On Recursive Refinement of Convex Polygons

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Abstract

It is known that one can improve the accuracy of the finite element solution of partial differential equations (PDE) by uniformly refining a triangulation. Similarly, one can uniformly refine a quadrangulation. Recently polygonal meshes have been used for numerical solution of partial differential equations based on virtual element methods, weak Galerkin methods, and polygonal spline methods. A refinement scheme of pentagonal partition was introduced in [6]. It is natural to ask if one can create a hexagonal refinement or general polygonal refinement schemes. In this short article, we show that one cannot refine a hexagon using hexagons of smaller size. In general, one can only refine an n-gon by n-gons of smaller size if $n \leq 5$.

1 Introduction

In the fields of computer aided geometry design of surfaces and numerical solutions of partial differential equations (PDE), triangulations have been the traditional way of partitioning spatial domains. Due to the recent development of the virtual element methods, weak Galerkin methods, and polygonal splines (see [2], [3], [12], [13], [15], [6]), one is able to use an arbitrary polygonal partition for numerical solutions of PDE. In addition, generalized barycentric coordinates (GBC) over arbitrary polygons of $n$ sides, $n$-gon for short, were invented for surface applications (cf. [5]). An excellent polygonal mesh generator can be found in [14]. It is known that we can uniformly refine a triangulation and a quadrangulation (cf. [11]) which is a common strategy to demonstrate the accuracy as well as the convergence of a numerical algorithm for solving a PDE. In particular, for polynomial finite elements or bivariate splines (cf. [1]), the uniform refinement of triangulations/quadrangulations enables the spline spaces to have the nestedness property which can be important for several applications, e.g. construction of a multi-resolution analysis which leads to wavelets or tight wavelet frames (cf. e.g. [7]) as well as construction of multi-grid methods for numerical solutions of PDE (cf. e.g. [4]).

Recently a refinement scheme of pentagonal partitions was introduced in [6], pictured in Fig. 1, and used to reduce the error in numerical solutions based on polygonal splines which consist of generalized Bernstein-Bézier functions in terms of GBC.

A natural question to ask is if one can create a hexagonal refinement, i.e. refine a hexagon by using hexagons of smaller size. In general, one can ask if one can create a general polygonal

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refinement scheme. In this short article, we will show that one cannot refine a hexagon by hexagons only. In fact, our arguments proved more. That is, one cannot refine an n-gon by n-gons of smaller size whenever \( n \geq 6 \). Hence, if one uses a polygonal mesh of single polygon type, then one cannot expect to generate the mesh starting from a few seeded n-gons with \( n \geq 6 \) by a recursive refinement scheme. This result will be shown in the next section. Then we shall discuss how to refine a general n-gon. We introduce a simple remedy refinement scheme of hexagons by using pentagons and one hexagon of smaller size. Similarly, a general n-gon can be refined by using pentagons and an n-gon of smaller size. In addition, we shall pose a few open questions about the possibility of refining a domain of general shape by using pentagons only. All these will be contained in §3.

2 Main Results and Proofs

2.1 Partitions of Polygons

**Definition 2.1.** Let \( V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^2 \) be a set of points. An edge \( e_k \) connecting \( v_{i_k} \) to \( v_{j_k} \) for some \( i_k, j_k \in \{1, 2, \ldots, n\} \) is defined as \( e_k = \{x \in \mathbb{R}^2 \mid x = tv_{i_k} + (1-t)v_{j_k}, 0 \leq t \leq 1\} \). Let \( E = \{e_k\}_{k=1}^N \) be a set of edges. We say \( P = (V, E) \) is a polygon with vertices \( V \) and edges \( E \) if

1. \( \forall v \in V \), there exists exactly two distinct edges \( e_{k_1}, e_{k_2} \in E \) such that \( e_{k_1} \cap e_{k_2} = v \)

2. \( \forall e_{k_1}, e_{k_2}, \text{ distinct, } e_{k_1} \cap e_{k_2} \text{ is either the empty set or exactly one vertex } v \in V \).

The somewhat technical definition is meant to eliminate “poorly” behaved polygons which self-intersect. With this definition, polygons serve to separate \( \mathbb{R}^2 \) into a clear interior and exterior piece.

**Definition 2.2.** The polygon \( P = (V, E) \) is degenerate if it contains two edges which intersect at a vertex \( v \) with an angle of 180°.

The remainder of this paper will require that \( P \) is nondegenerate. Any degenerate polygon can be made nondegenerate by simply fusing the two edges that form an angle 180° into a single edge and omitting the vertex where they intersect.

**Definition 2.3.** A partition of a polygon \( P = (V, E) \) is a planar graph \( G = (\hat{V}, \hat{E}, \hat{F}) \) with vertices \( \hat{V} \), edges \( \hat{E} \) and faces \( \hat{F} \) such that

1. \( V \subset \hat{V} \)
(2) \( \forall e \in E, \exists \hat{e}_{i_1}, \ldots, \hat{e}_{i_m} \in \hat{E} \text{ such that } e \subset \bigcup_{k=1}^{m} \hat{e}_{i_k} \)

(3) \( \forall \hat{v} \in \hat{V}, \text{ there exist at least two distinct edges } \hat{e}_{k_1}, \hat{e}_{k_2} \in \hat{E} \text{ such that } \hat{e}_{k_1} \cap \hat{e}_{k_2} = \hat{v} \)

(4) \( f \in F \) is a face if it is a polygon whose vertices and edges are elements of \( \hat{V} \) and \( \hat{E} \) respectively with the requirement that the interior of \( F \) contains no other vertices or edges of the graph \( \hat{G} \).

Requirements (1) and (2) guarantee the polygon \( P \) is still visible as a subset of the planar graph \( \hat{G} \).

Each of \( \hat{V} \), \( \hat{E} \) and \( \hat{F} \) is assumed to have finite cardinality. If \( |\hat{F}| \geq 2 \) we say the partition is nontrivial. Finally, let \( \hat{V}^i \subset \hat{V} \) denote the subset of interior vertices and \( \hat{V}^b \subset \hat{V} \) denote the subset of boundary vertices.

Figure 2: A hexagon can be partitioned into two nonconvex hexagons.

Figure 3: A hexagon can be partitioned into 4 degenerate hexagons.

2.2 Desirable Properties of Partitions

The interest in this topic was sparked by the idea of implementing finite element schemes subordinate to polygonal partitions instead of the traditional triangulations which are a special case. As a result, the following are some basic requirements for our needs.

- **Recursive refinement**: In traditional finite element schemes triangulations are recursively refined until a desired numerical accuracy is reached. We have a refinement scheme for pentagonal partitions, illustrated in Figure 1 which requires that all pentagons are convex. In order to refine nonconvex polygons, the refinement algorithm would have to deal with a number of corner cases, thus increasing code complexity considerably.
• **Shape regularity**: In the case of triangulations, the approximation power of finite elements depends on the quality of a triangulation $\Delta$, typically measured by $\sup_{T \in \Delta} \frac{|T|}{\rho_T}$ where $|T|$ is the length of the longest edge of the triangle $T$ and $\rho_T$ is the radius of the largest inscribed circle of $T$. The smaller this ratio, the better the approximation power. For a discussion of shape regularity of triangulations, see [4] and [11]. As for the approximation power of finite elements defined on polygons using Generalized Barycentric Coordinates, see [8].

As a simple illustration of a poor refinement scheme, Figure 2 shows that a hexagon can be refined into two nonconvex hexagons, but the resulting hexagons have the same diameter as the original which leads to poor shape regularity. The goal of a refinement scheme should be to reduce the maximum of the diameters of all polygons in the partition. A simple way to avoid this scenario is to require that all polygons in the partition be convex.

• **Nondegeneracy**: Degeneracy can give rise to trivial partitions as seen in Figure 3, where we have actually recovered a triangulation rather than some new partition scheme.

![Figure 4: Left: a pentagonal partition. Right: an illustration of the $\Delta$-complex construction. The interior vertices of the partition have been lifted in the $Z$-direction for clarity. The construction is topological and has no associated geometrical data.](image)

### 2.3 From Geometric Shapes to Topological Surfaces

Partitions of polygons should be regarded as geometric objects embedded in $\mathbb{R}^2$ where notions of convexity and angle are appropriate. The proof of our main result, however, relies on a topological argument and so our goal is to view the partition $\mathcal{G}$ as a $\Delta$-complex or a CW complex as discussed in [9]. Then $\hat{V}$ is the set of 0-cells, $\hat{E}$ is the set of 1-cells and $\hat{F}$ is the set of 2-cells.

Start with two identical partitions $\hat{\mathcal{G}}$ and identify (glue) the boundary edges of each copy of $\hat{\mathcal{G}}$ to produce the CW complex $\mathcal{G} = \{V, E, F\}$ with 0-cells $V$, 1-cells $E$ and 2-cells $F$. Since $|F| \geq 2$, we get $|F| \geq 4$. The result $\mathcal{G}$, which is visualized in Figure 4 is homeomorphic to a sphere and as a result its Euler characteristic is 2. We then have the following well-known theorem (see [9], p. 146).

**Theorem 2.1.** Let $\mathcal{G}$ be a finite CW complex. Then its Euler Characteristic is given by

$$\chi(\mathcal{G}) = \sum_n (-1)^n c_n$$

where $c_n$ is the number of $n$-cells of $\mathcal{G}$.
Theorem 2.1 asserts that our object $\mathcal{G}$ obeys the formula $|V| - |E| + |F| = 2$. Recall that the degree of a vertex is the number of incident edges to that vertex. We introduce the following notation.

\begin{align*}
V_k &= \text{number of vertices of degree } k \\
F_k &= \text{number of faces with } k \text{ vertices}
\end{align*}

By our definition of partition, specifically assumption (3), $V_1 = 0$. That is, each vertex should have at least two incident edges.

**Lemma 2.1.**

\begin{align*}
\sum_{k=2}^{\infty} V_k &= |V| \\
\sum_{k=2}^{\infty} F_k &= |F| \\
\sum_{k=2}^{\infty} kV_k &= 2|E| \\
\sum_{k=2}^{\infty} kF_k &= 2|E|
\end{align*}

**Proof.** This is a simple interpretation of our notation. Note that since we assume the partition is finite, all these sums are finite. 

Let us substitute the sums in Lemma 2.1 into this modified Euler characteristic equation

$2|V| - 2|E| + 2|F| = 4$

and obtain

\begin{align*}
\sum_{k=2}^{\infty} 2V_k - \sum_{k=2}^{\infty} kV_k + \sum_{k=2}^{\infty} 2F_k &= 4, \quad (2.1) \\
\sum_{k=2}^{\infty} 2V_k - \sum_{k=2}^{\infty} kF_k + \sum_{k=2}^{\infty} 2F_k &= 4. \quad (2.2)
\end{align*}

Add two copies of (2.1) to (2.2) to obtain the following equation:

\[ \sum_{k=2}^{\infty} (6 - 2k)V_k + \sum_{k=2}^{\infty} (6 - k)F_k = 12. \quad (2.3) \]

We now state the following Lemma without proof and then proceed to the main result in this paper.

**Lemma 2.2.** If each interior face in $\hat{\mathcal{G}}$ is convex and nondegenerate, then $\hat{V}^i$ contains no vertices of degree 2.

**Theorem 2.2.** If $\hat{\mathcal{G}}$ is a planar graph in which every face, including the exterior face, is a nondegenerate $n$-gon for some fixed $n$ and every interior face is convex, then $n \leq 5$.

**Proof.** We prove the result by contradiction. Assume such a partition $\hat{\mathcal{G}}$ exists for some $n \geq 6$. Let us construct a CW complex $\mathcal{G}$ as discussed earlier in the section. Specialize equation (2.3) to the assumptions of this theorem, namely, $F_k = 0$ for all $k \geq 2, k \neq n$. Thus, we have

\[ \sum_{k=2}^{\infty} (6 - 2k)V_k + (6 - n)F_n = 12. \quad (2.4) \]
Let us write out the first few terms of this sum.

$$ (6 - n)F_n + 2V_2 - 2V_4 - 4V_5 - 6V_6 - \cdots = 12. \quad (2.5) $$

Note that the coefficient of $V_3$ is zero and that $V_k \geq 0$, so all but a few of these terms are negative. If $n \geq 6$, then $(6 - n)F_n \leq 0$ as well. Throwing away all the negative terms from inequality $2.5$ we conclude $V_2 \geq 6$.

**Claim:** The only candidates for vertices of degree 2 in the CW complex $G$ are the vertices of the original convex $n$-gon $P$. In other words,

$$ V_2 \leq n. \quad (2.6) $$

We now prove the claim. We know from Lemma $2.2$ that no interior vertices of $\hat{G}$ have degree 2, so we must look at vertices on the boundary. Any vertex $v$ on the boundary of $\hat{G}$ is either a vertex of the original $n$-gon $P$ or is contained in an edge $(v_i, v_{i+1})$. In the latter case, the degree of $v$ would be at least 3 since there are edges leading to $v_i, v_{i+1}$ and to some interior vertex. Otherwise there would exist a degenerate face, which the assumptions of the theorem forbid. Thus, the only candidates for vertices of degree 2 are the $n$ vertices of the original $n$-gon $P$, which proves the claim.

We now examine two cases regarding the number of vertices of the original polygon, each of which leads to a contradiction. Thus, the implication is that no such partition $\hat{G}$ exists.

**Case 1:** $n \geq 7$. Since we assumed the partition is nontrivial, we know $F_n \geq 4$ so $(6 - n)F_n \leq 4(6 - n)$. Then starting with $(2.5)$ and using $(2.6)$ we get

$$ 12 = (6 - n)F_n + 2V_2 - 2V_4 - 4V_5 - 6V_6 - \cdots \leq (6 - n)F_n + 2V_2 \leq 4(6 - n) + 2n = 24 - 2n. $$

That is, $2n \leq 12$ or $n \leq 6$ which is a contradiction.

**Case 2:** All that remains is to examine the interesting edge case of $n = 6$, so the entire partition is composed of hexagons. In that case equation $(2.5)$ specializes to

$$ 2V_2 - 2V_4 - 4V_5 - 6V_6 - \cdots = 12 \quad (2.7) $$

It follows that $6 \leq V_2$. Also, from $(2.6)$, we have $V_2 \leq 6$ and hence, $V_2 = 6$. So we are forced to conclude that $V_k = 0$ for all $k \geq 4$. However, we can show that at least $V_4 \neq 0$.

Indeed, since the partition $\hat{G}$ is nontrivial, there must be an interior edge $E'$ leading from a boundary edge $E_i$ to the interior of the convex hull of $P$. But how is $E'$ attached to the boundary?
Figure 6: Left: An edge $E'$ connects an interior point to a newly created vertex $v$ on a boundary edge. Right: The resulting CW complex after gluing two copies of the partition along their boundaries. The degree of $v$ is at least 4.

If it were attached to one of the original vertices $v_i \in P$, then the degree of $v_i$ in the graph $\hat{G}$ would be at least 3 and upon constructing the CW complex $G$, the degree of $v_i$ would climb to at least 4. See Figure 5 for an illustration. As we asserted above that we need $V_2 = 6$ and the vertices $v_i \in P$ are the only candidates for this job, we cannot connect an interior edge to $v_i$.

Now suppose the edge $E'$ connects a vertex $v$ on the boundary edge $E_i$ and $v \notin P$. Then the degree of that vertex in the partition $\hat{G}$ is at least 3 and upon constructing the CW complex $G$, the degree of $v$ would climb to at least 4. Now we have $V_4 \geq 1$ which contradicts (2.7). See Figure 6 for an illustration.

From Theorem 2.2 we conclude the following

**Corollary 2.1.** If $n \geq 6$, an $n$-gon cannot be refined by using convex, nondegenerate $n$-gons of smaller size.

**Remark 2.1.** The inspiration for this proof was drawn from [10]. In his book the author uses topology to treat a number of popular problems concerning polyhedra. In particular, there is a discussion on why at least twelve pentagons are necessary for the construction of a soccer ball.

### 3 Conclusion and Remarks

Traditionally, people use triangulations for numerical solutions of PDE. Several groups of researchers have started exploring other options such as the viability of polygonal meshes for the purpose of solving PDE. We have shown in the previous section that one is not able to refine an $n$-gon using $n$-gons of smaller size for $n \geq 6$. This result is a roadblock preventing us from producing a mesh of single polygon type starting from one $n$-gon with $n \geq 6$. However, there are a few simple remedies. Let us explain a scheme for refining a hexagon by using pentagons together with a hexagon of smaller size. This is shown in Fig. 7. Similarly, we can refine an $n$-gon with $n \geq 7$ using pentagons of smaller size together with a small $n$-gon as shown in Figures 8 and 9. These imply that if one uses polygonal meshes to numerically solve PDE, one has to construct elements or basis functions over polygons of more than one type.

Indeed, we can use such a refinement of hexagons for numerical solutions of PDE, e.g. Poisson’s equation. In Fig. 10 we present a numerical solution based on a modified pentagonal partition (all pentagons except for one hexagon on the top of the dome and several quadrilaterals around the boundary). The graph is produced by using the MATLAB codes based on the polygonal spline basis functions of second order constructed in [6]. See [6] for polygonal splines for numerical solution of PDE.
Figure 7: A hexagon and its refinement using pentagons

Figure 8: A heptagon and its refinement using pentagons

Figure 9: An octagon and its refinement using pentagons
Finally, let us present some remarks in order.

**Remark 3.1.** Of course, there are many other ways to refine an $n$-gon if one uses pentagons. We present a different way to refine a convex $n$-gon using pentagons for $n = 6, 7, 8$ illustrated as in Fig. 11. It is clear that we can refine any convex $n$-gon for $n \geq 9$ using convex pentagons.

**Remark 3.2.** It is interesting to know if one can refine a triangle or quadrilateral by using pentagons of smaller sizes. To the best of our knowledge, no one knows how to do that so far. Such a refinement of triangles will be useful to partition any domain into pentagons. It is our belief that the method of proof in this paper can be adapted to answer this question.

**Remark 3.3.** Another open problem is to partition any polygonal domain $\Omega$ into convex pentagons. That is, $\Omega = \bigcup_{i} p_{i}$, where $p_{i}$ is a convex pentagon for each $i = 1, \ldots, n$ and the intersection of any two pentagons is either the empty set or their common edge or their common vertex. It would be interesting to create such a scheme. If the open problem in Remark 3.2 can be solved, one can first use a Voronoi diagram to partition a polygon and then convert all polygons into pentagons. Indeed, for each $n$-gon, we use Remark 3.2 for $n \leq 4$ and use Remark 3.1 for $n \geq 6$.

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Figure 11: Subdivision of hexagon, heptagon, and octagon into pentagonal partitions

References


