_a = x^a$ to a function defined on the real line. But how do we know it's "the right" extension? Is it continuous? Differentiable?

If we can answer the second question, we don't have to worry about the first: a function $f : \mathbb{Q} \to \mathbb{Q}$ has at most one continuous extension to a function $f : \mathbb{R} \to \mathbb{R}$.[2]

There are now several ways to prove continuity of f_a, the easiest being to reduce to the power series situation: when $a = e$, $f_e(x) = e^x$ is a function which agrees with $E(x)$ for all rational numbers. Moreover, like any continuous increasing function, the value of E at any number x_0 is the supremum of all rational values $\frac{p}{q} < x_0$. Thus in fact $E(x)$ does equal e^x for all real x.

Actually there is another way:

Exercise 80*: Let $f : \mathbb{Q} \to \mathbb{R}$ be a function defined only on the rational numbers which is uniformly continuous.
a) Show that f extends (necessarily uniquely) to a continuous function defined on all of $\mathbb{R}$. (Hint: this is a generalization of Theorems 3.29 and 3.30 in your text.)
b) If f is also increasing, then the unique extension is given by $f(x_0) := \sup_{\{\frac{p}{q} < x_0\}} f(\frac{p}{q})$.
c) Let $a > 1$ be areal number. Use parts a) and b) to give a direct proof that $f_a(x) = a^x$, defined *a priori* on $\mathbb{Q}$, extends to a continuous function on $\mathbb{R}$.

For bases a different from e, we still have not shown that $f(x) = a^x$ is infinitely differentiable (only that it is continuous, by Exercise X). For this, the idea is to reduce to the case $a = e$, which involves introducing the logarithm function.

Since $E : \mathbb{R} \to (0, \infty)$ is an increasing, infinitely differentiable function, it has an increasing infinitely differentiable inverse function $L : (0, \infty) \to \mathbb{R}$; recall that L is characterized by the properties $L(E(x)) = x$ and $E(L(y)) = y$. Differentiating the first equation using the chain rule, we get $L'(E(x))E'(x) = 1$, or, since

[1] The book does not literally use the word "supremum."

[2] I hope this was covered in the first semester, but in case not: if $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are both continuous functions extending f, then $g := f_1 - f_2$ is a continuous function which is zero on every rational number. Thus, if g took on any nonzero value, it would, by continuity, be nonzero on some open interval, so necessarily nonzero at some rational number, a contradiction.

$E'(x) = E(x)$, $L'(E(x)) = \frac{1}{E(x)}$. In other words, for all positive y, $L'(y) = \frac{1}{y}$. We also have $L(1) = 0$, since $E(0) = 1$. Thus $L : (0, \infty \to \mathbb{R}$ is a differentiable function with $L(1) = 0$ and $L'(x) = \frac{1}{x}$. There is a unique such function; in calculus courses it is called the (natural) logarithm $\ln x$ or $\ln_e x$. (In higher mathematics, we assume that the base of the logarithm is e unless otherwise mentioned, so we simply write $\log x$.)

From properties of the inverse function it follows that $L(xy) = L(x) + L(y)$, $\lim_{x \to 0^+} L(x) = -\infty$, $\lim_{x \to \infty} L(x) = +\infty$.

Now we can define the function $f_a(x) = a^x$ in terms of L and E: $F_a(x) := E(L(a)x)$. It is easy to see that F_a also has the fundamental property $F_a(x+y) = F_a(x)F_a(y)$ and that $F_a(r) = a^r$ whenever r is a rational number. As above, since F_a is an infinitely differentiable increasing function, it is the unique continuous extension of $f_a : \mathbb{Q} \to \mathbb{R}$, and agrees with the "supremum definition" of a^x given in calculus books.

3. The trigonometric functions

Let us now turn to the functions $\sin x$ and $\cos x$. Recall that we have already shown that any pair of differentiable functions $S(x)$ and $C(x)$ such that $S'(x) = C(x)$, $C'(x) = -S(x)$, $S(0) = 0$ and $C(0) = 1$ must be equal to their Taylor series and given by the following expansions:

$$S(x) := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

$$C(x) := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Of course we would like to say that $S(x) = \sin x$ and $C(x) = \cos x$, but we do not want to have to resort to discussions involving angles, lengths of arcs and other such things. We want to see how much can be derived directly from the power series expansions themselves. For instance, we would like the show that $C^2 + S^2 = 1$. Unfortunately, although this identity does hold, showing it directly from the power series expansions involves some rather unpleasant algebra (try it and see).

This is where complex numbers come in to save the day:

Proposition 2. *For all real x, we have the following identities:*

$$C(x) = \frac{1}{2}(E(ix) + E(-ix)),$$

$$S(x) = \frac{1}{2i}(E(ix) - E(-ix)),$$

$$E(ix) = C(x) + iS(x).$$

We leave the proof of these identities as an exercise.

Now we are in business: since the coefficients of $E(z)$ are real, we have $\overline{E(ix)} = E(\overline{ix}) = E(-ix)$ for all real x, hence

$$C(x)^2 + S(x)^2 = |E(ix)|^2 = E(ix)\overline{E(ix)} = E(ix)E(-ix) = E(ix - ix) = E(0) = 1.$$

We're not done yet: we'd like to prove that $S(x)$ and $C(x)$ are periodic functions, whose period is a mysterious number approximately equal to $2 \cdot 3.141592653\ldots$. This can also be worked out from the power series expansions, with some cleverness:

We first claim that there exists $x_0 > 0$ such that $C(x_0) = 0$. Otherwise, since $C(0) = 1 > 0$, we'd have $C(x) > 0$ for all x, hence $S'(x) = C(x) > 0$ for all x, hence S would be strictly increasing on the entire real line. Since $S(0) = 0$, it follows that $S(x) > 0$ for all $x > 0$. Now, if $0 < x < y$, we have

$$S(x)(y - x) < \int_x^y S(t)dt = C(x) - C(y) \leq 2.$$

But now for fixed x and $y > x + \frac{2}{S(x)}$, this gives a contradiction.

Observation: Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $f(0) > 0$ and $f(x) = 0$ for some $x > 0$. Then there is a *least* positive number x_0 such that $f(x_0) = 0$.

Proof: Let S be the set of positive numbers x such that $f(x) = 0$. Obviously this set is bounded below (by zero), so it has an infimum, say c. If $f(c)$ were positive, then, by continuity, there would be a small interval around c on which $f(c)$ is positive; similarly if $f(c)$ were negative. But by definition of c, every interval of the form $(c, c + \delta)$ for $\delta > 0$ must contain numbers x for which $f(x) = 0$. Thus the only possibility is that $f(c) = 0$.

Now we define the number π by $\pi := 2x_0$. where x_0 is the lesat positive number x such that $C(x) = 0$. The relation $C(x)^2 + S(x)^2 = 1$ together with $C(\frac{\pi}{2}) = 0$ shows that $S(\frac{\pi}{2}) = \pm 1$. On the other hand, since $C(x) = S'(x)$ is non-negative on $[0, \frac{\pi}{2}]$, $S(x)$ is increasing on this interval, so it must be that $S(\frac{\pi}{2}) = 1$. Thus $E(\frac{\pi i}{2}) = i$. Using the addition formula for $E(z)$ we recover Euler's amazing identity

$$e^{i\pi} = \left(e^{\frac{i\pi}{2}}\right)^2 = -1,$$

and also $e^{2\pi i} = 1$. So in general we have $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$, so E is periodic with period $2\pi i$.

Using the periodicity of E and the formula of Proposition 2, we get that for all x $C(x + 2\pi) = C(x)$ and $S(x + 2\pi) = S(x)$.

Finally, we can make contact with geometric considerations as follows: since for all real t, $|e^{it}| = 1$, the parameterized curve

$$r(t) = e^{it} = C(t) + iS(t) \iff (C(t), S(t)) = (x(t), y(t))$$

has image contained in the unit circle in the xy-plane. It is not too hard to show[3] that as t ranges from 0 to 2π every point on the unit circle is the image of a unique t (except that $r(0) = r(2\pi i) = 1$).

Finally, if we grant that by the arclength of the parameterized curve $r(t) = (x(t), y(t))$ from $t = a$ to $t = b$ we mean the integral

$$\int_{t=a}^{b} \sqrt{|\frac{dx}{dt}|^2 + |\frac{dy}{dt}|^2} dt$$

it is easy to show that $C(x) = \cos x$ and $S(x) = \sin x$. Indeed, for $r(t) = (C(t), S(t))$, the arclength integral is

$$\int_{t=0}^{\theta} S^2(t) + C^2(t) dt = \theta,$$

so the point $r(\theta) = (C(\theta), S(\theta))$ really is the point that we arrive at by starting at the point $(1, 0)$ on the unit circle and traversing θ units of arc.

You will find that knowing power series expansion of the sine and cosine functions replaces many ad hoc geometric arguments by routine calculations: for instance, the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ is immediate from the power series expansion: try it and see.

[3]Using, for instance, a suitable version of the intermediate value theorem for parameterized functions with range contained in the unit circle.