On Fourier restriction and the Newton polygon.

Akos Magyar

1. Introduction.

If $S \subset \mathbb{R}^{n+1}$ is a manifold a central question of harmonic analysis is to determine the pair of exponents $p, q \geq 1$ for which the Fourier restriction property holds on $S$, denoted by $R_S(p \to q)$, that is when one has a bound

$$
(1) \quad \left( \int_S |\hat{\varphi}|^q \, d\sigma \right)^{1/q} \leq C_{p,q,s} \|\varphi\|_{L^p(\mathbb{R}^{n+1})}
$$

where $d\sigma$ is a measure compactly supported on $S$. This problem is still open even case of the unit sphere $S^2 \subset \mathbb{R}^3$, however the case $q = 2$ for surfaces of everywhere non-vanishing curvature was answered by the classic works of Stein and Thomas, see [S], [T].

The aim of this note is bring into attention the relation between the range of exponents $p$ for which $R_S(p \to 2)$ holds and a numeric invariant, the so-called distance of the associated Newton polygon.

To be more precise, first note that the restriction property is local by nature, and is invariant under affine maps $x \to Ax + b$ (where $\det A \neq 0$). Thus in principle it is enough to study the local problem: $S = (x, f(x))$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a (germ of an) analytic function, such that $f(0) = \nabla f(0) = 0$. If $f(x) = \sum_{k \in \mathbb{Z}^n} a_k x^k$ is the Taylor expansion of $f(x)$ then its Newton polyhedron is defined by

$$
N_f = \text{Conv} \left( \bigcup_{k \in A} k + \mathbb{R}^n_+ \right)
$$

$A = \{k : a_k \neq 0\}$ is the support of the series. The smallest positive $d = d_f$ such that the point $d = (d, \ldots, d)$ is in $N_f$ is called the distance of $N_f$, and its reciprocal $\delta_f = 1/d_f$ the Newton decay rate.

First we give the following necessary condition

**Theorem 1.** Let $S = (x, f(x))$ where $f : \mathbb{R}^n \to \mathbb{R}$ is an analytic function, such that $f(0) = \nabla f(0) = 0$ and let $d\sigma = \psi \, dx$ be a measure on $S$ in local coordinates $x$ where $\psi$ is a smooth function of small support.

If the local $R_S(p \to 2)$ restriction property holds at $(0,0)$ then one has

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$p' \geq 2(1 + d_f)$ where $\frac{1}{p} + \frac{1}{p'} = 1$

and $d_f$ is the distance of the Newton polyhedron $N_f$.

The idea of the proof of (2) is doing the well-known Knapp example, in the general case of an analytic hypersurface. In other words to test the restriction property (1) on functions $\phi_\delta$ whose Fourier transform $\hat{\phi}_\delta$ is supported on a box $B_\delta$ best fitted to the surface $S$.

However the Knapp example is not always sharp, a fact hard to find in the literature, a simple example is the graph of the function:

$$f(x, y) = (y - x^2)^m$$

where $m \geq 3$. Thus one needs additional conditions on $f(x, y)$. One such condition is the so-called R - nondegeneracy of Varchenko, see [AV]. However as we shall see that may be too restrictive especially when the surface has curvatures vanishing to a high order, that is when $d_f$ is large. Our partial result of sufficiency is taking this into consideration.

Let $n = 2$ and let $\alpha$ be the edge of the boundary of the Newton polygon $N_f$ which contains the point $d_f$. Define the principle quasi-homogeneous part of $f$ by

$$f_\alpha(x, y) = x^{-A'}y^{-B} \sum_{(k,l) \in \alpha} a_{kl} x^k y^l$$

if $\alpha$ is connecting the points $(A, B)$ and $(A', B')$ where $A' < A$ and $B < B'$. We define a number, called the multiplicity, to $f_\alpha$.

**Definition 1.** The multiplicity $r_\alpha$ of $f_\alpha$ is defined to be the highest multiplicity of a real root of either $\partial_y f_\alpha(1, y)$ or $\partial_x f_\alpha(x, 1)$.

Note that by the quasi-homogeneity of both $\partial_y f_\alpha$ and $\partial_x f_\alpha$ it is easy to see that in the product decomposition $\partial_{xy}^2 f_\alpha$ either w.r.t. $x$ or $y$ there is a real factor of multiplicity $r - 1$. If $r > 2$ this implies that $\partial_{xy}^2 f_\alpha$ is R - degenerate in the sense of Varchenko, as the latter is equivalent of having distinct real factors in the product decomposition w.r.t. $x$ and $y$, see [AV]. We have

**Theorem 2.** Let $S = (x, y, f(x, y))$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is an analytic function, such that $f(0, 0) = \nabla f(0, 0) = 0$. Moreover assume that $d_f \geq 2$ and the multiplicity $r_\alpha$ of the principal part $f_\alpha$ satisfies

$$r_\alpha \leq d_f - 1$$

d$f$ being the Newton distance as before.

Then the local $R_S(p \to 2)$ restriction property holds at $(0,0)$ for

$$p' > 2(1 + d_f)$$
Note that the above result holds for functions with factors of high multiplicity, for example take: \( f(x, y) = (y - a x^2)^m (y - b x^2)^m (a \neq b) \) for any \( m \in \mathbb{N} \). The proof of (4) is based on an oscillatory integral estimate

**Lemma 1.** Let the function \( f(x, y) \) satisfy the conditions in (4). Then for a smooth function \( \psi(x, y) \) supported in a sufficiently small neighborhood of \((0,0)\) one has the bound for the associated oscillatory integral

\[
|I(\lambda)| = \left| \int_{\mathbb{R}^2} e^{i\lambda f(x,y)} \psi(x,y) \, dx \, dy \right| \leq C \varepsilon (1 + |\lambda|)^{-\delta_f + \varepsilon} \quad \forall \varepsilon > 0
\]

This compares to the result of Varchenko in dimension 2, which shows the same estimate, in any dimensions, under the \( \mathbb{R} - \text{nondegeneracy} \) condition. The proof of (6) exploits a factorization, called the Puiseux product, similarly as was done for oscillatory integral operators in [PS1] and [R].

The route from from (6) to (5) is standard, but based on a highly nontrivial result of Karpoushkin, see [K], [PS3]; namely the local stability of the decay rate of oscillatory integrals with analytic phase in 2 dimensions. In fact it implies that for \( \xi = \lambda(-1, u, v) \) the estimate

\[
|\hat{d}\sigma(\xi)| = \left| \int_{\mathbb{R}^2} e^{i\lambda f(x,y) - ux - vy} \psi(x,y) \, dx \, dy \right| \leq C \varepsilon (1 + |\lambda|)^{-\delta_f + \varepsilon}
\]

for all \( \varepsilon > 0 \) holds uniformly for \((u, v)\) being in a small neighborhood of \((0,0)\), and then also for all \((u, v)\), since outside the neighborhood the gradient of the phase: \( |(\partial_x f - u, \partial_y f - v)| \geq C \lambda \) thus the integral is rapidly and uniformly decreasing.

Also by Greenleaf’s theorem a uniform bound \( |\hat{d}\sigma(\xi)| \leq (1 + |\xi|)^{-\delta} \) implies that the local \( R_S(p \to 2) \) restriction property holds for \( p' \geq 2(1 + \beta^{-1}) \), see [G]. Note that we only need this without the endpoint, i.e. for \( p' > 2(1 + \beta^{-1}) \) which is much easy to prove.

2. **Necessary conditions.**

In this section we prove Theorem 1. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) is an analytic function, such that \( f(0,0) = \nabla f(0,0) = 0 \), and let \( \Gamma_f \) be the boundary and \( V_f \) be the set of vertices of its Newton polygon. We use the notation \( |x|^a = |x_1|^{a_1} \cdots |x_n|^{a_n} \) for \( x \in \mathbb{R}^n \) and \( a \in \mathbb{Z}^n \).

**Proposition 1.** One has for \( x \) being in a sufficiently small neighborhood of the origin:

\[
|f(x)| \leq C \max_{b \in V_f} |x|^b
\]

**Proof.** Let us introduce the ordering on \( \mathbb{Z}^n_+ : \ a < b \text{ if } b - a \in \mathbb{Z}^n_+ , \ b \neq a \), and set of minimal elements: \( A' = \{ a \in A : \exists! b \in A, \ b < a \} \). It is easy to see by an induction on \( n \) that \( A' \) is finite, thus \( V_f \) is finite as well.
Since the Taylor series of $f(x)$ converges one has for every $a \in A'$

$$\sum_{k \in A, \ a \leq k} |a_k| |x|^k \leq C_b |x|^b$$

it follows that

(8)  $$|f(x)| \leq C \max_{a \in A'} |x|^a$$

Moreover for every $a \in A'$ there are vertices $b_1, \ldots, b_n$ (not necessarily distinct) such that a convex for a combination of them: $b = \sum_i \delta_i b_i \leq a'$. Indeed every point on the boundary $\Gamma_f$ is of this form. The Proposition follows from (8).

**Definition 2.** We say that $a \in \mathbb{R}_+^n$ is admissible if

(9)  $$k \cdot a \geq 1 \quad \text{for every} \quad k \in \Gamma_f$$

We have

**Proposition 2.** Let

$$\delta = \min \{ \sum_{j=1}^n a_j : a \text{ is admissible} \}$$

Then $\delta = \delta_f = 1/d_f$.

**Proof.** If $a$ is admissible then $a \cdot d_f = d_f \sum_j a_j \geq 1$ hence $\delta \geq \delta_f$.

On the other hand, let $\alpha$ be the face of $N_f$ containing $d_f$ and let $a \in \mathbb{R}_+^n$ be defined such that $a \cdot k = 1$ for every $k \in \alpha$. Then by the convexity of $N_f$ the plane through $\alpha$ separates $0$ and $N_f$ thus $a \cdot l \geq 1$ for every $l \in N_f$. In particular $a \in \mathbb{R}_+^n$ and $a$ is admissible. Also $a \cdot d_f = d_f \sum_j a_j = 1$ and thus $\delta \leq \delta_f$.  

**Proof of Theorem 1.** Let $\psi(x)$ be smooth cut-off function of small support and let $a = (a_1, \ldots, a_n)$ be admissible. For $0 \leq \tau < 1$ define the function $\phi_{a, \tau}$ such that

$$\phi_{a, \tau}(x_1, \ldots, x_{n+1}) = \psi(\tau^{-a_1} x_1, \ldots, \tau^{-a_n} x_n, \tau^{-1} x_{n+1})$$

If $|x_j| \leq \tau^{a_j}$ for $1 \leq j \leq n$ then since $a$ is admissible and (7) it follows

$$|f(x)| \leq C \max_{k \in V_f} \tau^{k-a} \leq C \tau$$
and thus (choosing $\psi$ appropriately), $|\hat{\phi}_{a,\tau}(x, f(x))| \geq c > 0$. It follows

\[
(10) \quad \int_S |\hat{\phi}_{a,\tau}|^2 \, d\sigma = \int_{\mathbb{R}^n} |\hat{\phi}_{a,\tau}(x, f(x))|^2 \, dx \geq c \tau^{\delta_a}
\]

where $\delta_a = \sum_j a_j$. On the other hand by scaling one has $\|\phi_{a,\tau}\|_L^p \approx \tau^{(1+\delta_a)/p'}$ and if the $R_S(p \to 2)$ restriction property holds, then one must have

\[
\tau^{\delta_a/2} \leq C \tau^{(1+\delta_a)/p'}
\]

for every $0 < \tau < 1$ and admissible $a$. Thus

\[
p' \geq \max_a 2(1 + \delta_a^{-1}) = 2(1 + d_f)
\]

by Proposition 2, and this proves (2). \(\square\)

3. Sufficient conditions.

We will need the following Van der Corput type lemma for oscillatory integrals with polynomial type phases, proved in [PH], see Theorem. 1 there, and the remarks after it.

**Lemma 2.** Let $I = [a, b]$ be an interval of length at most 1, $\psi(y) \in C^1$ be a smooth function and let $f(y)$ be a real and $C^2$. Assume that $|f'(y)| \geq \delta > 0$ for all $y \in I$, and $f''(y)$ has at most $d$ roots in $I$. Then one has for $\lambda > 0$

\[
(11) \quad |\int_I e^{i\lambda p(y)}\psi(y) \, dy| \leq \lambda^{-1}\delta^{-1}(2d + 1)(\sup_I |\psi| + \sup_I |\psi'|)
\]

To be able to apply this for $f(x, y)$ in one variable, one needs

**Proposition 3.** Let $F(x, y)$ be a nonzero real and analytic function. Then there is an $\eta > 0$ and a positive integer $d$ such that for every $0 < |x| < \eta$, the function $y \to F(x, y)$ has at most $d$ roots in $[-\eta, \eta]$, and the same is true by interchanging the role of $x$ and $y$.

**Proof.** Let $K = \min \{k : a_{kl} \neq 0\}$ where $F(x, y) = \sum_{k,l} a_{kl} x^k y^l$ and write $F(x, y) = x^K G(x, y)$. The function $G(0, y)$ is not identically zero, thus by the Weierstrass preparation theorem, see [H] Sec. 7.5:

\[
G(x, y) = U(x, y)(c_d y^d + c_{d-1} x y^{d-1} + \ldots + c_0(x))
\]

where $U(x, y) \neq 0$ in a neighborhood of $(0,0)$ and $c_j(x)$ are analytic. If at $x$ there is a $j$ s.t. $c_j(x) \neq 0$ then $y \to F(x, y)$ can have at most $d$ roots. However the set of common zeroes of the
functions \( c_{d-1}(x), \ldots, c_0(x) \) cannot accumulate at \( x = 0 \) because then, by analyticity, all \( c_j(x) \) and thus \( F(x, y) \) would be identically 0. \( \square \)

We describe the Newton polygon \( N_f \) associated to \( f(x, y) \) in more detail. If \( \alpha \) runs through the compact edges of the boundary \( \Gamma_f \), connecting the vertices \((A_\alpha, B_\alpha)\) and \((A'_\alpha, B'_\alpha)\) where \( A_\alpha > A'_\alpha \) then put \( n_\alpha = B'_\alpha - B_\alpha \) and \( \gamma_\alpha = \frac{A_\alpha - A'_\alpha}{B'_\alpha - B_\alpha} \). Also let \( A \) be the \( x \) and \( B \) be the \( y \) coordinate of the vertical and horizontal infinite edges of \( \Gamma_f \). We say that \( \alpha < \beta \) if \( \gamma_\alpha < \gamma_\beta \). Note that

\[
A_\alpha = A + \sum_{\beta < \alpha} n_\beta \gamma_\beta \quad \text{and} \quad B_\alpha = B + \sum_{\beta > \alpha} n_\beta
\]

Also, \( N_f \) has some symmetry properties, if \( g(x, y) = f(\pm x, \pm y) \) then \( N_f = N_g \), if \( g(x, y) = f(y, x) \) then \( N_g \) is obtained by reflecting \( N_f \) to the bisector \( y = x \). We will denote by \( N_x \) and \( N_y \) the Newton polygons associated to \( \partial_x f \) and \( \partial_y f \), then \( N_x \) (resp. \( N_y \)) is obtained from \( N_f \) by shifting it to the left (resp. down) by 1, and may be replacing the infinite edges.

The germ of an analytic function \( F(x, y) \) admits a factorization called the Puiseux product, see [R], of the form

\[
F(x, y) = U(x, y)x^{A}y^{B} \prod_{\alpha = 1}^{n_\alpha} (y - y_{\alpha i}(x))
\]

where \( U(x, y) \) and \( c_{\alpha i} \) are nonzero, and \( y_{\alpha i}(x) \) is asymptotic to a fractional power series of the form: \( c_{\alpha i}x^{\gamma_0} + c_{\alpha i}'x^{\gamma_0 + \gamma} \ldots \), as \( x \to 0 \) in particular for any given \( \tau > 0 \) one has

\[
|y_{\alpha i}(x) - c_{\alpha i}x^{\gamma_0}| \leq \tau x^{\gamma_0}
\]

for \( x > 0 \) small enough w.r.t. \( \tau \). Moreover there is a fixed large constant \( D > 0 \) such that if \( 2^{-j-1} \leq x < 2^{-j} \) and \( 2^{-k-1} \leq y < 2^{-k} \) for \( j, k \geq J \) large enough such that

\[
|y - y_{\alpha i}(x)| \sim \begin{cases} 2^{-j\gamma_0} & \text{if } k \geq j\gamma_0 + D \\ 2^{-k} & \text{if } k \leq j\gamma_0 - D \end{cases}
\]

where \( x \sim y \) means that \( C^{-1}x \leq y \leqCx \) for some constant whose value is unimportant. By the notation \( k \gg j\gamma_0 \) it is meant that \( k - j\gamma_0 \geq D \). We remark that both conditions can be achieved by choosing the support of \( \psi(x, y) \) in (6) small enough.

Following [R], one decomposes the support of the integral in (6) into the four quadrants of \( \mathbb{R}^2 \) and notes that each of the resulting integrals are treated the exactly the same way because the Newton polygon is invariant w.r.t. coordinate changes: \( x \leftrightarrow -x, \ y \leftrightarrow -y \). Thus in (6) we assume that the integration is taking place for \( x > 0, \ y > 0 \) and then let: \( I_{jk}(\lambda) = \int_{R_{jk}} e^{i\lambda \psi} \) where \( R_{jk} = [2^{-j-1}, 2^{-j}] \times [2^{-k-1}, 2^{-k}] \). Note that one only have to consider rectangles \( R_{jk} \) intersecting the support of \( \psi \), thus one can assume \( j, k \geq J \). After these preparation we turn to the
Proof of Lemma 1. Consider the Newton polygon $N_y$ associated with $F(x, y) = \partial_y f(x, y)$. For a fixed compact edge $\alpha$ define

$$I_\alpha(\lambda) = \sum_{j, \gamma \ll k \ll j, \gamma} I_{jk}(\lambda)$$

and

$$I^\alpha(\lambda) = \sum_{|\gamma - k| < D} I_{jk}(\lambda)$$

where $\alpha < \alpha'$ are consecutive edges, also if $\alpha_0$ denote the minimal resp. $\alpha_s$ the maximal edge, then let

$$I_{-\alpha}(\lambda) = \sum_{k \ll j, \gamma} I_{jk}(\lambda)$$

and

$$I_{+\alpha}(\lambda) = \sum_{k \gg j, \gamma} I_{jk}(\lambda)$$

It is clear that it is enough to prove (6) for the integrals appearing in (16) and (17).

To estimate $I_\alpha(\lambda)$ observe that if $(x, y) \in R_{jk}$ the by (12) and (15)

$$|F(x, y)| \sim 2^{-jA_\alpha}2^{-kB_\alpha}\prod_{\beta < \alpha} 2^{-j\gamma_\beta} \prod_{\beta \geq \alpha} 2^{-kn_\beta} = 2^{-(jA_\alpha + kB_\alpha)}$$

At this point we are in a position to apply Lemma 2 for $f(x, y)$ as a function of $y$, $x$ being fixed. Indeed if $d_f \geq 2$ then $\partial_y f(x, y)$ is nonzero and analytic it has at most a fixed number roots for all $x \sim 2^{-j}$. If $\mu$ denotes the right side of (18) then:

$$|\partial_y f(x, y)| \sim \mu$$

thus by Lemma 2, and then integrating trivially in $x$

$$|I_{jk}(\lambda)| = \int_{R_{jk}} e^{i\lambda f(x,y)} \psi(x,y) \, dy \leq C \lambda^{-1}2^{iA_\alpha + kB_\alpha}$$

and also trivially $|I_{jk}(x)| \leq 2^{-(j+k)}$. Let us assume first that $A_\alpha > B_\alpha$ and write $k = j\gamma_\alpha + r$ (hence $r \geq D$). Substituting this for $k$ in (18) and then taking the geometric mean of the two estimates to cancel the factors depending on $j$, one obtains

$$|I_{jk}(\lambda)| \leq C \lambda^{-\frac{\delta}{\alpha}+\epsilon 2((B_\alpha+1)\delta-1-\epsilon)\lambda^{-j}} \leq 2^{-j\epsilon}$$

where $\delta = \frac{\gamma_\alpha+1}{A_\alpha + \gamma_\alpha(B_\alpha+1)}$. Here $\epsilon$ is a generic small parameter whose actual values can be different (but comparable) at its different occurrences even within the same formula, by abusing but simplifying the notation.

It is easy to see that $\delta = 1/d_\alpha$ where $(d_\alpha, d_\alpha)$ is on the line through $(A_\alpha, B_\alpha + 1)$ of normal slope $\gamma_\alpha$ which is the compact edge of the Newton polygon $N_f$, that is the one associated to $f(x, y)$. By convexity $d_\alpha \leq d_f$ and thus $\delta \geq \delta_f$. The other exponent in (20) is less then equal than 0, if $B_\alpha + 1 \leq d_\alpha$ that is if the vertex $(A_\alpha, B_\alpha + 1)$ of $N_f$ is on or below the bisector, which is the same as $A_\alpha > B_\alpha$. The other case is analogous by writing $k = j\gamma_\alpha' - r$ or by exploiting the symmetry in
\( x \) and \( y \). Indeed if the vertex \((A_\alpha, B_\alpha + 1)\) is above the bisector, then interchanging the role of \( x \) and \( y \) which amounts to reflecting \( N_f \) to the bisector \( y = x \), reduces the situation to the first case. Estimating \( I^+(\lambda) \) works exactly the same way, but here \((A_\alpha, B_\alpha + 1)\) is lowest vertex of \( N_f \), and bounds for \( I^+(\lambda) \) are obtained by interchanging the role of \( x \) and \( y \).

The multiplicity condition enters in the estimates for \( I^\alpha(\lambda) \). Again consider the Puiseux product of \( F(x, y) = \partial_y f(x, y) \), and assume first that all \( c_{\alpha i} \)'s are real. Write

\[
F_\alpha(x, y) = \prod_{i=1}^{n_\alpha} (y - c_{\alpha i} x^{\gamma_{\alpha i}}) = (y - c_1 x^{\gamma_1}) \cdots (y - c_\alpha x^{\gamma_\alpha}) r_s
\]

and let \( r_\alpha = \max_s r_s \). For each \( 1 \leq l \leq s \) for each \( x \) define the clusters:

\[ C_{x, l} = \{ y_{\alpha i}(x) : c_{\alpha i} = c_l \}. \]

Observe that by (14) the diameter of each cluster is at most \( \tau 2^{-j_\gamma} \) while the distance between them is \( \sim 2^{-j_\gamma} \).

For a fixed \( x \) we make use of the Whitney decomposition of the set: \([2^{-k_1}, 2^{-k_2}]/C_x\) where \( C_x = \{y_{\alpha 1}(x), \ldots, y_{\alpha n}(x)\}\). It is a collection of intervals \( I_m \) of length \( \sim 2^{-m} \) such that the distance of a point \( y \in I_m \) from the set \( C_x \) is again \( \sim 2^{-m} \). It is clear that one can assume \( m \geq j_\gamma \) and for a given \( m \) there are just a fixed number of intervals \( I_m \) independently of \( x \). We estimate the size of \( F(x, y) \) if \( y \in I_m \) for a given \( m \) (in fact the intervals \( I_m \) depend on \( x \) too, however all our estimates will be independent of \( x \)). If \( C_l \) is the closest cluster to \( I_m \) then for \( y \in I_m \) one has

\[
|y - y_{\alpha i}(x)| \geq \begin{cases} 2^{-m} & \text{if } y_{\alpha i}(x) \in C_l \\ 2^{-j_\gamma} & \text{if } y_{\alpha i}(x) \notin C_l \end{cases}
\]

where \( x \gtrsim y \) means that \( x \gtrsim Cy \) for a constant \( C > 0 \) whose value is unimportant. Thus in this case

\[
|F| \geq 2^{-j(A_\alpha + kB_\alpha)} \prod_{\beta < \alpha} 2^{-j n_{\beta} \gamma_{\beta}} \prod_{\beta > \alpha} 2^{-k n_{\beta}} 2^{-m r_{\alpha} - (n_{\alpha} - r_l) j_\gamma}
\]

\[
\geq 2^{-j(A_\alpha + \gamma_\alpha B_\alpha)} 2^{r_{\alpha} (m - j_\gamma)}
\]

because \( r_{\alpha} \geq r_l \) and \( m \geq j_\gamma \).

In case of complex roots note that \( |y - y_{\alpha i}(x)| \gtrsim 2^{-j_\gamma} \) holds for every \( y \) thus only the real roots enter with multiplicities. Thus in the general case of both complex and real roots estimate (23) is valid with the \( r_l \)'s denoting the multiplicities of the real roots.

Next, one uses by Lemma 2, which yields

\[
|I_{jk,m}(x, \lambda)| \leq C \lambda^{-1} 2^{j A_\alpha + k B_\alpha} 2^{r_{\alpha} (m - j_\gamma)}
\]

uniformly for \( x \in I_m \). As in the previous case, this can be balanced against the trivial estimate \( 2^{-m} \) giving
(25) \[ |I_{jk,m}(x, \lambda)| \leq C \lambda^{-\frac{1}{r+1} + \epsilon} 2^{-m\epsilon} 2^j \frac{\alpha + \gamma B_\alpha - r\gamma - \alpha}{r+1} - \epsilon \]

and can be summed in \( m \). Since the estimate obtained is uniform in \( x \), integrating in \( x \) gives

(26) \[ |I_{jk}(\lambda)| \leq C \epsilon \lambda^{-\frac{1}{r+1} + \epsilon} 2^j \frac{\alpha + \gamma B_\alpha - r\gamma - \alpha}{r+1} - 1 - \epsilon \]

which we balance against the trivial estimate \(|I_{jk}| \leq 2^{-(j+k)} \leq 2^{-j(1+\gamma)}\) to kill the \( j \) factors, by choosing \( \delta \) such that

\[ \delta \left( \frac{A_\alpha + \gamma B_\alpha - r\gamma}{r+1} \right) = (1-\delta)(1+\gamma) \]

which gives

(27) \[ \delta = (r+1) \frac{1 + \gamma}{A_\alpha + \gamma (B_\alpha + 1)} = (r+1)\delta_\alpha \]

where \( \delta_\alpha = 1/d_\alpha \) and \( d_\alpha \) is the distance to the compact edge of the Newton polygon \( N_f \) and \( \delta_\alpha \geq \delta_f \).

Substituting back into (26) one obtains

(28) \[ |I_{jk}(\lambda)| \leq C \lambda^{-\delta f + \epsilon} 2^{-j\epsilon} \]

which can be summed in \( j \), note that there are only at most \( D \) possible values of \( k \) for each \( j \), giving the desired estimate in (6). However one can only take the geometric mean of two estimates if \( 0 \leq \delta \) which is immediate from (27) and if \( \delta \leq 1 \). In case if \( \alpha \) is the principal edge containing the point \((d_f, d_f)\) this holds by our multiplicity condition: \( r + 1 \leq d_f = d_\alpha \). Otherwise the edge joining \((A_\alpha, B_\alpha + 1)\) and \((A'_\alpha, B'_\alpha + 1)\) of \( N_f \) corresponding to \( \alpha \), is either lies completely below the bisector or above it. In the former case since \( A'_\alpha < A_\alpha, B_\alpha < B'_\alpha \) it follows that \( d_\alpha \geq B'_\alpha + 1 = r + 1 + (n\alpha - r) + B_\alpha \) thus \( d_\alpha \geq r + 1 \) which was required. The other case can be reduced to this by interchanging the role of the variables \( x \) and \( y \), and thus by reflecting the Newton polygon to the bisector. This proves Lemma 1. \( \square \)
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