Finite Elements for Symmetric Tensors

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Outline

1. Linear Elasticity Problem and Mixed Finite Elements
2. Conforming Elements on Tetrahedral Meshes
   - Brezzi’s stability conditions
   - 2D triangular Arnold-Winther elements
   - 3D tetrahedral Arnold-Awanou-Winther elements
3. Relaxing Conformity and Symmetry
   - Nonconforming Elements
   - Elements with symmetry weakly imposed
4. Summary
Applications of finite element

FEM Model Details

FEM Model – Front Suspension

FEM Model – Vehicle Interior

FEM Model – Bottom View

http://www.epm.ornl.gov/SC98/car.html
Displacement \( u_i = x'_i - x_i \)

\[
dl'^2 = dl^2 + \sum_{i,k} 2(\epsilon u)_{ik} \, dx_i \, dx_k
\]

\[
(\epsilon u)_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \sum_l \frac{\partial u_l}{\partial x_k} \frac{\partial u_l}{\partial x_i} \right)
\]

For small deformations, the strain tensor is

\[
(\epsilon u)_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)
\]

\[\sum_k \sigma_{ik} n_k\] is the ith component of the force acting on the element of surface \( ds \) with normal \( \mathbf{n} \).
Linear Elasticity Problem

\[ A \sigma = \epsilon(u) \quad \text{in } \Omega \]
\[ \text{div } \sigma = f \quad \text{in } \Omega \]
\[ u = 0 \quad \text{on } \partial \Omega, \]

- \( u : \Omega \rightarrow \mathbb{R}^n \) measures the displacement
- \( \sigma : \Omega \rightarrow \text{Sym} \) measures the internal forces where Sym is the space of \( n \times n \) symmetric matrices.

\[ \epsilon(u) = \frac{1}{2} (\nabla u + (\nabla u)^T) \]

\( A \) is called compliance tensor and \( f \) encodes the body forces.
Variational Formulations

1. Primal variational principle: displacement over $v = 0$ on $\partial \Omega$

   $$\int_\Omega \frac{1}{2} A^{-1} \varepsilon(v) : \varepsilon(v) + f \cdot v$$

2. Dual Variational Principle: stress field over $\text{div } \tau = f$

   $$\int_\Omega \frac{1}{2} A \tau : \tau$$

3. Mixed variational principle: stress field and displacement

   $$\int_\Omega \left( \frac{1}{2} A \tau : \tau + \text{div } \tau \cdot v - f \cdot v \right) dx$$
Mixed Weak Formulation

Find $\sigma \in \Sigma = H(\text{div}, \Omega, \mathbb{S}) = \{ \sigma \in L^2(\Omega, \mathbb{S}), \text{div} \sigma \in L^2(\Omega, \mathbb{R}^n) \}$ and $u \in V = L^2(\Omega, \mathbb{R}^n)$ such that

\[
(A\sigma, \tau) + (\text{div} \tau, u) = 0, \quad \forall \tau \in \Sigma,
\]
\[
(\text{div} \sigma, v) = (f, v), \quad \forall v \in V.
\]

Discrete problem posed on $\Sigma_h \subset \Sigma$ and $V_h \subset V$

$\sigma = (\sigma_{ij})_{i,j=1,...,n} \quad \sigma_{ij} = \sigma_{ji} \quad u = (u_i)_{i=1,...,n}$

Stable approximations?
Mixed Weak Formulation

Find $\sigma \in \Sigma = H(\text{div}, \Omega, \mathbb{S}) = \{ \sigma \in L^2(\Omega, \mathbb{S}), \text{div} \sigma \in L^2(\Omega, \mathbb{R}^n) \}$ and $u \in V = L^2(\Omega, \mathbb{R}^n)$ such that

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$$\sigma = (\sigma_{ij})_{i,j=1,\ldots,n} \quad \sigma_{ij} = \sigma_{ji} \quad u = (u_i)_{i=1,\ldots,n}$$

Stable approximations?
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Discrete problem posed on $\Sigma_h \subset \Sigma$ and $V_h \subset V$

$$\sigma = (\sigma_{ij})_{i,j=1,...,n} \quad \sigma_{ij} = \sigma_{ji} \quad u = (u_i)_{i=1,...,n}$$

Stable approximations?
Finite element spaces

- $K$ is a closed subset of $\mathbb{R}^n$ with a nonempty interior and a Lipshitz continuous boundary
- $P_K$ is a finite dimensional space of vector valued or matrix valued functions defined over the set $K$
- $\Theta_K$ is a finite set of linearly independent linear functionals, $\theta_i$, $i = 1, \ldots, N$ referred to as degrees of freedom of the finite element, defined over the set $P_K$.

It is assumed that the set $\Theta_K$ is $P_K$-unisolvent in the sense that

$$\theta_i(p) = 0, i = 1, \ldots, N \implies p \equiv 0$$
Examples
A piecewise infinitely differentiable function belongs to $H^k$ if and only if it is $C^{k-1}$

A piecewise smooth vector field $v$ is in $H(\text{div})$ if and only if for each common face $F = K_1 \cap K_2$ of the triangulation, the trace on $F$ of the normal component $n \cdot v \mid_{K_1}$ and $n \cdot v \mid_{K_2}$ is the same

$$
\sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33}
\end{pmatrix}
$$

$\sigma$ symmetric  

$u = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33}
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\end{pmatrix}$

Stable approximations?
Surprisingly stable mixed finite elements for elasticity have been difficult to construct.


1. Linear Elasticity Problem and Mixed Finite Elements

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4. Summary
Abstract Mixed Formulation

Find \( \sigma \in \Sigma = H(\text{div}, \Omega, \mathbb{S}) \) and \( u \in V = L^2(\Omega, \mathbb{R}^n) \) such that

\[
(A\sigma, \tau) + (\text{div} \tau, u) = 0, \quad \forall \tau \in \Sigma,
\]

\[
(\text{div} \sigma, v) = (f, v), \quad \forall v \in V.
\]

\[
a(\sigma, \tau) + b(\tau, u) = 0, \quad \forall \tau \in \Sigma,
\]

\[
b(\sigma, v) = (f, v), \quad \forall v \in V.
\]

Discrete Problem

\[
a(\sigma_h, \tau_h) + b(\tau_h, u_h) = 0, \quad \forall \tau_h \in \Sigma_h, \quad \Sigma_h \subset \Sigma
\]

\[
b(\sigma_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad V_h \subset V.
\]
Brezzi’s stability conditions

\( \Sigma_h \subset \Sigma \) and \( V_h \subset V \)

Sufficient conditions for optimal error bounds

First Brezzi condition \( \exists \alpha > 0 \) independent of \( h \) such that

\[
a(\tau, \tau) \geq \alpha \|\tau\|_{\Sigma}^2
\]

for all \( \tau \) in \( K_h \) where

\[
K_h = \{ \tau \in \Sigma_h : b(\tau, v) = 0, \forall v \in V_h \}
\]

Second Brezzi condition \( \exists \beta > 0 \) independent of \( h \) such that

\[
\sup_{\tau \in \Sigma_h} \frac{b(\tau, v)}{\|\tau\|_{\Sigma}} \geq \beta \|v\|_V \quad \forall v \in V_h
\]

\[
\|\sigma - \sigma_h\|_{\Sigma} + \|u - u_h\|_V \leq \gamma \{ \inf_{\tau \in \Sigma_h} \|\sigma - \tau\|_{\Sigma} + \inf_{v_h \in V_h} \|u - v_h\|_V \}
\]

with \( \gamma \) independent of \( h \).

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Finite Elements for Symmetric Tensors
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\[ \Sigma_h \subset \Sigma \text{ and } V_h \subset V \]

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\[ \|\sigma - \sigma_h\|_{\Sigma} + \|u - u_h\|_V \leq \gamma \{ \inf_{\tau \in \Sigma_h} \|\sigma - \tau\|_{\Sigma} + \inf_{v_h \in V_h} \|u - v_h\|_V \} \]

with \( \gamma \) independent of \( h \).
Brezzi’s stability conditions

$$\Sigma_h \subset \Sigma$$ and $$V_h \subset V$$

Sufficient conditions for optimal error bounds

First Brezzi condition \( \exists \alpha > 0 \) independent of \( h \) such that

$$a(\tau, \tau) \geq \alpha \| \tau \|^2_\Sigma$$

for all \( \tau \) in \( K_h \) where

$$K_h = \{ \tau \in \Sigma_h : b(\tau, v) = 0, \forall v \in V_h \}$$

Second Brezzi condition \( \exists \beta > 0 \) independent of \( h \) such that

$$\sup_{\tau \in \Sigma_h} \frac{b(\tau, v)}{\| \tau \|_\Sigma} \geq \beta \| v \|_V \quad \forall v \in V_h$$

$$\| \sigma - \sigma_h \|_\Sigma + \| u - u_h \|_V \leq \gamma \{ \inf_{\tau \in \Sigma_h} \| \sigma - \tau \|_\Sigma + \inf_{v_h \in V_h} \| u - v_h \|_V \}$$

with \( \gamma \) independent of \( h \).
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\[
\| \sigma - \sigma_h \|_\Sigma + \| U - U_h \|_V \leq \gamma \{ \inf_{\tau \in \Sigma_h} \| \sigma - \tau \|_\Sigma + \inf_{v_h \in V_h} \| U - v_h \|_V \}
\]

with \( \gamma \) independent of \( h \).
Sufficient conditions for stability

- \( \text{div} \, \Sigma_h \subset V_h \).
- There exists a linear operator \( \Pi_h : H^1(\Omega, S) \to \Sigma_h \), bounded in \( \mathcal{L}(H^1, L^2) \) uniformly with respect to \( h \), and such that with \( P_h : L^2(\Omega, \mathbb{R}^n) \to V_h \) denoting the \( L^2 \)-projection

\[
\begin{align*}
H(\text{div}, \Omega, S) & \xrightarrow{\text{div}} L^2(\Omega, \mathbb{R}^n) \\
\downarrow \Pi_h & \quad & \downarrow P_h \\
\Sigma_h & \xrightarrow{\text{div}} & V_h
\end{align*}
\]
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Arnold-Winther 2D elements

Elasticity Differential Complex

\[ \mathcal{P}_1(\Omega) \xrightarrow{\subset} H^2(\Omega) \xrightarrow{J} H(\text{div}, \Omega, \mathbb{S}) \xrightarrow{\text{div}} L^2(\Omega, \mathbb{R}^2) \rightarrow 0 \]

\[ \downarrow I_h \quad \downarrow \Pi_h \quad \downarrow P_h \]

\[ \mathcal{P}_1(\Omega) \xrightarrow{\subset} Q_h \xrightarrow{J} \Sigma_h \xrightarrow{\text{div}} V_h \rightarrow 0 \]

\[ Jq := \left( \begin{array}{ccc} \frac{\partial^2 q}{\partial y^2} & -\frac{\partial^2 q}{\partial x \partial y} \\ -\frac{\partial^2 q}{\partial x \partial y} & \frac{\partial^2 q}{\partial x^2} \end{array} \right) \]

\[ \mathcal{P}_1(T) \xrightarrow{\subset} \mathcal{P}_5(T) \xrightarrow{J} \mathcal{P}_3(T, \mathbb{S}) \xrightarrow{\text{div}} \mathcal{P}_2(T, \mathbb{R}^2) \rightarrow 0 \]
Arnold-Winther 2D elements

Elasticity Differential Complex

\[ \mathcal{P}_1(\Omega) \xleftarrow{\subset} H^2(\Omega) \xrightarrow{J} H(\text{div}, \Omega, \mathbb{S}) \xrightarrow{\text{div}} L^2(\Omega, \mathbb{R}^2) \to 0 \]

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But which degrees of freedom? $\dim Q_h = 6V + E$

Condition for $\sigma$ to be in $H(\text{div})$ is that $\sigma n$ is continuous

From the diagram

$$\mathcal{P}_1(T) \subset \mathcal{P}_5(T) \overset{J}{\rightarrow} \mathcal{P}_3(T, S) \overset{\text{div}}{\rightarrow} \mathcal{P}_2(T, \mathbb{R}^2) \rightarrow 0$$

possible choices are $\Sigma_T = \mathcal{P}_3(T, S)$ and $V_T = \mathcal{P}_2(T, \mathbb{R}^2)$

$$3(V - E + T) - (6V + E) + (xV + yE + zT) - wT = 0$$

which gives $x = 3$, $y = 4$ and $z = w - 3$.

Degrees of freedom

1. the values of each component of $\tau(x)$ at the vertices of $T$ (9 degrees of freedom)
2. the first two moments of each component of $\tau n$ on each edge (12 degrees of freedom)
But which degrees of freedom? \( \dim Q_h = 6V + E \)

Condition for \( \sigma \) to be in \( H(\text{div}) \) is that \( \sigma n \) is continuous

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possible choices are \( \Sigma_T = \mathcal{P}_3(T, S) \) and \( V_T = \mathcal{P}_2(T, \mathbb{R}^2) \)

\[ 3(V - E + T) - (6V + E) + (xV + yE + zT) - wT = 0 \]

which gives \( x = 3, y = 4 \) and \( z = w - 3 \).

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Degrees of freedom

1. the values of each component of $\tau(x)$ at the vertices of $T$ (9 degrees of freedom)
2. the first two moments of each component of $\tau n$ on each edge (12 degrees of freedom)
But recall that for stability, need \( \text{div} \Sigma_h \subset V_h \) and commutativity property \( P_h \text{div} \tau = \text{div} \Pi_h \tau, \tau \in H(\text{div}, \Omega, \mathbb{S}) \) or

\[
\int_T \text{div} \left( \tau - \Pi_h \tau \right) \cdot \nu \, dx = - \int_T \left( \tau - \Pi_h \tau \right) : \varepsilon(\nu) \, dx + \int_{\partial T} \left( \tau - \Pi_h \tau \right) n \cdot \nu \, ds
\]

Sufficient conditions are \( \nu \in \text{span}\{1, x\} \) on each edge

\[
\Sigma_T = \left\{ \tau \in P_3(T, \mathbb{S}), \text{div} \tau \in V_T \right\} \text{ and } V_T = P_1(T, \mathbb{R}^2)
\]

So \( w = 6 \) and \( z = 3 \).

3 the values of \( \int_T \tau : \phi \) for all \( \phi \) in \( \varepsilon(V_T) \)
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3D elasticity sequence

\[
\mathcal{T} \quad \rightarrow \quad H^1(\Omega, \mathbb{R}^3) \quad \overset{\epsilon}{\rightarrow} \quad H(\text{curl curl}^*, \Omega, \mathbb{S}) \\
\overset{\text{curl curl}^*}{\longrightarrow} \quad H(\text{div}, \Omega, \mathbb{S}) \quad \overset{\text{div}}{\rightarrow} \quad L^2(\Omega, \mathbb{R}^3) \quad \rightarrow 0
\]

\[
\mathcal{T} \quad \overset{\subset}{\in} \quad \mathcal{P}_{k+4}(\Omega, \mathbb{R}^3) \quad \overset{\epsilon}{\rightarrow} \quad \mathcal{P}_{k+3}(\Omega, \mathbb{S}) \\
\overset{\text{curl curl}^*}{\longrightarrow} \quad \mathcal{P}_{k+1}(\Omega, \mathbb{S}) \quad \overset{\text{div}}{\rightarrow} \quad \mathcal{P}_{k}(\Omega, \mathbb{R}^3) \quad \rightarrow 0.
\]

\[
\mathcal{T} \quad \rightarrow \quad R_h \quad \overset{\epsilon}{\rightarrow} \quad Q_h \quad \overset{\text{curl curl}^*}{\rightarrow} \quad \Sigma_h \quad \overset{\text{div}}{\rightarrow} \quad V_h \quad \rightarrow 0
\]

But both \( Q_h \) and \( \Sigma_h \) are spaces of symmetric matrix fields.
3D elasticity sequence

\[ T \rightarrow H^1(\Omega, \mathbb{R}^3) \overset{\epsilon}{\rightarrow} H(\text{curl curl}^*, \Omega, S) \]

\[ \text{curl curl}^* \rightarrow H(\text{div}, \Omega, S) \overset{\text{div}}{\rightarrow} L^2(\Omega, \mathbb{R}^3) \rightarrow 0 \]

\[ T \overset{\subset}{\rightarrow} P_{k+4}(\Omega, \mathbb{R}^3) \overset{\epsilon}{\rightarrow} P_{k+3}(\Omega, S) \]

\[ \text{curl curl}^* \rightarrow P_{k+1}(\Omega, S) \overset{\text{div}}{\rightarrow} P_k(\Omega, \mathbb{R}^3) \rightarrow 0. \]

\[ T \rightarrow R_h \overset{\epsilon}{\rightarrow} Q_h \overset{\text{curl curl}^*}{\rightarrow} \Sigma_h \overset{\text{div}}{\rightarrow} V_h \rightarrow 0 \]

But both \( Q_h \) and \( \Sigma_h \) are spaces of symmetric matrix fields.
Features of the 2D elements

$V_K$ space of discontinuous piecewise polynomials, $\mathcal{P}_s(K; \mathbb{R}^2)$

$$\Sigma_K = \{ \tau \in \mathcal{P}_k(K; \mathbb{S}) \mid \text{div} \tau \in V_K \}$$

$\Sigma_K$ space of matrix fields with degrees of freedom

- vertex degrees of freedom
- degrees of freedom for $\tau n$
- $\int_K \tau : \epsilon(v)$, $v \in V_K$
- $\int_K \tau : \phi$, $\phi$ in

$$\{ \tau \in \Sigma_K, \text{div} \tau = 0, \tau n = 0 \text{ on } \partial K \}$$

$k = s + 2$, $s \geq 1$
Postulate $V_K = \mathcal{P}_s(K; \mathbb{R}^3)$, and

$$\Sigma_K = \{ T \in \mathcal{P}_k(K; \mathbb{S}) \mid \text{div } T \in V_K \}$$

$$\dim \mathcal{P}_k(T, \mathbb{R}) = \frac{(k+2)(k+1)}{2}$$

Need $6 \times 4 = 24$ vertex degrees of freedom

edge with normal $n_-$ and $n_+$, $n'_- Tn_+ \in \mathcal{P}_k(e, \mathbb{R})$, $k - 1$ d.o.f.

Same for $n'_- Tn_-$, $n'_- Tt$, $n'_+ Tn_+$, $n'_+ Tt$

$$\frac{(k+2)(k+1)}{2} - 3 - 3(k - 1) = \frac{(k-1)(k-2)}{2} = \dim \mathcal{P}_{k-3}(f, \mathbb{R})$$

Face degrees of freedom $\int_f Tn \cdot v \ dx$, $v \in \mathcal{P}_{k-3}(f, \mathbb{R}^3)$

$$\int_K Tn \cdot v \ dx = 0, \ v \in V_K \text{ Need } s \leq k - 3$$
Postulate $V_K = \mathcal{P}_s(K; \mathbb{R}^3)$, and

$$\Sigma_K = \{ \mathbf{T} \in \mathcal{P}_k(K; \mathbb{S}) | \text{div} \mathbf{T} \in V_K \}$$

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Face degrees of freedom $\int_f Tn \cdot \mathbf{v} \, dx$, $\mathbf{v} \in \mathcal{P}_{k-3}(f, \mathbb{R}^3)$

$\int_K Tn \cdot \mathbf{v} \, dx = 0$, $\mathbf{v} \in V_K$ Need $s \leq k - 3$
Postulate $V_K = \mathcal{P}_s(K; \mathbb{R}^3)$, and

$$\Sigma_K = \{ T \in \mathcal{P}_k(K; \mathbb{S}) | \ \text{div} \ T \in V_K \}$$

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$$\frac{(k+2)(k+1)}{2} - 3 - 3(k - 1) = \frac{(k-1)(k-2)}{2} = \dim \mathcal{P}_{k-3}(f, \mathbb{R})$$

Face degrees of freedom $\int_f Tn \cdot v \ dx,$ $v \in \mathcal{P}_{k-3}(f, \mathbb{R}^3)$

$$\int_K Tn \cdot v \ dx = 0, v \in V_K$$

Need $s \leq k - 3$
Low order elements

\[ \Sigma_K = \{ T \in \mathcal{P}_4(K; \mathbb{S}) | \text{div} \ T \in \mathcal{P}_1(K; \mathbb{R}^3) \} \quad \text{and} \quad V_K = \mathcal{P}_1(K; \mathbb{R}^3) \]

\[ \dim \Sigma_K \leq 162 \quad \dim V_K = 12 \]

Degrees of freedom

1. vertex d.o.f. \((4 \times 6 = 24)\)
2. edge d.o.f. \((6 \times 3 \times 5 = 90)\)
3. face d.o.f. \((4 \times 3 \times 3 = 36)\)
4. \(\int_K T : U, U \in \mathfrak{e}(V_K) \) (6 d.o.f.)
5. the value of the moments \(\int_K T : U \ dx, U \in M_4(K), \) (6 d.o.f.)

\[ M_4(K) := \{ T \in \mathcal{P}_4(K; \mathbb{S}) | \text{div} \ T = 0 \text{ and } Tn = 0 \text{ on } \partial K \} \]

Somewhat similar space, Adams-Cockburn
Low order elements

\[ \Sigma_K = \{ T \in \mathcal{P}_4(K; \mathbb{S}) \mid \text{div } T \in \mathcal{P}_1(K; \mathbb{R}^3) \} \quad \text{and} \quad V_K = \mathcal{P}_1(K; \mathbb{R}^3) \]

\[ \dim \Sigma_K \leq 162 \quad \text{dim } V_K = 12 \]

Degrees of freedom

1. vertex d.o.f. \( (4 \times 6 = 24) \)
2. edge d.o.f. \( (6 \times 3 \times 5 = 90) \)
3. face d.o.f. \( (4 \times 3 \times 3 = 36) \)
4. \( \int_K T : U, U \in \epsilon(V_K) \) (6 d.o.f.)
5. the value of the moments \( \int_K T : U \ dx, U \in M_4(K), \) (6 d.o.f.)

\[ M_4(K) := \{ T \in \mathcal{P}_4(K; \mathbb{S}) \mid \text{div } T = 0 \text{ and } Tn = 0 \text{ on } \partial K \}. \]

Somewhat similar space, Adams-Cockburn
Higher order elements

\[ \Sigma_K = \{ T \in P_{k+3}(K; \mathbb{S}) \mid \text{div } T \in P_k(K; \mathbb{R}^3) \} \quad \text{and} \quad V_K = P_k(K; \mathbb{R}^3) \]

\[ \| S - S_h \|_0 \leq C h^{k+2} \| S \|_{k+2} \]

\[ \| u - u_h \|_0 \leq C h^{k+1} \| u \|_{k+1} \]

Reduced space is \( O(h^3) \) for stress and \( O(h^2) \) for displacement

\[
\begin{align*}
\mathcal{T} & \quad \rightarrow \quad C^\infty(\Omega; \mathbb{R}^3) \quad \xleftarrow{\epsilon} \quad C^\infty(\Omega; \mathbb{S}) \quad \xrightarrow{L} \quad C^\infty(\Omega; \mathbb{S}) \quad \xrightarrow{\text{div}} \quad C^\infty(\Omega; \mathbb{R}^3) \rightarrow 0 \\
\end{align*}
\]

\[
\begin{align*}
\Pi_h^R \downarrow & \quad \Pi_h^Q \downarrow \quad \Pi_h^\Sigma \downarrow \quad P_h \\
\mathcal{T} & \quad \rightarrow \quad R_h \quad \xleftarrow{\epsilon} \quad Q_h \quad \xrightarrow{L} \quad \Sigma_h \quad \xrightarrow{\text{div}} \quad V_h \rightarrow 0
\end{align*}
\]
Higher order elements

\[ \Sigma_K = \{ T \in \mathcal{P}_{k+3}(K; \mathbb{S}) \mid \text{div} \ T \in \mathcal{P}_k(K; \mathbb{R}^3) \} \quad \text{and} \quad V_K = \mathcal{P}_k(K; \mathbb{R}^3) \]

\[ \| S - S_h \|_0 \leq C h^{k+2} \| S \|_{k+2} \]

\[ \| u - u_h \|_0 \leq C h^{k+1} \| u \|_{k+1} \]

Reduced space is O(h^3) for stress and O(h^2) for displacement

\[ \mathcal{T} \rightarrow C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\epsilon} C^\infty(\Omega; \mathbb{S}) \xrightarrow{L} C^\infty(\Omega; \mathbb{S}) \xrightarrow{\text{div}} C^\infty(\Omega; \mathbb{R}^3) \rightarrow 0 \]

\[ \downarrow \Pi_h^R \quad \downarrow \Pi_h^Q \quad \downarrow \pi_h^\Sigma \quad \downarrow P_h \]

\[ \mathcal{T} \rightarrow R_h \xrightarrow{\epsilon} Q_h \xrightarrow{L} \Sigma_h \xrightarrow{\text{div}} V_h \rightarrow 0 \]
1. Linear Elasticity Problem and Mixed Finite Elements

2. Conforming Elements on Tetrahedral Meshes
   - Brezzi’s stability conditions
   - 2D triangular Arnold-Winther elements
   - 3D tetrahedral Arnold-Awanou-Winther elements

3. Relaxing Conformity and Symmetry
   - Nonconforming Elements
   - Elements with symmetry weakly imposed

4. Summary
The divergence of the stress field is no longer $L^2$ integrable
Consistency error has to be shown to be bounded
No vertex degrees of freedom and much less d.o.f.

**Triangular** Arnold-Winther 02 (12-3)
**2D Rectangular** Awanou 09 (16-3), Yi 05 (19-6), Yi 06 (13-4), Hu-Shi 07 (12-4)
**Tetrahedral** Arnold-Awanou-Winther 09 (36-6)
**3D Rectangular** Yi 05 (60-12), Man-Hu-Shi 08 (54-12)

O($h$) for stress and displacement (except Yi)
1. Linear Elasticity Problem and Mixed Finite Elements

2. Conforming Elements on Tetrahedral Meshes
   - Brezzi’s stability conditions
   - 2D triangular Arnold-Winther elements
   - 3D tetrahedral Arnold-Awanou-Winther elements

3. Relaxing Conformity and Symmetry
   - Nonconforming Elements
   - Elements with symmetry weakly imposed

4. Summary
\( \mathbb{M} \) denotes the space of \( 2 \times 2 \) matrix fields as \( (\tau) = \tau_{21} - \tau_{12} \).

Find \( (\sigma, u, \gamma) \in \Sigma \times V \times Q \subset H(\text{div}, \Omega, \mathbb{M}) \times L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R}) \) such that

\[
(A\sigma, \tau) + (\text{div} \tau, u) + (a\sigma, \gamma) = 0, \quad \tau \in H(\text{div}, \Omega, \mathbb{M}),
\]
\[
(\text{div} \sigma, v) = (f, v), \quad v \in L^2(\Omega, \mathbb{R}^2),
\]
\[
(a\sigma, q) = 0, \quad q \in L^2(\Omega, \mathbb{R}).
\]

Discrete problem posed on \( \Sigma_h \subset \Sigma, \ V_h \subset V \) and \( Q_h \subset V \).

Allows piecewise constant functions for stress and rotation on triangular, tetrahedral and rectangular in two and three dimensions.
Advantages and Disadvantages

Conforming elements with symmetric stress fields
1. Vertex degrees of freedom
2. High number of degrees of freedom
3. But conformity useful in some situations

Nonconforming elements with symmetric stress fields
1. No vertex degrees of freedom and low number of d.o.f.
2. Can potentially be extended to arbitrary quadrilaterals

Elements with weakly imposed symmetry conditions
1. Typically simpler than conforming elements
2. But comparable to nonconforming elements

Fully nonconforming elements?