A Polygonal Spline Method for General 2nd-Order Elliptic Equations and Its Applications

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Abstract

We explain how to use polygonal splines to numerically solve second-order elliptic partial differential equations. The convergence of the polygonal spline method will be studied. Also, we will use this approach to numerically study the solution of some mixed parabolic and hyperbolic partial differential equations. Comparison with standard bivariate spline method will be given to demonstrate that our polygonal splines have some better numerical performance.

1 Introduction

Traditionally, people use triangulations to numerically solve partial differential equations (PDE). Recently, a new trend is to use more general polygonal meshes. That is, we can be more versatile and efficient than the standard finite element method when numerically solving PDEs. See [3], [4], and [5] for a so-called virtual element method to solve PDEs based on arbitrary polygonal meshes. In [15], the researchers use \( C^0 \) GBC elements to solve the standard Poisson equations in 2D and 3D settings over arbitrary convex polygonal partitions. In [18], the researchers construct \( C^0 \) quadratic finite elements based on GBC to solve PDEs. In [17], these researchers use their weak Galerkin method based on rectangular partitions to solve second order elliptic PDEs. In [10], the researchers constructed a class of continuous polygonal finite elements of arbitrary order \( d \) which allows for reproduction of polynomials of total degree \( d \). These elements were implemented for numerical solution of Poisson equations. In this paper, we shall explore how to use these polygonal elements to solve general second order elliptic PDEs and some mixed parabolic and hyperbolic partial differential equations.

A model PDE problem considered in this paper can be described as follows. Let \( \Omega \) be a bounded open polyhedral domain in \( \mathbb{R}^2 \), and let \( \Gamma = \partial \Omega \) be the boundary of \( \Omega \). We consider the following general 2nd order PDE:

\[
\begin{cases}
    \mathcal{L}(u) = f, & \mathbf{x} \in \Omega \\
    u = g, & \mathbf{x} \in \Gamma,
\end{cases}
\]

where \( \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \) and \( \mathcal{L} \) is a partial differential operator in the following form

\[
\mathcal{L}(u) := -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} \left( A_{ij} \frac{\partial}{\partial x_i} u \right) + \sum_{k=1}^{2} B_k \frac{\partial}{\partial x_k} u + Cu,
\]

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with $A_{ij} \in L_\infty(\Omega)$, $B_k \in L_\infty(\Omega)$, $C \in L_\infty(\Omega)$, and $f$ is a function in $L_2(\Omega)$.

When the matrix $A = [A_{ij}]_{1 \leq i, j \leq 2}$ is symmetric and positive definite over $\Omega$, the PDE in (1.1) is said to be elliptic. A typical PDE of this type can be given by defining the operator $\mathcal{L}$ with the following weight functions: Let

$$
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = 
\begin{bmatrix}
\varepsilon + x & xy \\
x y & \varepsilon + y
\end{bmatrix}
$$

(1.3)

with $\varepsilon > 0$, $\mathbf{B} = (B_1, B_2) = (0, 0)$, and $C = \exp(-x^2 - y^2)$. Then the corresponding PDE is elliptic. For for any functions $f \in L^2(\Omega)$ and $g \in L_\infty(\partial \Omega)$ in (1.1), the PDE has a unique solution. See Theorem 1 in a later section.

There is a standard approach to use methods for solution of 2nd-order elliptic PDE to study hyperbolic equations, transport equations, and mixed parabolic and hyperbolic equations. Indeed, consider a singularly-perturbed elliptic PDE:

$$
-\varepsilon \Delta u + (2 - y^2)D_x u + (2 - x)D_y u + (1 + (1 + x)(1 + y)^2)u = f, \quad (x, y) \in \Omega = (0, 1) \times (0, 1)
$$

(1.4)

with $u|_{\partial \Omega} = g$, where $f$ and $g$ are any appropriate functions. When $\varepsilon = 0$, this is a hyperbolic test problem considered in [2,12,13]. One can numerically solve (1.4) for $\varepsilon > 0$ very small to approximate the solution of the hyperbolic problem with $\varepsilon = 0$.

For another example, the following is a singularly perturbed advection-diffusion problem:

$$
-\varepsilon \Delta u + D_x u + D_y u = f, \quad (x, y) \in \Omega = (0, 1) \times (0, 1)
$$

(1.5)

with $u|_{\partial \Omega} = g$, where $f$ and $g$ are any appropriate functions. This example was studied in [13].

For another example, the following problem is parabolic for $y > 0$ and hyperbolic for $y \leq 0$:

$$
-\varepsilon D_{yy} u + D_x u + c_1 u = 0, \quad (x, y) \in (1, 1) \times (0, 1)
$$

$$
D_x u + c_2 u = 0, \quad (x, y) \in (1, 1) \times (0, 0]
$$

(1.6)

with $u|_{\partial \Omega} = g$, for any constants $c_1 > 0$ and $c_2 > 0$. It was also studied in [13]. We can use the following general elliptic PDE to study the problem by considering

$$
-\eta D_{xx} u - \varepsilon D_{yy} u + D_x u + c_1 u = f_1, \quad (x, y) \in (1, 1) \times (0, 1)
$$

$$
-\eta \Delta u + D_x u + c_2 u = f_2, \quad (x, y) \in (1, 1) \times (0, 0]
$$

(1.7)

with $u|_{\partial \Omega} = g$ and $\eta > 0$, where $f_1$, $f_2$ and $g$ are any appropriate functions. We can approximate the solution to (1.7) by letting $\eta > 0$ go to zero and use spline functions which are not necessarily continuous at $y = 0$.

These examples show the usefulness of a numerical solution to the model problem (1.1) in this paper. On the other hand, it is known that not all second order elliptic PDE has a unique solution because of Fredholm alternative theorem (cf. e.g. [8]). Although there are many sufficient conditions to ensure the existence and uniqueness of the solution (1.1), it is interesting to know when such a PDE can be numerically solved and admits a numerical solution even without satisfying any sufficient conditions. These are our motivations to study (1.1). In the next section, we provide a standard sufficient condition to ensure the existence and uniqueness of the weak solution of (1.1) based on Lax-Milgram theorem. The PDEs listed above may not satisfy the sufficient condition. However, we are still able to find their numerical solutions by using our polygonal splines.
This second order elliptic PDE in (1.1) has been studied by many other methods before. For example, in [1], the researchers used bivariate spline method to numerically solve (1.1). In particular, given a triangulation ∆ of a domain Ω, let

\[ S_d^r(\Delta) = \{ s \in C^r(\Omega) : s|_T \in P_d, \forall T \in \Delta \} \] (1.8)

be the spline space of smoothness r and degree d ≥ r. In general, we need to use d ≥ 3r + 2 to have the spline space \( S_d^r(\Delta) \) to be nonempty (cf. [14]). The researchers use \( S_d^r(\Delta) \) and add the smoothness conditions \( Hc = 0 \) as side constraints to approximate the following weak solution:

\[
\sum_{i,j=1}^{2} \int_{\Omega} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{k=1}^{2} \int_{\Omega} [B_k \frac{\partial u}{\partial x_k}] v + \int_{\Omega} C u v = \int_{\Omega} f v 
\] (1.9)

for all test function \( v \in S_d^{-1}(\Delta) \). For another example, the researchers in [17] use the weak Galerkin method to solve (1.1). In this paper, we shall use the polygonal splines invented in [10] to solve (1.9). Polygonal splines have been shown to be more efficient than polynomial spline functions for numerical solution of the Poisson equation (cf. [10]). We shall use the polygonal splines to solve the general second order elliptic PDE and compare the numerical solutions with the solutions by using bivariate spline functions. We will continue to demonstrate that the numerical solution by using the polygonal splines are generally better than that by using splines of the same degrees.

The paper is organized as follows. We shall review polygonal splines in §2. Then we review the ellipticity concept to explain when the PDE in (1.1) has a solution in §3. Implementation and convergence of polygonal spline solution will be explained in §4. These are generalizations of the standard results for Poisson equation: the Céa lemma and the Aubin-Nitsche technique for the optimal convergence rate in \( L_2 \) norm over convex domain. Finally, we present many numerical results using the polygonal splines to solve the PDEs given above.

2 Preliminary on Polygonal Splines

Let us begin with generalized barycentric coordinates (GBCs). There are many ways to define barycentric coordinates in a polygon with \( n \) sides, \( n \geq 3 \); see [7]. We will restrict our attention to convex polygons in this paper.

Let \( P_n = (v_1, \ldots, v_n) \) be a convex polygon. Any functions \( \phi_i, i = 1, \ldots, n \), will be called generalized barycentric coordinates (GBCs) if, for all \( x \in P_n \), \( \phi_i(x) \geq 0 \) and

\[
\sum_{i=1}^{n} \phi_i(x) = 1, \quad \text{and} \quad \sum_{i=1}^{n} \phi_i(x)v_i = x. \] (2.1)

When \( P_n \) is a triangle, the coordinates \( \phi_1, \phi_2, \phi_3 \) are the usual barycentric coordinates. For \( n > 3 \), the \( \phi_i \) are not uniquely determined by (2.1), but they share the basic property that they are piecewise linear on the boundary of \( P_n \):

\[
\phi_i(v_j) = \delta_{ij} \quad \text{and} \quad \phi_i((1-\mu)v_j + \mu v_{j+1}) = (1-\mu)\phi_i(v_j) + \mu \phi_i(v_{j+1}), \quad \mu \in [0, 1]. \] (2.2)

Example 2.1 (Wachspress (rational) coordinates). For general \( n \geq 3 \), let \( n_i \in \mathbb{R}^2 \) be the outward unit normal to the edge \( e_i = [v_i, v_{i+1}] \), \( i = 1, \ldots, n \), and for any \( x \in P_n \) let \( h_i(x) \) be the perpendicular distance of \( x \) to the edge \( e_i \), so that

\[
h_i(x) = (v_i - x) \cdot n_i = (v_{i+1} - x) \cdot n_i. \] (2.3)
Let
\[ w_i(x) = d_i \prod_{j=1, j \neq i-1}^n h_j(x), \quad \text{and} \quad W = \sum_{j=1}^n w_j, \quad (2.4) \]
where \( d_i \) is the cross product
\[ d_i = n_{i-1} \times n_i = \begin{vmatrix} n_{i-1}^x & n_i^x \\ n_{i-1}^y & n_i^y \end{vmatrix}, \]
and \( n_j = (n_j^x, n_j^y) \) is the normal of edge \([v_i, v_{i+1}]\). Then the functions \( \phi_i = w_i/W, \ i = 1, \ldots, n \) are GBCs, which are rational functions of degree \((n-2, n-3)\). See [9] for several other representations of these coordinates.

For a convex polygon \( P_n \) with \( n \geq 3 \) sides, let \( \phi_1, \ldots, \phi_n \) be a set of GBCs. For any \( d \geq 0 \), and any multi-index \( j = (j_1, \ldots, j_n) \in \mathbb{N}_0^n \) with \(|j| := j_1 + \cdots + j_n = d\), let
\[ B^d_j(x) = \frac{d!}{j_1! \cdots j_n!} \phi_1^{j_1}(x) \cdots \phi_n^{j_n}(x), \quad x \in P_n, \quad (2.5) \]
which we will call a Bernstein-Bézier function. Note that for \( n > 3 \), \( B^d_j \) is no longer a polynomial in general. For any \( n \), define \( \Phi^d(P_n) \) as the linear space of functions of the form
\[ s(x) = \sum_{|j|=d} c_j B^d_j(x), \quad x \in P_n, \quad c_j \in \mathbb{R}. \quad (2.6) \]
The following properties are known:

1. \( \Pi_d \subset \Phi^d(P_n) \), where \( \Pi_d \) is the space of polynomials of degree \( \leq d \).

2. Due to (2.2), the function \( s(x) \) in (2.6) is a univariate polynomial of degree \( \leq d \) on each edge of the polygon.

3. When \( n \geq 4 \), the functions \( B^d_j \in \Phi^d(P_n) \) are not linearly independent.

Based on the polynomial blossom property, when \( d = 2 \) we can construct a basis for a subspace which still contains the space of quadratic polynomials \( \Pi_2 \):

**Theorem 2.2.** Let
\[ F_i = \phi_i \lambda_{i,0}, \quad F_{i,1} = \phi_i \lambda_{i,1} + \phi_{i+1} \lambda_{i+1,-1}, \quad i = 1, \ldots, n, \]
and
\[ \Psi_2(P_n) = \text{span}\{F_i, F_{i,1}, i = 1, \ldots, n\}. \quad (2.7) \]
Then \( \Pi_2 \subset \Psi_2(P_n) \subset \Phi_2(P_n) \).

By specializing to specifically Wachspress coordinates, we can do the same for \( d \geq 3 \):

**Theorem 2.3.** For \( d \geq 3 \) and with Wachspress coordinates \( \phi_i \), let
\[ F_i = \phi_i \lambda_{i,0}^{d-1}, \quad i = 1, \ldots, n, \]
\[ F_{i,k} = \binom{d-1}{k} \phi_i \lambda_{i,1}^{d-1-k} + \binom{d-1}{k-1} \phi_{i+1} \lambda_{i+1,0}^{d-1-k}, \quad i = 1, \ldots, n, \quad k = 1, \ldots, d-1, \]
\[
\Psi_d(P_n) := \text{span}\{F_i, \ i = 1, \ldots, n\} \oplus \text{span}\{F_{i,k}, \ i = 1, \ldots, n, \ k = 1, \ldots, d-1\} \oplus \frac{b}{\sum w_j} \Pi_{d-3}, \tag{2.8}
\]

where \(b\) is the bubble function given by

\[
b := \prod_{k=1}^{n} h_k.
\]

Then \(\Pi_d \subset \Psi_d(P_n) \subset \Phi_d(P_n)\).

Given a domain \(\Omega\), let \(\Delta = \{P_n\}\) be a partition of \(\Omega\) into convex polygons, we can use the basis elements of \(\Psi_d(P_n)\) over each polygon \(P_n\) to define spline functions. For any polygons \(P_i, P_j \in \Delta\), we assume that the intersection \(P_i \cap P_j\) is either empty, a common edge, or a common vertex. Then define

\[
S_d(\Delta) := \{s \in C(\Omega), \ s|_{P_n} \in \Psi_d(P_n), \forall P_n \in \Delta\}. \tag{2.9}
\]

This is the spline space we will use to numerically solve PDEs of the form (1.1). In particular, our numerical trials were performed when \(d = 2\) and \(d = 3\). In [10], it was shown that one can form an interpolatory basis \(\{L_j, j = 1, \ldots, dn\}\) for the part of \(\Psi_d(P_n)\) associated with the boundary of \(P_n\), \(\text{span}\{F_i, \ i = 1, \ldots, n\} \oplus \text{span}\{F_{i,k}, \ i = 1, \ldots, n, \ k = 1, \ldots, d-1\}\), as follows:

\[
L_{di-k} = \sum_{s=1}^{d-1} r_{ks} F_{i,s}, \ i = 1, \ldots, n, \ k = 1, \ldots, d-1,
\]

\[
L_i = F_i - \sum_{k=1}^{d-1} (1 - k/d)^d (L_{di-k} + L_{d(i-2)+k}), \ i = 1, \ldots, n,
\]

where \(R = (r_{ks})_{k,s=1}^{d}\) is the inverse of a matrix built from coefficients of Lagrange interpolation of univariate Bernstein polynomials of degree \(d\). When \(d > 3\), there is additional work to be done, as there is still another component of the space \(\Psi_d(P_n)\), namely \(\frac{b}{W} \Pi_{d-3}\). Since we focus on degree 3 and less in this paper, the interested reader should refer to [10] for more information on this case.

3 Existence, Uniqueness, Stability, and Convergence of Solutions

We start to review some sufficient conditions such that the elliptic PDE in (1.1), i.e.

\[
-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} \left( A_{ij} \frac{\partial u}{\partial x_i} \right) + \sum_{k=1}^{2} B_k \frac{\partial u}{\partial x_k} + C u = f
\]

admits a unique weak solution with zero boundary condition, i.e. \(g = 0\) on \(\partial \Omega\), where \(u, f, A_{ij}, B_k, C : \Omega \rightarrow \mathbb{R}\) for \(\Omega \subset \mathbb{R}^2\), and \(A_{ij}, B_k, C \in L_\infty(\Omega)\). The weak formulation of this PDE is given by (1.9) for all \(v \in H_0^1(\Omega)\). To do so, we shall use the following norm and semi-norm on \(H^1(\Omega)\) for convenience: \(\|u\|_{2,\Omega} = \|u\|_{L^2(\Omega)}, \|u\|_{1,2,\Omega} = \|\nabla u\|_{L^2(\Omega)},\) and \(\|u\|_{2,2,\Omega} = \|u\|_{H^2(\Omega)}\). Similarly \(\|u\|_{d+2,2,\Omega} = \|u\|_{H^{d+2}(\Omega)}\).

Define by \(a(u, v)\) the bilinear form in the left-hand side of the equation in (1.9). To find the weak solution in \(H_0^1(\Omega)\), we mainly use the Lax-Milgram theorem to show \(a(u, v)\) is bounded above and coercive. We need the following definition

**Definition 3.1.** We say the PDE in (1.1) is uniform elliptic if the coefficient matrix \([A_{ij}]\) is symmetric and positive definite with the smallest eigenvalue \(\geq \alpha > 0\) over \(\Omega\) for a positive number \(\alpha\). \(\alpha\) is called ellipticity of the PDE.
Theorem 3.2. Suppose that the second order PDE in (1.1) is uniform elliptic with ellipticity $\alpha > 0$. Let $\beta := \|B\|_{\infty, \Omega} < \infty$ and $C \geq \gamma > 0$. Suppose that there exists a positive constant $c$ such that

$$\alpha > \frac{\beta}{2c} \text{ and } \gamma \geq \frac{c\beta}{2}.$$  \hspace{1cm} (3.1)

Then the PDE (1.1) has a unique weak solution $u$ in $H^1_0(\Omega)$ satisfying the weak formulation (1.9) for $v \in H^1_0(\Omega)$.

PROOF. It is easy to see $a(u, v)$ is bounded above. We mainly need to prove $a(u, u)$ is bounded from below. By the uniform ellipticity, the first term in $a(u, u)$ is bounded from below by $\alpha \|u\|^2_{1,2,\Omega}$. The third term in $a(u, u)$ is bounded by $\gamma \|u\|^2_{1,2,\Omega}$. Let us take a close look at the middle term in $a(u, u)$. We use Cauchy-Schwarz inequality to have

$$\left| \int_{\Omega} \sum_{k=1}^{2} B_k u \frac{\partial u}{\partial x_k} \right| \leq \beta \int_{\Omega} |u| |\nabla u| \, dx \, dy \leq \beta \|u\|_{2,\infty} \|u\|_{1,2,\Omega} \leq \frac{\beta}{2} \|u\|^2_{2,\infty} + \frac{\beta}{2c} \|u\|^2_{1,2,\Omega}$$

for any positive number $c$. Thus,

$$a(u, u) \geq \alpha \|u\|^2_{1,2,\Omega} - \frac{\beta}{2c} \|u\|^2_{1,2,\Omega} - \frac{\beta}{2} \|u\|^2_{2,\infty} + \gamma \|u\|^2_{2,\infty}.$$  

By choosing $c > 0$ such that $\alpha - \beta/(2c) > 0$ and $\gamma - \beta/2 > 0$. For example, $c = \beta/\alpha$. Then $a(u, u)$ is coercive. By using the well-known Lax-Milgram theorem, we conclude that for bounded linear functional $F(v) = \int_{\Omega} f v \, dx \, dy$, there exists a unique $u \in H^1_0(\Omega)$ such that $a(u, v) = F(v)$. \hspace{1cm} $\square$

When $B_1$ is a function of $y$ only and $B_2$ is a function of $x$ only, we note that for all $u \in H^1_0(\Omega)$,

$$\int_{\Omega} B_1 \left( \frac{\partial}{\partial x_1} u \right) u \, dx \, dy = - \int_{\Omega} B_1 \left( \frac{\partial}{\partial x_1} u \right) u \, dx \, dy$$  \hspace{1cm} (3.2)

by using integration by parts and zero boundary condition. Thus, $\int_{\Omega} B_1 \left( \frac{\partial}{\partial x_1} u \right) u \, dx \, dy = 0$. Similarly,

$$\int_{\Omega} B_2 \left( \frac{\partial}{\partial x_2} u \right) u \, dx \, dy = 0.$$  

Hence, the terms involving first order derivatives in $a(u, u)$ are zero and

$$a(u, u) = \int_{\Omega} \left[ \sum_{i,j=1}^{2} A_{ij} \frac{\partial}{\partial x_i} u \frac{\partial}{\partial x_j} u + C u^2 \right] \, dx \, dy$$

$$\geq \alpha \|u\|^2_{1,2,\Omega} + \gamma \|u\|^2_{2,\infty}$$  \hspace{1cm} (3.3)

which implies that $a(u, u)$ is coercive. Thus, we have established the following

Theorem 3.3. Suppose that the second order PDE in (1.1) is uniform elliptic with ellipticity $\alpha > 0$. Suppose that $B_1$ is a function of $y$ only and $B_2$ is a function of $x$ only. If $C \geq 0$, then the PDE (1.1) has a unique weak solution $u$ in $H^1_0(\Omega)$ satisfying the weak formulation (1.9) for $v \in H^1_0(\Omega)$.

A new result for the existence and uniqueness of the weak solution for PDE in (1.1) is the following
Corollary 3.4. Suppose that the second order PDE in (1.1) is uniform elliptic with ellipticity \( \alpha > 0 \). Suppose that \( B_1(x, y) = \tilde{B}_1(x, y) + B'_1(y) \) and \( B_2(x, y) = \tilde{B}_2(x) + B'_2(x) \), where \( B'_1(y) \) is a function of \( y \) only and \( B'_2 \) is a function of \( x \) only. Let \( \hat{\beta} := \max\{\|\tilde{B}_1\|_{\infty, \Omega}, \|\tilde{B}_2\|_{\infty, \Omega}\} < \infty \) and \( C \geq \gamma > 0 \). Suppose that there exists a positive constant \( c \) such that
\[
\alpha > \hat{\beta}^2 c \quad \text{and} \quad \gamma \geq c \hat{\beta}^2.
\]
Then the PDE (1.1) has a unique weak solution \( u \) in \( H^1_0(\Omega) \) satisfying the weak formulation (1.9) for \( v \in H^1_0(\Omega) \).

In particular, when \( B_1 = B_2 \equiv 0 \), the PDE in (1.1) has a unique weak solution according to Theorem 3.2 and Theorem 3.4. In fact, we can establish the existence, uniqueness and stability of the solution of (1.1) without using Lax-Milgram theorem. Indeed, it is easy to see that the weak form \( a(u, v) = \langle f, v \rangle \) is the Euler-Lagrange equation of the following minimization
\[
\min_{u \in H^1(\Omega)} J_f(u),
\]
where \( J_f(u) \) is a functional defined by
\[
J_f(u) = \frac{1}{2} a(u, u) - \langle f, u \rangle.
\]
To approximate the exact solution \( u \in H^1(\Omega) \) with \( u|_{\partial \Omega} = 0 \), we solve the following minimization:
\[
\min_{u \in H^1_0(\Omega)} J_f(u).
\]
We shall replace \( H^1_0(\Omega) \) in (3.7) by \( S_d = H^1_0(\Omega) \cap S_d(\Delta) \) to numerically solve the PDE, where \( S_d(\Delta) \) is the space of polygonal splines of order \( d \) which are defined with respect to a polygonal partition \( \Delta \) of \( \Omega \) as explained in the previous section. For convenience, we simply explain the existence, uniqueness, and stability of the minimizer of the following
\[
\min_{u \in S_d} J_f(u).
\]

Theorem 3.5. Suppose that \( [A_{ij}]_{1 \leq i, j \leq 2} \) is symmetric and positive definite. Suppose that \( B_1 = B_2 \equiv 0 \). If \( C \geq \gamma \geq 0 \), then \( J_f \) is strongly convex and hence has a unique minimizer \( u_f \). Hence, there exists a unique weak solution \( u_f \) satisfying (1.9).

Proof. We will find conditions to ensure that (3.6) above is strongly convex, and hence has a unique minimizer. Indeed, fix some nonzero \( v \in S_d \) with \( v|_{\partial \Omega} = 0 \), and define a function \( F_v : \mathbb{R} \rightarrow \mathbb{R} \) by
\[
F_v(t) = \frac{1}{2} a(u + tv, u + tv) - \langle f, u + tv \rangle.
\]
We will show that \( F''_v(t) > 0 \) for any choice of \( v \). Note that
\[
F'_v(t) = \frac{1}{2} \sum_{i,j=1}^2 \int_{\Omega} A_{ij} \left( \frac{\partial v}{\partial x_i} \frac{\partial}{\partial x_j} (u + tv) + \frac{\partial v}{\partial x_j} \frac{\partial}{\partial x_i} (u + tv) \right) + \int_{\Omega} Cv(u + tv) - \int_{\Omega} fv.
\]
Hence we have
\[
F''_v(t) = \sum_{i,j=1}^{2} \int_{\Omega} A_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} + \int_{\Omega} C v^2 \\
= \int_{\Omega} \nabla v^T A \nabla v dx dy + \int_{\Omega} C v^2 dx dy \\
\geq \alpha \|\nabla v\|_{L^2(\Omega)}^2 + \gamma \|v\|_{L^2(\Omega)}^2 \\
\geq \min\{\alpha, \gamma\} \|v\|_{L^2(\Omega)}^2 \\
= \mu \|v\|_{L^2(\Omega)}^2 > 0.
\]

So \( F_v \) is strongly convex. Since this is true regardless of choice of \( v \), and our space is finite-dimensional, we have that \( J_f \) must in fact be \( \mu \)-strongly convex for some \( \mu > 0 \). Thus, the minimizer \( u_S \) is unique. The Euler-Lagrange equation implies that \( u_S \) satisfies the weak solution (1.9) for \( v \in S_d \).

Let \( \mu \) be the convexity coefficient of \( J_f \), i.e. the Hessian matrix of \( J_f \) is positive definite with small eigenvalue bigger than or equal to \( \mu \). Notice that, since \( F''_v(t) \) does not depend on \( f \), that \( J_f \) is \( \mu \)-strongly convex for any choice of \( f \). We can retrieve the following result regarding the stability of the minimizer of \( J_f \):

**Theorem 3.6.** Suppose that the PDE in (1.1) satisfies the uniform ellipticity conditions in the hypotheses of Theorem 3.2. For two functions \( f \) and \( g \), denote the minimizer of \( J_f \) by \( u_f \) and the minimizer of \( J_g \) by \( u_g \). Then \( \|u_f - u_g\|_{L^2(\Omega)} \leq \mu^{-1} \|f - g\|_{L^2(\Omega)} \).

**Proof.** Since \( J_f \) and \( J_g \) are both \( \mu \)-strongly convex, we can say that
\[
J_f(u_g) - J_f(u_f) \geq \langle \nabla J_f(u_f), u_g - u_f \rangle + \frac{\mu}{2} \|u_f - u_g\|_{L^2(\Omega)}^2 = \frac{\mu}{2} \|u_f - u_g\|_{L^2(\Omega)}^2 
\]
and
\[
J_g(u_f) - J_g(u_g) \geq \langle \nabla J_g(u_g), u_f - u_g \rangle + \frac{\mu}{2} \|u_f - u_g\|_{L^2(\Omega)}^2 = \frac{\mu}{2} \|u_f - u_g\|_{L^2(\Omega)}^2,
\]
where the last equalities in each equation come from the fact that \( u_f \) and \( u_g \) minimizing \( J_f \) and \( J_g \), respectively, implies that both \( \nabla J_f(u_f) \) and \( \nabla J_g(u_g) \) are 0.

Notice that, for any function \( u \),
\[
J_g(u) - J_f(u) = \frac{1}{2} a(u, u) - \langle g, u \rangle - \frac{1}{2} a(u, u) + \langle f, u \rangle = \langle f - g, u \rangle,
\]
so if we sum equations (3.9) and (3.10), then after some simplification we see that
\[
\langle f - g, u_f - u_g \rangle \geq \mu \|u_f - u_g\|_{L^2(\Omega)}^2.
\]
Using Cauchy-Schwarz on the left side of this inequality gives us
\[
\|f - g\|_{L^2(\Omega)} \|u_f - u_g\|_{L^2(\Omega)} \geq \mu \|u_f - u_g\|_{L^2(\Omega)}^2.
\]
A simple division by \( \mu \|u_f - u_g\|_{L^2(\Omega)} \) yields the desired result.

Finally we discuss convergence of the numerical solutions. The following results are a straightforward extension of the similar results for Poisson equation based on finite elements to a general second order elliptic PDE (1.1) using the polygonal splines.
Theorem 3.7. Suppose that the PDE in (1.1) satisfies the assumptions in Theorem 3.2. Suppose that the weak solution \( u \) of the PDE in (1.1) is in \( H^{d+1}(\Omega) \). Letting \( u_S \in S_d \) be the weak solution satisfying \( a(u_S, v) = \langle f, v \rangle \) for all \( v \in S_d \). Then

\[
|u - u_S|_{1,2,\Omega} \leq K |u|_{d+1,2,\Omega} |\Delta|^d, \tag{3.11}
\]

where \(|\Delta|\) is the length of the longest edge in \( \Delta \), and \( K = K(\Omega, \Delta, A, B, C) \) is a positive constant depending only on the domain \( \Omega \), the partition \( \Delta \), the largest eigenvalue \( \Lambda \) of \( A \) and \( \|C\|_{\infty,\Omega} \).

Proof. We must prove some preliminary results in order to prove the theorem itself. First, notice that in the proof of Theorem 3.2, we actually have

\[
a(v, v) \geq \mu |v|_{1,2,\Omega}^2, \tag{3.12}
\]

where \( \mu = \alpha - \frac{c\beta}{2} \) for \( c > 0 \) such that \( \gamma - \frac{\beta}{2c} \geq 0 \). In addition, we can show that \( a(u, v) \) is bounded. Indeed,

\[
a(u, v) = \int_\Omega \sum_{i,j=1}^2 A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \int_\Omega \sum_{k=1}^2 B_k \frac{\partial u}{\partial x_k} v + \int_\Omega C u v
\]

\[
\leq \Lambda |\nabla u|_{L^2} |\nabla v|_{L^2} + \beta |\nabla u|_{L^2} |\nabla v|_{L^2} + \|C\|_{\infty} |u|_{L^2} |v|_{L^2}
\]

\[
\leq M_1 (|\nabla u|_{L^2} |\nabla v|_{L^2} + |\nabla u|_{L^2} |\nabla v|_{L^2} + |u|_{L^2} |v|_{L^2} + |u|_{L^2} |v|_{L^2})
\]

\[
= M_1 |u|_{1,2,\Omega} |v|_{1,2,\Omega} + |u|_{1,2,\Omega} |v|_{1,2,\Omega} + |u|_{1,2,\Omega} K_1 |v|_{1,2,\Omega}
\]

where \( \Lambda > 0 \) is the largest eigenvalue of \( [A_{ij}]_{1 \leq i, j \leq 2} \), \( M_1 = \max\{\Lambda, \beta, \|C\|_{\infty,\Omega} \} \), and Poincaré’s inequality have been used. That is,

\[
a(u, v) \leq M |u|_{1,2,\Omega} |v|_{1,2,\Omega}. \tag{3.13}
\]

for another positive constant \( M \). By definition of weak solution, we know that for all \( v \in H^1_0(\Omega) \), \( a(u, v) = \langle f, v \rangle \), and for all \( v \in S_d \), \( a(u_S, v) = \langle f, v \rangle \). Since \( S_d \subset H^1_0(\Omega) \), we can say that for all \( v \in S_d \),

\[
a(u - u_S, v) = 0, \quad \forall v \in S_d. \tag{3.14}
\]

Finally, recall that the space \( S_d \) contains the d-degree polynomial space \( \Pi_d(\Omega) \) over each convex polygon in \( \Delta \). Hence, by the Bramble-Hilbert lemma, we know that there exists a piecewise polynomial function \( v \in S_d \) such that

\[
|u - v|_{1,2,\Omega} \leq C |\Delta|^d |u|_{H^{d+1}} \tag{3.15}
\]

for some constant \( C \) depending on the shape parameter of \( \Omega \) and partition \( \Delta \) (cf. [6]).

Now, define \( u_{\text{best}} := \arg \min_{s \in S} |u - s|_{1,2,\Omega} \). Then we have

\[
\mu |u_{\text{best}} - u_S|_{1,2,\Omega}^2 \leq a(u_{\text{best}} - u_S, u_{\text{best}} - u_S) \leq a(u_{\text{best}} - u, u_{\text{best}} - u_S) \leq M |u_{\text{best}} - u|_{1,2,\Omega} |u_{\text{best}} - u_S|_{1,2,\Omega} \tag{by (3.12)}
\]

\[
= M |u_{\text{best}} - u|_{1,2,\Omega} |u_{\text{best}} - u_S|_{1,2,\Omega} \leq M |u - u_{\text{best}}|_{1,2,\Omega} + \mu |u - u_{\text{best}}|_{1,2,\Omega} \tag{by (3.13)}
\]

\[
\Rightarrow |u - u_S|_{1,2,\Omega} \leq \frac{\mu + \mu M}{\mu} |u - u_{\text{best}}|_{1,2,\Omega} \leq \frac{\mu + \mu M}{\mu} C |u|_{d+1,2,\Omega} |\Delta|^d \tag{by (3.15)}.
\]
These complete the proof.

We next explain the convergence in $L_2$ norm. When $\Omega$ is a convex domain, the convergence rate $\|u - u_{\text{best}}\|_{L_2(\Omega)}$ should be optimal based on a generalization of the well-known Aubin-Nitsche technique (cf. [1]) for Poisson equation. That is, we have

**Theorem 3.8.** Suppose that the elliptic PDE in (1.1) satisfies the elliptic condition in Theorem 3.2.

Suppose that the underlying Lipschitz domain $\Omega$ is convex. Let $u_S$ be the weak solution of (1.1). Then

$$\|u - u_S\|_{L_2(\Omega)} \leq C|\Delta|^{d+1}|u|_{d+1,2,\Omega}$$

(3.16)

for $d \geq 1$.

**Proof.** For $u - u_S \in L_2(\Omega)$, we can find the weak solution $w \in H^1_0(\Omega)$ satisfying

$$a(v, w) = \langle u - u_S, v \rangle, \quad \forall v \in H^1_0(\Omega).$$

(3.17)

Indeed, let $\hat{a}(u, v) = a(v, u)$ be a new bilinear form. By using the same proof of Theorem 3.2, we can show $\hat{a}(u, v)$ is a bounded bilinear form and $\hat{a}(u, u)$ is coercive since $\hat{a}(u, u) = a(u, u)$. By the Lax-Milgram theorem, there exists a weak solution $w$ satisfying (3.17). It is known that $w \in H^2(\Omega)$ when $\Omega$ is convex (cf. [1]) and satisfies $|w|_{2,2,\Omega} \leq C\|u - u_S\|_{L_2(\Omega)}$ for a positive constant $C > 0$ independent of $u$ and $u_S$.

Thus, we use (3.14) with an appropriate $v \in S_d$,

$$\|u - u_S\|_{L_2(\Omega)} = \langle u - u_S, w - u_S \rangle = a(u - u_S, w)$$

$$= a(u - S, w - v) \leq M|u - u_S|_{1,2,\Omega} |w - v|_{1,2,\Omega}$$

$$\leq MC|\Delta|^{d}|u|_{d+1,2,\Omega} C|\Delta||w|_{2,2,\Omega}$$

$$\leq MC|\Delta|^{d+1}|u|_{d+1,2,\Omega} C\|u - u_S\|_{L_2(\Omega)}.$$

for positive constants $C$ which are different in different lines. It now follows that

$$\|u - u_S\|_{L_2(\Omega)} \leq C|\Delta|^{d+1}|u|_{d+1,2,\Omega}$$

for another positive constant $C$. This completes the proof.

**4 Numerical Results**

**4.1 Explanation of Our Implementation**

In this section we explain our implementation to numerically solve general second-order elliptic PDEs. It is an adaptation of the method detailed in [10] which uses polygonal splines to solve Poisson equations, which is in turn based on the method in [1], which details how to solve both linear and nonlinear PDEs with multivariate splines.

Our goal will be to solve for a vector of coefficients, $u$. We can begin in the same place as in [10], first constructing a matrix $H$ to determine continuity conditions. If two polygons share an edge, then there are $d + 1$ pairs of elements supported on that edge between the two polygons, and each pair share values on the edge. Hence we force their respective coefficients to match, resulting in a linear system summarized by $H u = 0$. We can similarly represent our boundary conditions by a linear system $B u = g$. 

---

**Notes:**

- Use appropriate styles and formatting for mathematical expressions.
- Ensure all symbols and equations are correctly typeset.
- Check for consistency in notation and terminology.

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**End Notes:**

- Review the document for any formatting issues or clarity enhancements.
- Ensure all references and citations are correctly formatted and included.

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**End Document:**

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**End Page:**

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An important difference arises from here: we will need to form a different "stiffness" matrix than in the simpler Poisson case. In particular, in \( \mathbb{R}^D \), using degree \( d \) polygonal splines, the new left-hand side of the weak form of the problem can be simplified to the following:

\[
\sum_{P_n \in \Delta} \sum_{k=1}^{d_n} u_k \left[ \sum_{i,j=1}^{D} \int_{P_n} A_{ij} \frac{\partial v}{\partial x_i} \frac{\partial L_k}{\partial x_j} + \int_{P_n} c v L_k \right]
\]

where we have expressed \( u \approx u_S = \sum_{k=1}^{d_n} u_k L_k \) for some coefficients \( u_k \), where \( L_k \) is an ordering of the interpolatory basis of \( S_d(\Delta) \) (which, when restricted to the domain \( P_n \), is simply \( \Psi_d(P_n) \)). Similarly write \( f \approx f_S = \sum_{k=1}^{d_n} f_k L_k \) and notice that the right-hand side of the weak form will be equal to \( \sum_{P_n \in \Delta} \sum_{k=1}^{d_n} f_k \int v L_k \) for any \( v \in S_d(\Delta) \cap H^1_0(\Omega) \). Hence, it must be true for \( v = L_m \) for \( m = 1, 2, ..., d_n \), so we construct the following matrices:

\[
M = [M_{P_n}]_{P_n \in \Delta}, \text{ where } M_{P_n} = (M_{P_n,p,q})_{p,q=1}^D, \text{ and } M_{P_n,p,q} = \int_{P_n} L_p L_q ;
\]

\[
K = [K_{P_n}]_{P_n \in \Delta}, \text{ where } K_{P_n} = \sum_{i,j=1}^{D} K_{i,j}^{P_n} \text{ and }
K_{i,j}^{P_n} = (K_{i,j}^{P_n,p,q})_{p,q=1}^D \text{ where } K_{i,j}^{P_n,p,q} = \int_{P_n} A_{ij} \frac{\partial L_p}{\partial x_i} \frac{\partial L_q}{\partial x_j} ;
\]

\[
\mathcal{M}_{P_n} = (\mathcal{M}_{P_n,p,q})_{p,q=1}^D \text{ where } \mathcal{M}_{P_n,p,q} = \int_{P_n} C L_p L_q ;
\]

\[
K = [K_{P_n}]_{P_n \in \Delta} + [M_{P_n}]_{P_n \in \Delta};
\]

\[
\mathbf{u} = (u_k)_{k=1}^D ;
\]

\[
\mathbf{f} = (f_k)_{k=1}^D ,
\]

where the integrals are computed using the tensor product of the Gauss quadrature formula of high order, say order \( 5 \times 5 \).

Then notice that we can rewrite our weak formulation as

\[
K \mathbf{u} = M \mathbf{f} .
\]

Our minimization problem in (3.8) can be recast in terms of polygonal splines as

\[
\min_{\mathbf{u}} \frac{1}{2} \mathbf{u}^T K \mathbf{u} - \mathbf{f}^T M \mathbf{u}
\]

which is a constrained minimization problem which can be solved by using the well-known iterative method described in [1]. We have implemented the computational scheme in MATLAB and experimented with many second order elliptic PDEs. Some numerical results will be shown in the next section.
4.2 Numerical Results of Elliptic PDEs

This section is divided into two parts. In the first part, we demonstrate the power of polygonal splines as a tool for numerically solving some general second-order elliptic PDEs. In the second part, we show the potential to use these numerical solutions to approximate the solutions of parabolic equations and hyperbolic equations. In all the following examples, we denote by \( u_S \) the spline solution, and \( u \) as the true solution. To approximate the \( L^2 \) error, we report the root mean squared (RMS) error \( E_{RMS} = ||u - u_S||_{RMS} \) of the spline solution based on 1001 \( \times \) 1001 equally-spaced points over \( \Omega \). Since \( \nabla(u - u_S) = \left( \frac{\partial}{\partial x}(u - u_S), \frac{\partial}{\partial y}(u - u_S) \right) \), we report the RMS error \( \nabla E_{RMS} = ||\nabla(u - u_S)||_{RMS} \), which is the average of the RMS error of \( \frac{\partial}{\partial x}(u - u_S) \) and \( \frac{\partial}{\partial y}(u - u_S) \).

Let us begin with numerical solutions of some standard second-order elliptic PDEs.

**Example 4.1.** We return to example (1.3) on the unit square \( \Omega = (0,1) \times (0,1) \) to demonstrate convergence of the method. Recall that for this example we let

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = \begin{bmatrix}
\varepsilon + x & xy \\
xy & \varepsilon + y
\end{bmatrix}
\]

and \( C = \exp(-x^2 - y^2) \), with \( \varepsilon > 0 \) to ensure ellipticity. We’ll set \( \varepsilon = 10^{-5} \) and choose \( f \) and \( g \) so that

\[
u(x,y) = \frac{(1 + x)^2}{4} \sin(2\pi xy)
\]

is the exact solution. We use the following polygonal partition:

![Image of partition and refinements]

Figure 1: A partition of the unit square and a few refinements

We employ our polygonal spline method to solve (1.1) with exact solution in (4.1). Our numerical results are shown below in Tables 1 and 2.
The numerical results in Tables 1 and 2 show that the polygonal spline method works very well. We shall compare with the solution using degree-2 and degree-3 bivariate splines based on a triangulation of the same domain:

From Tables 1, 2, 3, and 4, we can see that polygonal splines can produce more accurate solution with comparably sized polygonal partitions.

**Example 4.2.** Here is another example of an elliptic second order PDE: let

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\]

and \(C = 1\), and solve the PDE given by

\[-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} \left( A_{ij} \frac{\partial u}{\partial x_i} \right) + \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + Cu = f \text{ in } \Omega; \]
\[u = g \text{ on } \partial \Omega.\]  

To test our method, we shall choose \(f\) and \(g\) so that

\[u = (1 + x^2 + y^2)^{-1}\]

is the exact solution.

According to Theorem 3.4, this elliptic PDE has a unique weak solution. Although this PDE technically does not fit with our computational scheme in the previous section because the PDE involves the first-order derivatives, our minimization scheme 3.8 with additional first-order derivative terms still produces good results. We use the same partition as in Example 4.1 to solve this PDE. Tables 5 and 6 show the results of our minimization:
Similarly, the minimization \((3.8)\) with first order derivatives based on bivariate splines can also produce good numerical results. For comparison, Tables 7 and 8 tabulates the results of the same computation using bivariate splines of degree 2 and degree 3 over triangulations of the same domain:

<table>
<thead>
<tr>
<th># mesh</th>
<th>(E_{RMS})</th>
<th>rate</th>
<th>(\nabla E_{RMS})</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>3.54e-01</td>
<td>4.80e-04</td>
<td>0.00</td>
<td>1.14e-02</td>
</tr>
<tr>
<td>160</td>
<td>1.77e-01</td>
<td>5.23e-05</td>
<td>3.20</td>
<td>2.70e-03</td>
</tr>
<tr>
<td>640</td>
<td>8.84e-02</td>
<td>6.21e-06</td>
<td>3.07</td>
<td>7.14e-04</td>
</tr>
<tr>
<td>2560</td>
<td>4.42e-02</td>
<td>8.53e-07</td>
<td>2.86</td>
<td>2.29e-04</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th># mesh</th>
<th>(E_{RMS})</th>
<th>rate</th>
<th>(\nabla E_{RMS})</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2.43e-05</td>
<td>0.00</td>
<td>9.87e-04</td>
</tr>
<tr>
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<td>1.77e-01</td>
<td>1.81e-06</td>
<td>3.75</td>
<td>1.44e-04</td>
</tr>
<tr>
<td>640</td>
<td>8.84e-02</td>
<td>1.29e-07</td>
<td>3.81</td>
<td>2.01e-05</td>
</tr>
<tr>
<td>2560</td>
<td>4.42e-02</td>
<td>9.69e-09</td>
<td>3.74</td>
<td>2.97e-06</td>
</tr>
</tbody>
</table>

Table 7: Degree-2 Bivariate spline approximation of solution to Example 4.2 with exact solution in (4.4)

Table 8: Degree-3 Bivariate spline approximation of solution to Example 4.2 with exact solution in (4.4)

4.3 Numerical Solutions of Parabolic and Hyperbolic PDEs

Example 4.3. We return again to example (1.3) on the unit square \(\Omega = (0, 1) \times (0, 1)\), but this time with \(\varepsilon = 0\); let

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = \begin{bmatrix} x & xy \\ xy & y \end{bmatrix}
\]

and \(C = \exp(-x^2 - y^2)\). We’ll choose \(f\) and \(g\) so that

\[
u(x, y) = \frac{(1 + x)^2}{4} \sin(2\pi xy)
\]

is the exact solution. Notice that, with this new condition of \(\varepsilon = 0\), the PDE is not elliptic. However, our method still approximates the true solution quite well. We’ll show the convergence of our approximations for decreasing values of \(\varepsilon\):
For comparison, let us show the results of the same PDE using bivariate splines over a triangulation of the same domain instead:
The numerical results in the tables above show that the polygonal spline method works very well to approximate the solutions of non-elliptic PDEs.

**Example 4.3.** Let

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = \begin{bmatrix}
xy & 0 \\
0 & xy
\end{bmatrix}
\]

and \( C = 0 \). Choose \( f \) and \( g \) so that

\[
u = x(1 - x)y(1 - y)
\]

is the exact solution. This was studied in [16]. As in Example 4.3, this is nearly an elliptic PDE, but with some degeneracy at the origin. We shall use a different partition of the unit square this time, simply using a uniform grid of squares, as was the case in the original paper [16]. Their Weak Galerkin method retrieved the following results:
We use our method with polygonal splines to solve the PDE above and find that our method can produce much better results.

Table 21: Weak Galerkin approximation of solution to Example 4.4

<table>
<thead>
<tr>
<th># Polys</th>
<th>mesh</th>
<th>$|u - u_{WG}|_{L^2}$</th>
<th>rate</th>
<th>$|\nabla u - \nabla u_{WG}|_{H^1}$</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.25e-01</td>
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<td>2.52e-02</td>
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<tr>
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<td>6.25e-02</td>
<td>3.74e-04</td>
<td>1.96</td>
<td>1.23e-02</td>
<td>9.98e-01</td>
</tr>
<tr>
<td>1024</td>
<td>3.13e-02</td>
<td>9.47e-05</td>
<td>1.98</td>
<td>6.31e-03</td>
<td>9.98e-01</td>
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<td>2.39e-05</td>
<td>1.99</td>
<td>3.16e-03</td>
<td>9.98e-01</td>
</tr>
</tbody>
</table>

Table 22: Degree-2 Polygonal spline approximation of solution to Example 4.4

<table>
<thead>
<tr>
<th># P</th>
<th>mesh</th>
<th>$E_{RMS}$</th>
<th>rate</th>
<th>$\nabla E_{RMS}$</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>1.25e-01</td>
<td>1.83e-06</td>
<td>0.00</td>
<td>1.39e-04</td>
<td>0.00</td>
</tr>
<tr>
<td>256</td>
<td>6.25e-02</td>
<td>9.85e-08</td>
<td>4.22</td>
<td>1.60e-05</td>
<td>3.12</td>
</tr>
<tr>
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<td>3.13e-02</td>
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<td>4.12</td>
<td>1.91e-06</td>
<td>3.07</td>
</tr>
<tr>
<td>4096</td>
<td>1.56e-02</td>
<td>3.42e-10</td>
<td>4.05</td>
<td>2.33e-07</td>
<td>3.04</td>
</tr>
</tbody>
</table>

Table 23: Degree-3 Polygonal spline approximation of solution to Example 4.4

<table>
<thead>
<tr>
<th># P</th>
<th>mesh</th>
<th>$E_{RMS}$</th>
<th>rate</th>
<th>$\nabla E_{RMS}$</th>
<th>rate</th>
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<td>5.61e-10</td>
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</tr>
</tbody>
</table>

Comparison of Tables 21, 22, and 23 shows that our polygonal spline method produces much more accurate solution. These results call for some remarks. First, it is worth pointing out that our MATLAB code can only achieve $10^{-11}$ accuracy. In Table 23 the rates of convergences become negative because that the numerical results reach the accuracy and are not able to improve. That is, polygonal splines of degree-3 converged to the solution virtually instantly. Some improved performance has been observed before when the solution is a polynomial, but we would not normally expect immediate retrieval of the solution of a degree-4 polynomial using only degree-3 splines. The degree-2 splines also appear to have an increased order of convergence $O(h^4)$. We are interested in why the performance has been increased here, and our investigation seems to show that the partition plays a role. If we run a few iterations to solve the same problem over the unit square based on the partition from Example 4.1, we retrieve the following results:

Table 24: Degree-2 Polygonal spline approximation of solution to Example 4.4 over non-grid partition

<table>
<thead>
<tr>
<th># P</th>
<th>mesh</th>
<th>$E_{RMS}$</th>
<th>rate</th>
<th>$\nabla E_{RMS}$</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>2.50e-01</td>
<td>3.66e-05</td>
<td>0.00</td>
<td>1.19e-03</td>
<td>0.00</td>
</tr>
<tr>
<td>219</td>
<td>1.25e-01</td>
<td>3.09e-06</td>
<td>3.57</td>
<td>2.29e-04</td>
<td>2.38</td>
</tr>
<tr>
<td>1251</td>
<td>6.25e-02</td>
<td>2.75e-07</td>
<td>3.49</td>
<td>4.59e-05</td>
<td>2.32</td>
</tr>
</tbody>
</table>

Table 25: Degree-3 Polygonal spline approximation of solution to Example 4.4 over non-grid partition

<table>
<thead>
<tr>
<th># P</th>
<th>mesh</th>
<th>$E_{RMS}$</th>
<th>rate</th>
<th>$\nabla E_{RMS}$</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>2.50e-01</td>
<td>3.10e-06</td>
<td>0.00</td>
<td>9.17e-05</td>
<td>0.00</td>
</tr>
<tr>
<td>219</td>
<td>1.25e-01</td>
<td>1.22e-07</td>
<td>4.66</td>
<td>7.07e-06</td>
<td>3.70</td>
</tr>
<tr>
<td>1251</td>
<td>6.25e-02</td>
<td>4.59e-09</td>
<td>4.74</td>
<td>5.96e-07</td>
<td>3.57</td>
</tr>
</tbody>
</table>

We can see that this time the numerical solutions are closer to the expected convergence. Thus, the grid partition plays a role to the solution of this problem. We would like to invite the interested reader to investigate it further.

**Example 4.5.** Consider the following example:

$$-\varepsilon \Delta u + (2 - y^2) D_x u + (2 - x) D_y u + (1 + (1 + x)(1 + y)^2) u = f, \quad (x, y) \in \Omega = (0, 1) \times (0, 1) \quad (4.6)$$
with \( u|_{\partial\Omega} = g \), where \( f \) is so chosen that the exact solution is

\[
u(x, y) = 1 + \sin(\pi(1 + x)(1 + y)^2/8).
\]

When \( \varepsilon = 0 \), this is a hyperbolic test problem considered in [2] [12] [13]. We show that we are able to well-approximate this solution using our method by using decreasing positive values of \( \varepsilon \):

<table>
<thead>
<tr>
<th># P</th>
<th>mesh</th>
<th>( E_{RMS} )</th>
<th>rate</th>
<th>( \nabla E_{RMS} )</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>2.50e-01</td>
<td>1.28e-03</td>
<td>0.00</td>
<td>5.58e-02</td>
<td>0.00</td>
</tr>
<tr>
<td>219</td>
<td>1.25e-01</td>
<td>4.22e-04</td>
<td>1.60</td>
<td>2.38e-02</td>
<td>1.23</td>
</tr>
<tr>
<td>1251</td>
<td>6.25e-02</td>
<td>4.04e-04</td>
<td>0.07</td>
<td>2.28e-02</td>
<td>0.06</td>
</tr>
<tr>
<td>7251</td>
<td>3.13e-02</td>
<td>3.99e-04</td>
<td>0.02</td>
<td>2.15e-02</td>
<td>0.09</td>
</tr>
</tbody>
</table>

Table 26: Degree-2 Polygonal spline approximation of solution to Example 4.5 with \( \varepsilon = 10^{-3} \)

<table>
<thead>
<tr>
<th># P</th>
<th>mesh</th>
<th>( E_{RMS} )</th>
<th>rate</th>
<th>( \nabla E_{RMS} )</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>2.50e-01</td>
<td>1.83e-03</td>
<td>0.00</td>
<td>7.70e-02</td>
<td>0.00</td>
</tr>
<tr>
<td>219</td>
<td>1.25e-01</td>
<td>2.97e-04</td>
<td>2.62</td>
<td>3.03e-02</td>
<td>1.35</td>
</tr>
<tr>
<td>1251</td>
<td>6.25e-02</td>
<td>4.51e-05</td>
<td>2.72</td>
<td>1.25e-02</td>
<td>1.28</td>
</tr>
<tr>
<td>7251</td>
<td>3.13e-02</td>
<td>6.32e-06</td>
<td>2.84</td>
<td>3.67e-03</td>
<td>1.77</td>
</tr>
</tbody>
</table>

Table 27: Degree-3 Polygonal spline approximation of solution to Example 4.5 with \( \varepsilon = 10^{-3} \)

<table>
<thead>
<tr>
<th># P</th>
<th>mesh</th>
<th>( E_{RMS} )</th>
<th>rate</th>
<th>( \nabla E_{RMS} )</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>2.50e-01</td>
<td>2.50e-01</td>
<td>0.00</td>
<td>2.02e-02</td>
<td>0.00</td>
</tr>
<tr>
<td>219</td>
<td>1.25e-01</td>
<td>1.25e-01</td>
<td>0.18</td>
<td>2.24e-02</td>
<td>-0.15</td>
</tr>
<tr>
<td>1251</td>
<td>6.25e-02</td>
<td>6.25e-02</td>
<td>0.02</td>
<td>2.18e-02</td>
<td>0.04</td>
</tr>
<tr>
<td>7251</td>
<td>3.13e-02</td>
<td>3.13e-02</td>
<td>0.01</td>
<td>1.81e-02</td>
<td>0.26</td>
</tr>
</tbody>
</table>

Table 28: Degree-2 Polygonal spline approximation of solution to Example 4.5 with \( \varepsilon = 10^{-5} \)

<table>
<thead>
<tr>
<th># P</th>
<th>mesh</th>
<th>( E_{RMS} )</th>
<th>rate</th>
<th>( \nabla E_{RMS} )</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>2.50e-01</td>
<td>4.62e-04</td>
<td>0.00</td>
<td>2.02e-02</td>
<td>0.00</td>
</tr>
<tr>
<td>219</td>
<td>1.25e-01</td>
<td>4.07e-04</td>
<td>0.18</td>
<td>2.24e-02</td>
<td>-0.15</td>
</tr>
<tr>
<td>1251</td>
<td>6.25e-02</td>
<td>4.00e-04</td>
<td>0.02</td>
<td>2.18e-02</td>
<td>0.04</td>
</tr>
<tr>
<td>7251</td>
<td>3.13e-02</td>
<td>3.99e-04</td>
<td>0.01</td>
<td>1.81e-02</td>
<td>0.26</td>
</tr>
</tbody>
</table>

Table 29: Degree-3 Polygonal spline approximation of solution to Example 4.5 with \( \varepsilon = 10^{-5} \)

<table>
<thead>
<tr>
<th># P</th>
<th>mesh</th>
<th>( E_{RMS} )</th>
<th>rate</th>
<th>( \nabla E_{RMS} )</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>2.50e-01</td>
<td>4.62e-04</td>
<td>0.00</td>
<td>2.02e-02</td>
<td>0.00</td>
</tr>
<tr>
<td>219</td>
<td>1.25e-01</td>
<td>4.07e-04</td>
<td>0.18</td>
<td>2.24e-02</td>
<td>-0.15</td>
</tr>
<tr>
<td>1251</td>
<td>6.25e-02</td>
<td>4.00e-04</td>
<td>0.02</td>
<td>2.18e-02</td>
<td>0.04</td>
</tr>
<tr>
<td>7251</td>
<td>3.13e-02</td>
<td>3.99e-04</td>
<td>0.01</td>
<td>1.81e-02</td>
<td>0.26</td>
</tr>
</tbody>
</table>

Table 30: Degree-2 Polygonal spline approximation of solution to Example 4.5 with \( \varepsilon = 10^{-10} \)

<table>
<thead>
<tr>
<th># P</th>
<th>mesh</th>
<th>( E_{RMS} )</th>
<th>rate</th>
<th>( \nabla E_{RMS} )</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>2.50e-01</td>
<td>4.62e-04</td>
<td>0.00</td>
<td>2.02e-02</td>
<td>0.00</td>
</tr>
<tr>
<td>219</td>
<td>1.25e-01</td>
<td>4.07e-04</td>
<td>0.18</td>
<td>2.24e-02</td>
<td>-0.15</td>
</tr>
<tr>
<td>1251</td>
<td>6.25e-02</td>
<td>4.00e-04</td>
<td>0.02</td>
<td>2.18e-02</td>
<td>0.04</td>
</tr>
<tr>
<td>7251</td>
<td>3.13e-02</td>
<td>3.99e-04</td>
<td>0.01</td>
<td>1.81e-02</td>
<td>0.26</td>
</tr>
</tbody>
</table>

Table 31: Degree-3 Polygonal spline approximation of solution to Example 4.5 with \( \varepsilon = 10^{-10} \)

For comparison, here are the results of the same computation using bivariate splines over a triangulation of the same domain:
We can see that the polygonal spline solutions approximate the exact solution very well. However, as in Example 4.4, we see that this PDE has a unique weak solution, but does not fit with our minimization scheme. Nevertheless, our method works well as shown in Tables 34 and 35.

Example 4.6. For another example, the following problem is parabolic for \( y > 0 \) and hyperbolic for \( y \leq 0 \):

\[
-\varepsilon D_{yy}u + D_xu + c_1u = 0, \quad (x, y) \in (-1, 1) \times (0, 1) \\
D_xu + c_2u = 0, \quad (x, y) \in (-1, 1) \times (-1, 0)
\]  

with \( u|_{\partial\Omega} = g \), for any constants \( c_1 > 0 \) and \( c_2 > 0 \). This PDE was studied in [13]. Note that the solution is discontinuous at \( y = 0 \). We can solve the following general elliptic PDE to estimate the solution to this problem

\[
-\eta D_{xx}u - \varepsilon D_{yy}u + D_xu + c_1u = 0, \quad (x, y) \in (-1, 1) \times (0, 1) \\
-\eta \Delta u + D_xu + c_2u = 0, \quad (x, y) \in (-1, 1) \times (-1, 0)
\]
with \( u|_{\partial \Omega} = g \) and \( \eta > 0 \), We can approximate the solution to (4.7) by letting \( \eta > 0 \) go to zero and use spline functions which are not necessarily continuous at \( y = 0 \). Let the exact solution to (4.7) be given by

\[
u(x, y) = \begin{cases} 
\sin(\pi(1 + y)/2) \exp(-c_1 + \varepsilon \pi^2/4)(1 + x)), & -1 \leq x \leq 1, 0 \leq y \leq 1 \\
\sin(\pi(1 + y)/2) \exp(-c_2(1 + x)), & -1 \leq x \leq 1, -1 < y \leq 0.
\end{cases}
\]  

(4.9)

We use a similar partition to the one from Example 4.1, scaled to cover the domain \( \Omega = [-1, 1]^2 \) and with an added edge to account for the discontinuity at \( y = 0 \):

![Partition of domain Ω = [-1, 1]² and a few refinements](image)

Although this PDE does not technically fit our computational scheme due to its inclusion of first-order terms, we can still get a fair good estimate of the true solution using degree-2 polygonal splines. Numerical results are shown in Table 39.

<table>
<thead>
<tr>
<th># P</th>
<th>mesh</th>
<th>( E_{RMS} )</th>
<th>rate</th>
<th>( \nabla E_{RMS} )</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>6.67e-01</td>
<td>6.80e-03</td>
<td>0.00</td>
<td>2.13e-01</td>
<td>0.00</td>
</tr>
<tr>
<td>208</td>
<td>3.33e-01</td>
<td>2.45e-03</td>
<td>1.46</td>
<td>2.10e-01</td>
<td>0.02</td>
</tr>
<tr>
<td>1120</td>
<td>1.67e-01</td>
<td>1.15e-03</td>
<td>1.10</td>
<td>2.03e-01</td>
<td>0.05</td>
</tr>
<tr>
<td>6208</td>
<td>8.33e-02</td>
<td>4.98e-04</td>
<td>1.21</td>
<td>1.76e-01</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Table 38: Degree-2 Polygonal spline approximation of solution to (4.7) with exact solution (4.9) when \( \eta = 10^{-10}, c_1 = c_2 = 0.1 \)

If we change the value of \( c_2 \) to \( 0.1 + \varepsilon \pi^2/4 \), so that the solution is continuous, we retrieve the following results (without forcing continuity over the line \( y = 0 \)):
Table 39: Degree-2 Polygonal spline approximation of solution to (4.7) with exact solution (4.9) when \( \eta = 10^{-10}, C_1 = 0.1, c_2 = c_1 + \varepsilon \pi^2/4 \)

<table>
<thead>
<tr>
<th># P</th>
<th>mesh</th>
<th>( E_{RMS} )</th>
<th>rate</th>
<th>( \nabla E_{RMS} )</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>6.67e-01</td>
<td>1.64e-03</td>
<td>0.00</td>
<td>2.82e-02</td>
<td>0.00</td>
</tr>
<tr>
<td>208</td>
<td>3.33e-01</td>
<td>2.61e-04</td>
<td>2.65</td>
<td>1.03e-02</td>
<td>1.45</td>
</tr>
<tr>
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<td>1.67e-01</td>
<td>3.86e-05</td>
<td>2.76</td>
<td>3.60e-03</td>
<td>1.52</td>
</tr>
<tr>
<td>6208</td>
<td>8.33e-02</td>
<td>5.68e-06</td>
<td>2.76</td>
<td>1.23e-03</td>
<td>1.55</td>
</tr>
</tbody>
</table>

Enforcing continuity over the line \( y = 0 \) leads to the results in Table 40:

Table 40: Degree-2 Polygonal spline approximation of solution to (4.7) with exact solution (4.9) when \( \eta = 10^{-10}, C_1 = 0.1, c_2 = c_1 + \varepsilon \pi^2/4 \)

<table>
<thead>
<tr>
<th># P</th>
<th>mesh</th>
<th>( E_{RMS} )</th>
<th>rate</th>
<th>( \nabla E_{RMS} )</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>6.67e-01</td>
<td>1.65e-03</td>
<td>0.00</td>
<td>2.62e-02</td>
<td>0.00</td>
</tr>
<tr>
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<td>8.87e-03</td>
<td>1.56</td>
</tr>
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<tr>
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<td>2.75</td>
<td>1.20e-03</td>
<td>1.47</td>
</tr>
</tbody>
</table>

From Tables 39 and 40 we can see that the computational results are very similar.

References


