MATH 2400 FIRST MIDTERM EXAM

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Directions: You will have 75 minutes to complete the exam. Calculators are not permitted (nor would they be helpful...). Please attempt all problems. When in doubt, err on the side of writing a little too much rather than a little too little.

1) a) Find all real numbers \( x, y \) such that \( y^2 > 5x^2 \).

Solution: \( y^2 > 5x^2 \iff |y| > \sqrt{5}|x| \).

b) Sketch a graph of the set of all points in the plane satisfying \( y^2 > 5x^2 \).

Solution: I will describe the picture to (in less than a thousand words). Consider the two lines \( y = \pm \sqrt{5}x \). These lines divide the plane into four different regions. The desired set of points consists of the top and bottom regions and not the left and right regions. This can be seen for instance by first taking \( x \) and \( y \) to be non-negative and noting that in the first quadrant we want all points \((x, y)\) with \( y > \sqrt{5}x \). Then, since the locus \(|y| > \sqrt{5}|x|\) is unchanged by \( x \mapsto -x \) and \( y \mapsto -y \) we can simply reflect this “45 degree sector” through the \( x \)-axis, through the \( y \)-axis, and through both the \( x \)- and \( y \)-axes to get the full set.

2) Let \( x \) and \( y \) be nonzero real numbers.

a) Prove or disprove: if \( x \) and \( y \) are both rational, then \( xy \) is rational.

Solution: It’s true: if \( x = \frac{a}{b}, y = \frac{c}{d} \) with \( a, b, c, d \in \mathbb{Z} \), \( b, d \neq 0 \), then \( x + y = \frac{ad + bc}{bd} \) and \( ad + bc, bd \in \mathbb{Z} \) with \( bd \neq 0 \).

b) Prove or disprove: if \( x \) is rational and \( y \) is irrational, then \( xy \) is irrational.

Solution: It’s true. Let \( x = \frac{a}{b} \) with \( a, b \in \mathbb{Z}, b \neq 0 \). Seeking a contradiction, we assume that \( xy \) is rational, say \( xy = \frac{c}{d} \) with \( c, d \neq 0 \). Then \( y = \frac{ad}{bc} \) would be rational, contradiction.

c) Prove or disprove: if \( x \) and \( y \) are both irrational, then \( xy \) is rational.

Solution: This is not true, i.e., not always true. For instance we could take \( x = \sqrt{2}, y = \sqrt{3} \), so \( xy = \sqrt{6} \).

d) Prove or disprove: if \( x \) and \( y \) are both irrational, then \( xy \) is irrational.

Solution: Again this is not true, i.e., not always true. For instance we could take \( x = y = \sqrt{2} \), so \( xy = 2 \).
(Conclusion: if \( x, y \) are nonzero irrational numbers, \( xy \) may be either rational or irrational!)

3) Prove that for every integer \( n \geq 2 \),
\[
\sum_{k=1}^{n-1} (2k + 1) = n^2 - 1.
\]

First solution: We already know that \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \), so
\[
\sum_{k=1}^{n} (2k + 1) = 2 \sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} 1 = 2 \left( \frac{n-1}{2} \right) (n-1+1) + n-1 = n^2 - n + (n-1) = n^2 - 1.
\]

Second solution: by induction on \( n \).
Base Case \((n = 2): \sum_{k=1}^{1} 2k + 1 = 3 = 2^2 - 1, \) ok.
Induction Step: Let \( n \geq 2 \) be an integer; assume \( \sum_{k=1}^{n-1} (2k + 1) = n^2 - 1 \). Then
\[
\sum_{k=1}^{n} (2k + 1) = \sum_{k=1}^{n-1} (2k + 1) + 2n + 1 = n^2 - 1 + 2n + 1 = (n+1)^2 - 1.
\]

4)a) State the \( \epsilon - \delta \) definition of \( \lim_{x \to c} f(x) = L \).

Solution: For all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for all \( x \) with \( 0 < |x - c| < \delta \), \( f(x) \) is defined and \( |f(x) - L| < \epsilon \).

b) State the \( \epsilon - \delta \) definition of \( f(x) \) is continuous at \( x = c \).

Solution: For all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for all \( x \) with \( |x - c| < \delta \), \( |f(x) - f(c)| < \epsilon \).

5)a) Use \( \epsilon - \delta \) to show that \( f(x) = 19x - 3 \) is continuous at \( c = -2 \).

Solution: For any \( \epsilon > 0 \), let \( \delta = \frac{\epsilon}{19} \). Then, if \( |x - (-2)| = |x + 2| < \delta \),
\[
|f(x) - f(-2)| = |19x - 3 - (19(-2) - 3)| = 19|x + 2| < 19\delta = 19 \left( \frac{\epsilon}{19} \right) = \epsilon.
\]

b) Find \( \lim_{x \to 6} x^2 + x + 1 \), and prove that your answer is correct directly from the \( \epsilon - \delta \) definition of limits.

Solution: Since polynomials are continuous, the desired limiting value is \( f(6) = 6^2 + 6 + 1 = 43 \). Now we require \( \delta \leq 1 \) and compute:
\[
|f(x) - 43| = |x^2 + x + 1 - 43| = |x^2 + x - 42| = |x - 6||x + 7|.
\]
It will suffice to get a bound on \( |x + 7| \) for \( x \) in the interval \([6 - 1, 6 + 1]\). Here everything in sight is positive so it is clear that the maximum value \( x + 7 \) takes on this interval is at the right endpoint, where it takes the value \( (6 + 1) + 7 = 14 \). Thus we take \( \delta = \min(1, \frac{1}{14}) \) and then for \( 0 < |x - 6| < \delta \),
\[
|f(x) - 43| = |x + 7||x - 6| < 14\delta \leq 14 \left( \frac{\epsilon}{14} \right) = \epsilon.
\]