Spline Element Method for Partial Differential Equations

Gerard Awanou

Department of Mathematical Sciences
Northern Illinois University

2009 Multivariate Splines Summer School, Summer 2009
Outline

1. Why multivariate splines for PDEs?
   - Motivation
   - Features of the method
   - Approximation properties

2. Refinement of tetrahedra and saddle point problems
   - Uniform refinement of a tetrahedron
   - Numerical methods for saddle point problems

3. Linear problems

4. Robustness of the method for singular perturbation problems
   - Fourth order singular problem
   - Darcy-Stokes equation

5. Nonlinear problems
   - Navier-Stokes equations
1. Why multivariate splines for PDEs?
   - Motivation
   - Features of the method
   - Approximation properties

2. Refinement of tetrahedra and saddle point problems
   - Uniform refinement of a tetrahedron
   - Numerical methods for saddle point problems

3. Linear problems

4. Robustness of the method for singular perturbation problems
   - Fourth order singular problem
   - Darcy-Stokes equation

5. Nonlinear problems
   - Navier-Stokes equations
Finite element implementation with Lagrange multipliers

The finite element method is the most widely used method for solving numerically partial differential equations. A collection of methods falls under the designation f.e.m.

Model problem

\[
\begin{cases}
-\Delta u = f \text{ in } \Omega \\
u = g \text{ on } \partial\Omega
\end{cases}
\]

where $\partial\Omega$ will denote the boundary of the bounded domain $\Omega$ and $\Delta$ denotes the Laplace operator, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

Green’s identity

\[
\int_{\partial\Omega} (- \text{div } \nabla u) v \ dx = \int_{\partial\Omega} \nabla u \cdot \nabla v \ dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v. \]
Find $u$ in $H^1_0(\Omega)$ such that

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v, \quad \forall v \in H^1_0(\Omega),$$

Biharmonic equation

$$\begin{cases}
\Delta^2 u &= f \text{ in } \Omega \\
u &= g \text{ on } \partial \Omega \\
\frac{\partial u}{\partial n} &= h \text{ in } \partial \Omega,
\end{cases}$$

$$\int_{\Omega} \Delta u \Delta v \, dx = \int_{\Omega} f v, \quad \forall v \in H^2_0(\Omega), \quad (1)$$

Abstract variational problem Find $u \in V$ such that

$$a(u, v) = \langle l, v \rangle \quad \text{for all } v \in V$$

Lax Milgram lemma
Discrete approximations Find \( u \in V_h \) such that

\[
a(u, v) = \langle l, v \rangle \quad \text{for all } v \in V_h
\]

Cea’s lemma

\[
\| u - u_h \|_V \leq C \min_{v \in V_h} \| u - v \|_V
\]

for a constant \( C \) independent of \( h \).

Requirements for conforming approximations

Let \( k \geq 1 \) and suppose \( \Omega \) is bounded. Then a piecewise infinitely differentiable function \( v : \overline{\Omega} \to \mathbb{R} \) belongs to \( H^k(\Omega) \) if and only if \( v \in C^{k-1}(\overline{\Omega}) \).

Even in two dimensions, conforming finite element spaces can be very complicated and there are no satisfactory answer in three dimensions.

Nonconforming approximations are not very popular.
Discrete approximations Find $u \in V_h$ such that

$$a(u, v) = \langle l, v \rangle \quad \text{for all } v \in V_h$$

Cea’s lemma

$$\|u - u_h\|_V \leq C \min_{v \in V_h} \|u - v\|_V$$

for a constant $C$ independent of $h$.

Requirements for conforming approximations

Let $k \geq 1$ and suppose $\Omega$ is bounded. Then a piecewise infinitely differentiable function $v : \overline{\Omega} \rightarrow \mathbb{R}$ belongs to $H^k(\Omega)$ if and only if $v \in C^{k-1}(\overline{\Omega})$.

Even in two dimensions, conforming finite element spaces can be very complicated and there are no satisfactory answer in three dimensions.

Nonconforming approximations are not very popular.
Discrete approximations

Find \( u \in V_h \) such that

\[
a(u, v) = \langle l, v \rangle \quad \text{for all } v \in V_h
\]

Cea’s lemma

\[
\| u - u_h \|_V \leq C \ \min_{v \in V_h} \| u - v \|_V
\]

for a constant \( C \) independent of \( h \).

Requirements for conforming approximations

Let \( k \geq 1 \) and suppose \( \Omega \) is bounded. Then a piecewise infinitely differentiable function \( v : \overline{\Omega} \rightarrow \mathbb{R} \) belongs to \( H^k(\Omega) \) if and only if \( v \in C^{k-1}(\overline{\Omega}) \).

Even in two dimensions, conforming finite element spaces can be very complicated and there are no satisfactory answer in three dimensions.

Nonconforming approximations are not very popular.
Stokes equations Find \((u, p) \in H^1(\Omega)^n \times L^2_0(\Omega)\) such that

\[
\begin{aligned}
-\nu \Delta u + \nabla p &= f \quad \text{in } \Omega \\
\text{div } u &= 0 \quad \text{in } \Omega \\
u \Delta u &= g \quad \text{on } \partial \Omega
\end{aligned}
\]

Linear elasticity equations Find \((\sigma, u)\) in \(H(\text{div}, \Omega, S) \times L^2(\Omega, \mathbb{R}^n)\) such that

\[
\begin{aligned}
A\sigma &= \varepsilon(u) \\
\text{div } \sigma &= f \\
u \Delta u &= g \quad \text{on the boundary}
\end{aligned}
\]

Equilibrium conditions are very difficult to enforce in the finite element method

G. Awanou, Spline element method for elasticity, Coming soon
Stokes equations Find \((u, p) \in H^1(\Omega)^n \times L^2_0(\Omega)\) such that

\[
\begin{align*}
-\nu \Delta u + \nabla p &= f \quad \text{in } \Omega \\
\text{div } u &= 0 \quad \text{in } \Omega \\
u \quad u &= g \quad \text{on } \partial \Omega
\end{align*}
\]

Linear elasticity equations Find \((\sigma, u)\) in \(H(\text{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^n)\) such that

\[
\begin{align*}
A\sigma &= \varepsilon(u) \\
\text{div } \sigma &= f
\end{align*}
\]

\(u = g\) on the boundary and \(A\sigma = \frac{1}{2\mu} \sigma - \frac{\lambda}{4\mu(\mu+\lambda)} \text{tr} \sigma \mathbf{1}\),

Equilibrium conditions are very difficult to enforce in the finite element method

G. Awanou, Spline element method for elasticity, Coming soon
1. Why multivariate splines for PDEs?
   - Motivation
   - Features of the method
     - Approximation properties

2. Refinement of tetrahedra and saddle point problems
   - Uniform refinement of a tetrahedron
   - Numerical methods for saddle point problems

3. Linear problems

4. Robustness of the method for singular perturbation problems
   - Fourth order singular problem
   - Darcy-Stokes equation

5. Nonlinear problems
   - Navier-Stokes equations
Why multivariate splines for PDEs?

Refinement of tetrahedra and saddle point problems

Linear problems

Robustness of the method for singular perturbation problems

Nonlinear problems

Motivation

Features of the method

Approximation properties

\[ V_h = \{ c \in \mathbb{R}^N, Rc = 0 \}, \]

\[ W_h = \{ c \in \mathbb{R}^N, Rc = G \} \]

The condition \( a(u, v) = \langle l, v \rangle \) for all \( v \in V_h \) becomes

\[ K(c)d = L^T d \quad \forall d \in V_h, \text{ that is for all } d \text{ with } Rd = 0 \]

\[ K(c, c) + \lambda^T R = L^T. \]

\[
\begin{bmatrix}
K^T & R^T \\
R & 0
\end{bmatrix}
\begin{bmatrix}
c \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
L \\
G
\end{bmatrix}
\]
\[ V_h = \{ c \in \mathbb{R}^N, Rc = 0 \}, \]
\[ W_h = \{ c \in \mathbb{R}^N, Rc = G \} \]

The condition \( a(u, v) = \langle l, v \rangle \) for all \( v \in V_h \) becomes

\[ K(c)d = L^T d \quad \forall d \in V_h, \text{ that is for all } d \text{ with } Rd = 0 \]

\[ K(c, c) + \lambda^T R = L^T. \]

\[
\begin{bmatrix}
K^T & R^T \\
R & 0
\end{bmatrix}
\begin{bmatrix}
c \\
\lambda
\end{bmatrix}
= 
\begin{bmatrix}
L \\
G
\end{bmatrix}
\]
Advantages of the method

- Can be applied to a wide range of PDEs in science and engineering in both two and three dimensions.
- Constraints and Smoothness are enforced exactly and there is no need to implement basis functions with the required properties. Particularly suitable for higher order PDEs.
- No inf-sup condition
- One gets in a single implementation approximations of variable order.
- The mass and stiffness matrices are assembled easily and this can be done in parallel.
- Easy implementation of p-adaptive approximation and simplicity of a posteriori error estimate.
Possible Disadvantages

- Large size matrices for 3D problems and high order approximations

\[
\begin{bmatrix}
K^T & R^T \\
R & 0
\end{bmatrix}
\begin{bmatrix}
c \\
\lambda
\end{bmatrix}
= \begin{bmatrix}
L \\
G
\end{bmatrix}
\begin{bmatrix}
K^T & R^T \\
R & -\mu I
\end{bmatrix}
\begin{bmatrix}
c^{(l+1)} \\
\lambda^{(l+1)}
\end{bmatrix}
= \begin{bmatrix}
L \\
G - \mu \lambda^{(l)}
\end{bmatrix}
\]

Computing \(c^{(1)}\) from \(\lambda^{(0)}\), one solves

\[
(K^T + \frac{1}{\mu} R^T R)c^{(l+1)} = K^T c^{(l)} + \frac{1}{\mu} R^T G, \quad l = 1, 2, \ldots
\]

\[
\|c - c^{(l+1)}\| \leq C\mu \|c - c^{(l)}\|
\]
Why multivariate splines for PDEs?

- Motivation
- Features of the method
- Approximation properties

Refinement of tetrahedra and saddle point problems

- Uniform refinement of a tetrahedron
- Numerical methods for saddle point problems

Linear problems

Robustness of the method for singular perturbation problems

- Fourth order singular problem
- Darcy-Stokes equation

Nonlinear problems

- Navier-Stokes equations
Recall Cea’s lemma $\|u - u_h\|_V \leq C \min_{v \in V_h} \|u - v\|_V$

Assume $V_h$ is a spline space $S^r_d(T_h)$

Bivariate splines For $d \geq 3r + 2$ and $0 \leq m \leq d$ and for $0 \leq k \leq m$.

$$|f - Qf|_{p,k} \leq C h^{m+1-k} |f|_{p,m+1}$$

The constant $C$ depends on the smallest angle in $\triangle$

Trivariate splines For $0 \leq k \leq d$, $f \in W^{d+1}_p(\Omega)$

$$|f - Qf|_{p,k} \leq C h^{d+1-k} |f|_{p,d+1}$$

The constant $C$ depends on the shape parameter $\sigma_K = h_K / \rho_K$
Recall Cea’s lemma \(|u - u_h|_V \leq C \min_{v \in V_h} |u - v|_V\)
Assume \(V_h\) is a spline space \(S_d^r(\mathcal{T}_h)\)

**Bivariate splines** For \(d \geq 3r + 2\) and \(0 \leq m \leq d\) and for \(0 \leq k \leq m\).

\[
|f - Qf|_{p,k} \leq Ch^{m+1-k}|f|_{p,m+1}
\]

The constant \(C\) depends on the smallest angle in \(\triangle\)

**Trivariate splines** For \(0 \leq k \leq d\), \(f \in W_d^{d+1}(\Omega)\)

\[
|f - Qf|_{p,k} \leq Ch^{d+1-k}|f|_{p,d+1}
\]

The constant \(C\) depends on the shape parameter \(\sigma_K = h_K/\rho_K\)
Why multivariate splines for PDEs?

1. **Motivation**
2. **Features of the method**
3. **Approximation properties**

2. **Refinement of tetrahedra and saddle point problems**
   - Uniform refinement of a tetrahedron
   - Numerical methods for saddle point problems

3. **Linear problems**

4. **Robustness of the method for singular perturbation problems**
   - Fourth order singular problem
   - Darcy-Stokes equation

5. **Nonlinear problems**
   - Navier-Stokes equations
Non degenerate tetrahedron \( \sigma_T = \frac{h_T}{\rho_T} < \infty \)

Quasi-uniform tetrahedral partition \( \sigma_T = \frac{h_T}{\rho_T} \leq \sigma < \infty, \ \forall T \in \mathcal{T}_h \)

Care must be taken for uniform refinement

<table>
<thead>
<tr>
<th>Tetrahedra</th>
<th>Sigma</th>
<th>Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.1815</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>6.8102</td>
<td>2</td>
</tr>
<tr>
<td>64</td>
<td>10.1948</td>
<td>4</td>
</tr>
<tr>
<td>512</td>
<td>16.6254</td>
<td>10</td>
</tr>
<tr>
<td>4096</td>
<td>26.8399</td>
<td>24</td>
</tr>
</tbody>
</table>

Example of uniform refinement

<table>
<thead>
<tr>
<th>Tetrahedra</th>
<th>Sigma</th>
<th>Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.1815</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>4.1815</td>
<td>1</td>
</tr>
<tr>
<td>64</td>
<td>4.1815</td>
<td>1</td>
</tr>
<tr>
<td>512</td>
<td>4.1815</td>
<td>1</td>
</tr>
<tr>
<td>4096</td>
<td>4.1815</td>
<td>1</td>
</tr>
</tbody>
</table>
1. **Why multivariate splines for PDEs?**
   - Motivation
   - Features of the method
   - Approximation properties

2. **Refinement of tetrahedra and saddle point problems**
   - Uniform refinement of a tetrahedron
   - Numerical methods for saddle point problems

3. **Linear problems**

4. **Robustness of the method for singular perturbation problems**
   - Fourth order singular problem
   - Darcy-Stokes equation

5. **Nonlinear problems**
   - Navier-Stokes equations
\[
\begin{pmatrix}
A & L^T \\
L & 0
\end{pmatrix}
\begin{pmatrix}
c \\
\lambda
\end{pmatrix}
= \begin{pmatrix}
F \\
G
\end{pmatrix},
\]

Assume the system above has a unique solution \(c\).

\[
\begin{pmatrix}
A & L^T \\
L & -\epsilon I
\end{pmatrix}
\begin{pmatrix}
c^{(l+1)} \\
\lambda^{(l+1)}
\end{pmatrix}
= \begin{pmatrix}
F \\
G - \epsilon \lambda^{(l)}
\end{pmatrix},
\] \hspace{1cm} (2)

where \(\lambda^{(0)}\) is a suitable initial guess e.g. \(\lambda^{(0)} = 0\).

Assume that \(A_s = \frac{1}{2}(A + A^T)\) the symmetric part of \(A\) is positive definite with respect to \(L\), i.e., \(x^T A_s x \geq 0\) and \(x^T A_s x = 0\) with \(Lx = 0\) implies \(x = 0\). Then, the sequence \((c^{(l+1)})\) defined by the iterative method converges to the solution \(c\) for any \(\epsilon > 0\).

Furthermore,

\[
\|c - c^{(l+1)}\| \leq C\epsilon \|c - c^{(l)}\|
\]

for some constant \(C\) independent of \(\epsilon\) and \(l\).
Why multivariate splines for PDEs?

Refinement of tetrahedra and saddle point problems

Linear problems

Robustness of the method for singular perturbation problems

Nonlinear problems

Numerical methods for saddle point problems

\[ A c^{(l+1)} + L^T \lambda^{(l+1)} = F \quad \text{and} \quad (1) \]

\[ L c^{(l+1)} - \epsilon \lambda^{(l+1)} = G - \epsilon \lambda^{(l)} \quad (2). \]

Multiplying (2) on the left by \( L^T \) and substituting \( L^T \lambda^{(l+1)} \) into (1) and rewriting (2), we get

\[ (A + \frac{1}{\epsilon} L^T L) c^{(l+1)} = -L^T \lambda^{(l)} + F + \frac{1}{\epsilon} L^T G \quad (3) \]

\[ \lambda^{(l+1)} + \frac{1}{\epsilon} L c^{(l+1)} = \lambda^{(l)} + \frac{1}{\epsilon} G. \]

\( A + \frac{1}{\epsilon} L^T L \) is invertible

\[ (A + \frac{1}{\epsilon} L^T L)x = 0 \Rightarrow x = 0. \]
\[ 0 = x^T (A + \frac{1}{\epsilon} L^T L) x = x^T (A_s + \frac{1}{\epsilon} L^T L) x = x^T A_s x + \frac{1}{\epsilon} (Lx)^T (Lx) \]

It follows that \( x^T A_s x = 0 \) and \( (Lx)^T (Lx) = 0 \). so \( x = 0 \).

\( c^{(l+1)} \) converges to \( c \)

Put \( u^{(l+1)} = c^{(l+1)} - c \) and \( p^{(l+1)} = \lambda^{(l+1)} - \lambda \)

\[ \begin{cases} 
(A + \frac{1}{\epsilon} L^T L) u^{(l+1)} + L^T p^{(l)} = 0 \\
p^{(l+1)} = p^{(l)} + \frac{1}{\epsilon} Lu^{(l+1)}. 
\end{cases} \]

\[ \|p^{(l)}\|^2 - \|p^{(l+1)}\|^2 = \frac{2}{\epsilon} (A_s u^{(l+1)}, u^{(l+1)}) + \frac{1}{\epsilon^2} \|L u^{(l+1)}\|^2. \]

\( \{\|p^{(l)}\|\} \) is seen to be decreasing, bounded below by 0 so cvges

Use positive definiteness of \( A_s + \frac{1}{\epsilon} L^T L \) to conclude
We prove that \( \| c - c^{(l+1)} \| \leq C \epsilon \| c - c^{(l)} \| \)

\[
\begin{aligned}
(A + \frac{1}{\epsilon} L^T L)u^{(l+1)} + L^T p^{(l)} &= 0 \\
p^{(l+1)} &= p^{(l)} + \frac{1}{\epsilon} Lu^{(l+1)},
\end{aligned}
\]

\[
Au^{(l+1)} + L^T p^{(l+1)} = 0
\]

We write \( u^{(l+1)} = \hat{u}^{(l+1)} + \overline{u}^{(l+1)} \) with \( \hat{u}^{(l+1)} \in \text{Ker}(L) \) and \( \overline{u}^{(l+1)} \in \text{Im}(L^T) \). Note that \( L : \text{Im}(L^T) \rightarrow \text{Im}(L) \) has a bounded inverse, so there exists \( k_0 > 0 \) such that

\[
\| \overline{u}^{(l+1)} \| \leq \frac{1}{k_0} \| Lu^{(l+1)} \|,
\]

from which it follows that

\[
\| u^{(l+1)} \| \leq \frac{2\epsilon}{k_0} \| p^{(l)} \|
\]
To get a bound on \( \| \hat{u}^{(l+1)} \| \), we notice that \( A \) is invertible on \( \text{Ker}(L) \) since \( A + \frac{1}{\varepsilon} L^T L \) is invertible. This gives for some \( \alpha_0 > 0 \),

\[
\| \hat{u}^{(l+1)} \| \leq \frac{1}{\alpha_0} \sup_{\nu_0 \in \text{Ker}(L)} \frac{(\nu_0, A\hat{u}^{(l+1)})}{\| \nu_0 \|} = \sup_{\nu_0 \in \text{Ker}(L)} \frac{-\nu_0^T A\bar{u}^{(l+1)}}{\| \nu_0 \|} \leq \| A \| \| \bar{u}^{(l+1)} \|.
\]

Putting together, we obtain

\[
\| u^{(l+1)} \| \leq C\varepsilon \| p^{(l)} \|, \quad \text{for some constant} \ C > 0
\]

To finish, we need a bound on \( \| p^{(l)} \| \) in terms of \( \| u^{(l)} \| \). It can be shown that one can choose \( \lambda_0 \) such that \( p^{(l)} \in \text{Im}(L) \) and since \( L^T : \text{Im}(L) \to \text{Im}(L^T) \) has a bounded inverse,

\[
\| p^{(l)} \| \leq \frac{1}{k_0} \| L^T p^{(l)} \|.
\]

This completes the proof since \( L^T p^{(l)} = -Au^{(l)} \).
Trivariate splines

Let $d \geq 1$ and $r \geq 0$

$$S_d^r(\Omega) = \{ p \in C^r(\Omega), \ p|_t \in P_d, \ \forall t \in T \}.$$

$$B_{ijkl}^d(v) = \frac{d!}{i!j!k!l!} b_1^i b_2^j b_3^k b_4^l, \quad i + j + k + l = d.$$  

The set $B^d = \{ B_{ijkl}^d(x, y, z), \ i + j + k + l = d \}$ is a basis for $P_d$.

$$S|_T = \sum_{i+j+k+l=d} c_{ijkl}^T B_{ijkl}^d,$$
Interpolation

On the tetrahedron $T = \langle v_1, v_2, v_3, v_4 \rangle$ at $\xi_{ijkl} = \frac{iv_1 + jv_2 + kv_3 + lv_4}{d}$

On the edge $\langle v_1, v_2 \rangle$ at $\xi_{ij} = \frac{iv_1 + jv_2}{d}$

\[
\sum_{i+j=d} \tilde{c}_{ij} \tilde{B}^d_{ij}(v), \quad \tilde{B}^d_{ij} = \frac{d!}{i!j!} b_1^i b_2^j.
\]

\[
p = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}, \quad q = \sum_{i+j=d} c_{ij0} B^d_{ij0}.
\]

\[
p = \sum_{i+j+k+l=d} c_{ijkl} B^d_{ijkl}, \quad q = \sum_{i+j+k=d} c_{ijk0} B^d_{ijk0}.
\]

\[Rc = c_b\]
Derivatives

\[ D_i c, \ i = 1, 2 \] encode respectively the B-net of \( \frac{\partial s}{\partial x_i} \).

Integration

\[ \int_\Omega pq = c^T ld \]

Smoothness conditions

This shows that there's a \((l, N)\) matrix \(H\) such that \(s\) is in \(C^r(\Omega)\) if and only if

\[ Hc = 0. \]
Why multivariate splines for PDEs?
Refinement of tetrahedra and saddle point problems
Linear problems
Robustness of the method for singular perturbation problems
Nonlinear problems

Poisson equation

<table>
<thead>
<tr>
<th>Tetrahedra</th>
<th>d=1</th>
<th>d=2</th>
<th>d=3</th>
<th>d=4</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>6.4017e+00</td>
<td>1.3922e+00</td>
<td>2.3880e-01</td>
<td>3.2070e-02</td>
</tr>
<tr>
<td>6*8=48</td>
<td>2.7623e+00</td>
<td>2.9100e-01</td>
<td>2.5136e-02</td>
<td>1.7554e-03</td>
</tr>
<tr>
<td>48*8=384</td>
<td>9.5226e-01</td>
<td>4.9066e-02</td>
<td>1.5654e-03</td>
<td>5.0882e-05</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Tetrahedra</th>
<th>d=5</th>
<th>d=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>4.0221e-03</td>
<td>3.9298e-04</td>
</tr>
<tr>
<td>6*8=48</td>
<td>7.9726e-05</td>
<td>4.6256e-06</td>
</tr>
</tbody>
</table>

\[ u = \exp(x + y + z) \]
### Biharmonic equation

<table>
<thead>
<tr>
<th>Tetrahedra</th>
<th>6</th>
<th>48</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=2</td>
<td>1.5571e-01</td>
<td>5.9921e-02</td>
</tr>
<tr>
<td>d=3</td>
<td>6.4337e-02</td>
<td>9.1870e-03</td>
</tr>
<tr>
<td>d=4</td>
<td>1.0803e-02</td>
<td>5.9974e-03</td>
</tr>
<tr>
<td>d=5</td>
<td>1.2830e-02</td>
<td>Out of memory</td>
</tr>
</tbody>
</table>

\[ u = \exp(-x^2 - y^2 - z^2) \]
Stokes equations

\[ u_1 = -\exp(x + 2y + 3z), \quad u_2 = 2 \exp(x + 2y + 3z), \quad u_3 = -\exp(x + 2y + 3z), \quad p = x(1 - x)z(1 - z) \]

Table 1 Approximation Errors from Trivariate Spline Spaces on \( I_1 \)

<table>
<thead>
<tr>
<th>degrees</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( 3.3633 \times 10 )</td>
<td>( 5.9431 \times 10 )</td>
<td>( 4.0397 \times 10 )</td>
<td>( 1.3466 \times 10^3 )</td>
</tr>
<tr>
<td>4</td>
<td>( 1.7010 \times 10 )</td>
<td>( 4.4374 \times 10 )</td>
<td>( 3.5368 \times 10 )</td>
<td>( 3.8562 \times 10^2 )</td>
</tr>
<tr>
<td>5</td>
<td>( 2.3804 )</td>
<td>( 7.3711 )</td>
<td>( 5.9629 )</td>
<td>( 9.8470 \times 10^1 )</td>
</tr>
<tr>
<td>6</td>
<td>( 3.9620 \times 10^{-1} )</td>
<td>( 1.2238 )</td>
<td>( 1.0311 )</td>
<td>( 2.7404 \times 10^1 )</td>
</tr>
<tr>
<td>7</td>
<td>( 6.7456 \times 10^{-2} )</td>
<td>( 1.9789 \times 10^{-1} )</td>
<td>( 1.6260 \times 10^{-1} )</td>
<td>( 6.8411 )</td>
</tr>
<tr>
<td>Rate</td>
<td>( 1.56 \times 10^{7} d^{-9.8294} )</td>
<td>( 3.22 \times 10^{7} d^{-9.6203} )</td>
<td>( 2.32 \times 10^{7} d^{-9.5463} )</td>
<td>( 8.50 \times 10^6 d^{-7.1353} )</td>
</tr>
</tbody>
</table>

Table 2 Approximation Errors from Trivariate Spline Spaces on \( I_2 \)

<table>
<thead>
<tr>
<th>degrees</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( 1.5083 \times 10 )</td>
<td>( 1.8709 \times 10 )</td>
<td>( 1.5222 \times 10 )</td>
<td>( 4.4382 \times 10^2 )</td>
</tr>
<tr>
<td>4</td>
<td>( 9.4142 \times 10^{-1} )</td>
<td>( 2.2094 )</td>
<td>( 1.8373 )</td>
<td>( 3.5278 \times 10^1 )</td>
</tr>
<tr>
<td>5</td>
<td>( 9.1619 \times 10^{-2} )</td>
<td>( 2.2322 \times 10^{-1} )</td>
<td>( 2.0176 \times 10^{-1} )</td>
<td>( 5.8199 )</td>
</tr>
<tr>
<td>6</td>
<td>( 8.5128 \times 10^{-3} )</td>
<td>( 2.3520 \times 10^{-2} )</td>
<td>( 1.9276 \times 10^{-2} )</td>
<td>( 7.1884 \times 10^{-1} )</td>
</tr>
<tr>
<td>Rate</td>
<td>( 9.31 \times 10^6 d^{-11.5631} )</td>
<td>( 1.24 \times 10^7 d^{-11.1692} )</td>
<td>( 1.09 \times 10^7 d^{-11.1901} )</td>
<td>( 1.05 \times 10^7 d^{-9.1064} )</td>
</tr>
</tbody>
</table>
Why multivariate splines for PDEs?
- Motivation
- Features of the method
- Approximation properties

Refinement of tetrahedra and saddle point problems
- Uniform refinement of a tetrahedron
- Numerical methods for saddle point problems

Linear problems

Robustness of the method for singular perturbation problems
- Fourth order singular problem
- Darcy-Stokes equation

Nonlinear problems
- Navier-Stokes equations
**Biharmonic Poisson**

\[
\epsilon^2 \Delta^2 u - \Delta u = f \quad \text{in } \Omega
\]

\[
u = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{in } \partial \Omega
\]

\[
V = W = H_0^2(\Omega), \quad \epsilon^2 \int_\Omega \Delta u \Delta v + \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv.
\]

\[
V_h = \{u \in S_d^1, u = 0 \text{ and } \partial u/\partial n = 0 \text{ on } \partial \Omega\}
\]

\[
\equiv \{c \in \mathbb{R}^N, Hc = 0, Rc = 0, Nc = 0\}
\]
Why multivariate splines for PDEs?
Refinement of tetrahedra and saddle point problems
Linear problems
Robustness of the method for singular perturbation problems
Nonlinear problems

Fourth order singular problem
Darcy-Stokes equation

\[
\begin{bmatrix}
\epsilon^2 B + K & H^T & R^T & N^T \\
H & 0 & 0 & 0 \\
R & 0 & 0 & 0 \\
N & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
c \\
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
MF \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

\[
\|\| u \|\|^2 = \epsilon^2 \sum_{T \in \mathcal{T}_h} \int_T D^2 u : D^2 v \, dx + \sum_{T \in \mathcal{T}_h} \int_T Du : Dv \, dx
\]

\[
\frac{\|\| u_h^l - u_h \|\|}{\|\| u_h^l \|\|}
\]
Theorem

For \( d \geq 3r + 2 \) and \( 0 \leq m \leq d \) and for \( 0 \leq k \leq m \).

\[
|f - Qf|_{p,k} \leq C|\Delta|^{m+1-k} |f|_{p,m+1}
\]

For \( u \in H^{d+1}(\Omega) \), for \( d \geq 5 \),

\[
\inf_{v \in S_d^1} \|u - v\|_\epsilon^2 \leq \epsilon^2 h^{2(d-1)} \|u\|_{d+1}^2 + h^{2d} \|u\|_{d+1}^2
\]

\[
= h^{2(d-1)}(h^2 + \epsilon^2) \|u\|_{d+1}^2
\]
Why multivariate splines for PDEs?
Refinement of tetrahedra and saddle point problems
Linear problems
Robustness of the method for singular perturbation problems
Nonlinear problems

Using cubic polynomials

<table>
<thead>
<tr>
<th>$\epsilon/h$</th>
<th>$2^{-3}$</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
<th>$2^{-6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^0$</td>
<td>2.1265 $10^{-1}$</td>
<td>1.0271 $10^{-1}$</td>
<td>1.9464 $10^{-2}$</td>
<td>3.3889 $10^{-3}$</td>
</tr>
<tr>
<td>$2^{-2}$</td>
<td>1.8980 $10^{-1}$</td>
<td>9.4724 $10^{-2}$</td>
<td>4.5625 $10^{-2}$</td>
<td>8.5138 $10^{-3}$</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>9.6005 $10^{-2}$</td>
<td>4.5406 $10^{-2}$</td>
<td>2.2429 $10^{-2}$</td>
<td>1.0786 $10^{-2}$</td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>4.0374 $10^{-2}$</td>
<td>1.4142 $10^{-2}$</td>
<td>6.3389 $10^{-3}$</td>
<td>3.0774 $10^{-3}$</td>
</tr>
<tr>
<td>$2^{-8}$</td>
<td>3.2036 $10^{-2}$</td>
<td>7.5016 $10^{-3}$</td>
<td>2.2452 $10^{-3}$</td>
<td>8.6681 $10^{-4}$</td>
</tr>
<tr>
<td>$2^{-10}$</td>
<td>3.1423 $10^{-2}$</td>
<td>6.8540 $10^{-3}$</td>
<td>1.6684 $10^{-3}$</td>
<td>4.4247 $10^{-4}$</td>
</tr>
<tr>
<td>Poisson (S)</td>
<td>2.3071 $10^{-2}$</td>
<td>5.2191 $10^{-3}$</td>
<td>1.2678 $10^{-3}$</td>
<td>3.1412 $10^{-4}$</td>
</tr>
<tr>
<td>Poisson</td>
<td>1.9685 $10^{-3}$</td>
<td>2.6203 $10^{-4}$</td>
<td>3.3516 $10^{-5}$</td>
<td>4.2260 $10^{-6}$</td>
</tr>
<tr>
<td>Biharmonic</td>
<td>2.1450 $10^{-1}$</td>
<td>1.0367 $10^{-1}$</td>
<td>1.965410 $^{-2}$</td>
<td>4.3614 $10^{-3}$</td>
</tr>
</tbody>
</table>

$$u(x, y) = (\sin(\pi x) \sin(\pi y))^2$$
Using polynomials of degree 4

<table>
<thead>
<tr>
<th>$\epsilon/h$</th>
<th>$2^{-3}$</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
<th>$2^{-6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^0$</td>
<td>$2.4378 \times 10^{-2}$</td>
<td>$5.8674 \times 10^{-3}$</td>
<td>$1.0959 \times 10^{-3}$</td>
<td>$2.7988 \times 10^{-4}$*</td>
</tr>
<tr>
<td>$2^{-2}$</td>
<td>$2.1598 \times 10^{-2}$</td>
<td>$5.1899 \times 10^{-3}$</td>
<td>$1.2694 \times 10^{-3}$</td>
<td>$2.3759 \times 10^{-4}$*</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>$1.0534 \times 10^{-2}$</td>
<td>$2.4702 \times 10^{-3}$</td>
<td>$6.0576 \times 10^{-4}$</td>
<td>$9.4533 \times 10^{-5}$</td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>$4.0397 \times 10^{-3}$</td>
<td>$7.6437 \times 10^{-4}$</td>
<td>$1.7134 \times 10^{-4}$</td>
<td>$4.1365 \times 10^{-5}$</td>
</tr>
<tr>
<td>$2^{-8}$</td>
<td>$2.9540 \times 10^{-3}$</td>
<td>$3.9977 \times 10^{-4}$</td>
<td>$6.1923 \times 10^{-5}$</td>
<td>$1.1852 \times 10^{-5}$</td>
</tr>
<tr>
<td>$2^{-10}$</td>
<td>$2.8664 \times 10^{-3}$</td>
<td>$3.6170 \times 10^{-4}$</td>
<td>$4.6177 \times 10^{-5}$</td>
<td>$6.2184 \times 10^{-6}$</td>
</tr>
<tr>
<td>Poisson (S)</td>
<td>$1.9956 \times 10^{-3}$</td>
<td>$2.5712 \times 10^{-4}$</td>
<td>$3.2459 \times 10^{-5}$</td>
<td>$4.5161 \times 10^{-6}$</td>
</tr>
<tr>
<td>Poisson</td>
<td>$2.4134 \times 10^{-4}$</td>
<td>$1.5286 \times 10^{-5}$</td>
<td>$9.5869 \times 10^{-7}$</td>
<td>$6.2866 \times 10^{-10}$</td>
</tr>
<tr>
<td>Biharmonic</td>
<td>$2.4605 \times 10^{-2}$</td>
<td>$5.9116 \times 10^{-3}$</td>
<td>$1.2668 \times 10^{-3}$</td>
<td>$3.0533 \times 10^{-4}$*</td>
</tr>
</tbody>
</table>

Gerard Awanou | Spline Element Method
1. Why multivariate splines for PDEs?
   - Motivation
   - Features of the method
   - Approximation properties

2. Refinement of tetrahedra and saddle point problems
   - Uniform refinement of a tetrahedron
   - Numerical methods for saddle point problems

3. Linear problems

4. Robustness of the method for singular perturbation problems
   - Fourth order singular problem
   - Darcy-Stokes equation

5. Nonlinear problems
   - Navier-Stokes equations
Why multivariate splines for PDEs?
Refinement of tetrahedra and saddle point problems
Linear problems
Robustness of the method for singular perturbation problems
Nonlinear problems

Fourth order singular problem
Darcy-Stokes equation

\[ u - \epsilon^2 \cdot u - \nabla p = f \text{ in } \Omega \]
\[ \text{div } u = g \text{ in } \Omega \]
\[ u = 0 \text{ on } \partial \Omega. \]

\[ W = \{ u \in H^1_0(\Omega)^2, \text{div } u = g \} \]
\[ V = \{ u \in H^1_0(\Omega)^2, \text{div } u = 0 \} \]

Find \( u \in W \), \( \epsilon^2 \int_{\Omega} \nabla u \cdot \nabla u + \int_{\Omega} u \cdot v = \int_{\Omega} f \cdot v, \forall v \in V. \)

\[ W_h = \{ c \in \mathbb{R}^N, Hc = 0, Rc = 0, Dc = G \} \]
Why multivariate splines for PDEs?
Refinement of tetrahedra and saddle point problems
Linear problems
Robustness of the method for singular perturbation problems
Nonlinear problems

Fourth order singular problem
Darcy-Stokes equation

\[
\begin{bmatrix}
\epsilon^2 K + M & H^T & R^T & D^T \\
H & 0 & 0 & 0 \\
R & 0 & 0 & 0 \\
D & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
c \\
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
MF \\
0 \\
0 \\
G \\
\end{bmatrix}
\]

\[|||v|||^2_{\epsilon} = ||v||^2 + \epsilon^2 ||\nabla v||^2\]

\[\tilde{V} = \{ v \in L^2(\Omega)^2, \text{div } v = 0, v \cdot n = 0 \text{ on } \partial \Omega \}.\]
Divergence-free vector fields

\[ \mathcal{V}^d = \{ \mathbf{f} \in (H^d(\Omega))^2, \mathbf{f} = \text{curl} \, \phi, \phi \in H^{d+1}(\Omega) \} \]

and

\[ \mathcal{V}_d = \{ \mathbf{s} \in (S^0_0(\Omega))^2, \mathbf{s} = \text{curl} \, S, S \in S^1_{d+1}(\Omega) \} \]

\[ \inf_{\mathbf{s} \in \mathcal{V}_d} \| \mathbf{f} - \mathbf{s} \|_2 \leq h^d \| \mathbf{f} \|_d \text{ and } \inf_{\mathbf{s} \in \mathcal{V}_d} \| \nabla \mathbf{f} - \nabla \mathbf{s} \|_2 \leq h^{d-1} \| \mathbf{f} \|_d, \]

\[ \inf_{\mathbf{s} \in \mathcal{V}_d} \| \mathbf{u} - \mathbf{s} \|_\epsilon \leq (\epsilon h^{d-1} + h^d) \| \mathbf{u} \|_d, d \geq 4. \]
Using cubic polynomials

<table>
<thead>
<tr>
<th>$\epsilon/h$</th>
<th>$2^{-2}$</th>
<th>$2^{-3}$</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-0}$</td>
<td>$1.0061 \times 10^{-1}$</td>
<td>$2.3718 \times 10^{-2}$</td>
<td>$5.8212 \times 10^{-3}$</td>
<td>$1.4467 \times 10^{-3}$</td>
</tr>
<tr>
<td>$2^{-2}$</td>
<td>$8.9696 \times 10^{-2}$</td>
<td>$2.1014 \times 10^{-2}$</td>
<td>$5.1490 \times 10^{-3}$</td>
<td>$1.2791 \times 10^{-3}$</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>$4.6793 \times 10^{-2}$</td>
<td>$1.0240 \times 10^{-2}$</td>
<td>$2.4505 \times 10^{-3}$</td>
<td>$6.0662 \times 10^{-4}$</td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>$2.3921 \times 10^{-2}$</td>
<td>$3.8922 \times 10^{-3}$</td>
<td>$7.5706 \times 10^{-4}$</td>
<td>$1.8105 \times 10^{-4}$</td>
</tr>
<tr>
<td>$2^{-8}$</td>
<td>$2.0934 \times 10^{-2}$</td>
<td>$2.8315 \times 10^{-3}$</td>
<td>$3.9537 \times 10^{-4}$</td>
<td>$1.3380 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.00</td>
<td>$2.0708 \times 10^{-2}$</td>
<td>$2.7403 \times 10^{-3}$</td>
<td>$3.5598 \times 10^{-4}$</td>
<td>$5.8378 \times 10^{-5}$*</td>
</tr>
<tr>
<td>Darcy</td>
<td>$4.3368 \times 10^{-3}$</td>
<td>$8.6596 \times 10^{-4}$</td>
<td>$1.7278 \times 10^{-5}$*</td>
<td>$1.9652 \times 10^{-2}$+</td>
</tr>
</tbody>
</table>

\[ u = \text{curl}\left(\sin(\pi x)^2 \sin(\pi y)^2\right), \quad \text{and} \ p = \sin(\pi x) \]
Pressure $P_3/P_4$

<table>
<thead>
<tr>
<th>$\epsilon/h$</th>
<th>$2^{-2}$</th>
<th>$2^{-3}$</th>
<th>$2^{-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^0$</td>
<td>7.9689</td>
<td>3.1380</td>
<td>8.9906 $10^{-1}$</td>
</tr>
<tr>
<td>$2^{-2}$</td>
<td>5.0369 $10^{-1}$</td>
<td>1.9471 $10^{-1}$</td>
<td>5.6019 $10^{-2}$</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>3.3121 $10^{-2}$</td>
<td>1.1486 $10^{-2}$</td>
<td>3.4025 $10^{-3}$</td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>2.9191 $10^{-3}$</td>
<td>6.8701 $10^{-4}$</td>
<td>1.9454 $10^{-4}$</td>
</tr>
<tr>
<td>$2^{-8}$</td>
<td>1.5185 $10^{-3}$</td>
<td>9.0396 $10^{-5}$</td>
<td>1.1731 $10^{-5}$</td>
</tr>
<tr>
<td>0.00</td>
<td>1.4690 $10^{-3}$</td>
<td>6.9076 $10^{-5}$</td>
<td>2.7063 $10^{-6}$</td>
</tr>
<tr>
<td>Darcy</td>
<td>1.4690 $10^{-3}$</td>
<td>6.9076 $10^{-5}$</td>
<td>2.7063 $10^{-6}$</td>
</tr>
</tbody>
</table>
Using polynomials of degree 4

<table>
<thead>
<tr>
<th>$\epsilon/h$</th>
<th>$2^{-2}$</th>
<th>$2^{-3}$</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-0}$</td>
<td>$1.7486 \times 10^{-2}$</td>
<td>$1.0465 \times 10^{-3}$</td>
<td>$6.0180 \times 10^{-5}$</td>
<td>$4.0017 \times 10^{-6}$*</td>
</tr>
<tr>
<td>$2^{-2}$</td>
<td>$1.5541 \times 10^{-2}$</td>
<td>$9.2597 \times 10^{-4}$</td>
<td>$5.3232 \times 10^{-5}$</td>
<td>$1.7016 \times 10^{-5}$+</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>$7.8996 \times 10^{-3}$</td>
<td>$4.4325 \times 10^{-4}$</td>
<td>$2.5327 \times 10^{-5}$</td>
<td>$1.5584 \times 10^{-4}$+</td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>$3.6643 \times 10^{-3}$</td>
<td>$1.4529 \times 10^{-4}$*</td>
<td>$2.0257 \times 10^{-5}$</td>
<td>$1.2314 \times 10^{-4}$+</td>
</tr>
<tr>
<td>$2^{-8}$</td>
<td>$3.0271 \times 10^{-3}$</td>
<td>$8.5252 \times 10^{-5}$</td>
<td>$8.3301 \times 10^{-6}$*</td>
<td>$1.0183 \times 10^{-2}$+</td>
</tr>
<tr>
<td>0.00</td>
<td>$2.9751 \times 10^{-3}$</td>
<td>$7.8673 \times 10^{-5}$*</td>
<td>$1.7288 \times 10^{-3}$+</td>
<td>$1.7788 \times 10^{-2}$+</td>
</tr>
<tr>
<td>Darcy</td>
<td>$6.4139 \times 10^{-4}$</td>
<td>$2.1203 \times 10^{-5}$*</td>
<td>$1.2037 \times 10^{-4}$+</td>
<td>$1.5822 \times 10^{-2}$+</td>
</tr>
</tbody>
</table>

Gerard Awanou  Spline Element Method
Pressure $P_4/P_5$

<table>
<thead>
<tr>
<th>$\epsilon/h$</th>
<th>$2^{-2}$</th>
<th>$2^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-0}$</td>
<td>7.5061</td>
<td>9.0088 $10^{-1}$</td>
</tr>
<tr>
<td>$2^{-2}$</td>
<td>4.7198 $10^{-1}$</td>
<td>5.6434 $10^{-2}$</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>2.9984 $10^{-2}$</td>
<td>3.5811 $10^{-3}$</td>
</tr>
<tr>
<td>$2^{-6}$</td>
<td>2.5980 $10^{-3}$</td>
<td>2.2229 $10^{-4}$</td>
</tr>
<tr>
<td>$2^{-8}$</td>
<td>1.1138 $10^{-3}$</td>
<td>1.7635 $10^{-5}$</td>
</tr>
<tr>
<td>0.00</td>
<td>1.0343 $10^{-3}$</td>
<td>9.3820 $10^{-8}$</td>
</tr>
<tr>
<td>Darcy</td>
<td>1.0343 $10^{-3}$</td>
<td>9.3820 $10^{-8}$</td>
</tr>
</tbody>
</table>
Why multivariate splines for PDEs?

- Motivation
- Features of the method
- Approximation properties

Refinement of tetrahedra and saddle point problems

- Uniform refinement of a tetrahedron
- Numerical methods for saddle point problems

Linear problems

Robustness of the method for singular perturbation problems

- Fourth order singular problem
- Darcy-Stokes equation

Nonlinear problems

- Navier-Stokes equations

Gerard Awanou
Spline Element Method
Why multivariate splines for PDEs?
Refinement of tetrahedra and saddle point problems
Linear problems
Robustness of the method for singular perturbation problems
Nonlinear problems

Spline Element Method

Navier-Stokes equations

\[
\begin{cases}
-\nu \, \Delta \mathbf{u} + \sum_{j=1}^{3} u_j \frac{\partial \mathbf{u}}{\partial x_j} + \nabla p = \mathbf{f} & \text{in } \Omega, \\
\text{div } \mathbf{u} = 0 & \text{in } \Omega.
\end{cases}
\]

\[V_0 = \{ \mathbf{v} \in H^1_0(\Omega)^3, \text{div } \mathbf{v} = 0 \}\]

\[L^2_0(\Omega) = \{ u \in L^2(\Omega), \int_{\Omega} u = 0 \}\]

and

\[H^{1/2}(\partial \Omega) = \{ \tau(u), u \in H^1(\Omega) \},\]
Weak formulation: Find $u \in H^1(\Omega)^3$ such that

$$
\nu \int_{\Omega} \nabla u \cdot \nabla v + \sum_{j=1}^{3} \int_{\Omega} u_j \frac{\partial u}{\partial x_j} \cdot v = \int_{\Omega} f \cdot v \quad \forall v \in V_0
$$

$$
\text{div } u = 0 \quad \text{in } \Omega
$$

$$
u K c + B(c)c + L^T \lambda = \bar{M}F
$$

$$
Lc = \bar{G}
$$
Why multivariate splines for PDEs?
Refinement of tetrahedra and saddle point problems
Linear problems
Robustness of the method for singular perturbation problems
Nonlinear problems

Navier-Stokes equations

Lid driven Cavity Flow

Fig.: 3D fluid profile in the $x - y$ plane
Why multivariate splines for PDEs?
Refinement of tetrahedra and saddle point problems
Linear problems
Robustness of the method for singular perturbation problems
Nonlinear problems
Gerard Awanou
Spline Element Method

Fig.: 3D fluid profile in the $y - z$ plane
Why multivariate splines for PDEs?
Refinement of tetrahedra and saddle point problems
Linear problems
Robustness of the method for singular perturbation problems
Nonlinear problems

Navier-Stokes equations

**Fig.** 3D fluid profile in the $x-z$ plane
References