Note Simpler auxilary function $g_\varepsilon(z) = 1 + \varepsilon(z-a)$

If $a \le \mathrm{Re}\, z \le b$, then $\mathrm{Re}(g_\varepsilon(z)) = 1 + \varepsilon \mathrm{Re}(z-a) \ge 1$

$\Longrightarrow |g_\varepsilon(z)| \ge 1 \Longrightarrow \left|\frac{f(z)}{g_\varepsilon(z)}\right| \le f(z) \le 1$ if $\mathrm{Re}\, z = a$ OR $\mathrm{Re}\, z = b$.

$|\mathrm{Im}\, g_\varepsilon(z)| = |\mathrm{Im}\, z| \longrightarrow \infty$ as $z \to \infty$ ($a \le \mathrm{Re}\, z \le b$).

Pf of HY Suppose $f \in L^p$. Let f_n be simple s.t. $f_n \xrightarrow{L^p} f$ and

$f = f_1 + f_2$, $f_1 = f\,\mathbb{1}_{\{|f| \le 1\}} \in L^2$, $f_2 = f\,\mathbb{1}_{\{|f| > 1\}} \in L^1$

By def $\hat{f} = \hat{f_1} + \hat{f_2}$. Note that $f_{n,1} \xrightarrow{L^p} f_1 \Rightarrow f_{n,1} \xrightarrow{L^2} f_1$

$\Rightarrow \hat{f}_{n,1} \xrightarrow{L^2} \hat{f_1}$ and $\hat{f}_{n,2} \xrightarrow{L^\infty} \hat{f}$ $\Rightarrow$ (passing to a subseq) $\exists f_{n_k}$ simple s.t. $\hat{f}_{n_k} \to \hat{f}$ pointwise

Now; f_{n_k} $\Rightarrow |\hat{f}_{n_k}|^q \to |\hat{f}|^q$ pointw.

$\Rightarrow \liminf_{k\to\infty} |\hat{f}_{n_k}|^q = |\hat{f}|_q$

Fatou's Lemma $\Rightarrow$ $\|\hat{f}\|_q^q = \int \underline{\lim} |\hat{f}_{n_k}|^q \le \underline{\lim} \|\hat{f}_{n_k}\|^q \le \underline{\lim} \|f_{n_k}\|_p^p = \|f\|_p^p$

(as $\|f_{n_k}\|_p \to \|f\|_p$)

Note If $1 < p < 2$, then $\exists\, C_p < 1$ s.t. $\|\hat{f}\|_q \le C_p \|f\|_p$.

Optimal C_p is known and comes from $\|\hat{g}\|_q = C_p \|g\|_p$ for $g(x) = e^{-\pi|x|^2}$

(1) $\left| \sum_{n=1}^{N} \phi(x-n) \right| \leq C$ for all x

2) If $|x| \geq 2N$ then $\sum_{n=1}^{N} |\phi(x-n)| \leq |x|^{-2}$

Pf: (ii) $|x-n| \geq |x| - N \geq \frac{|x|}{2} \geq \frac{|x|+N}{2}$ $x \geq 2N$

$|x-n| \geq |x| \geq \frac{|x|+N}{2}$ if $x \leq -2N$

$$\Rightarrow |\phi(x-n)| \leq 4(|x|+N)^{-2} \leq 4|x|^{-2} N^{-2}$$

$$\sum |\phi(x-n)| \leq C N^{-1} |x|^{-2}$$

$$\Rightarrow \int_{|x| \geq 2N} \left(\sum |\phi(x-n)| \right)^p \leq N^{-1} \int_{|x| \geq 2N} N^{-2p} \leq N^{-1} \text{ negligible}$$

(i) Let n_0 be the closest to x; so $|x-n_0| \leq |x-n|$ $\forall 1 \leq n \leq N$.

Then $|\phi(x)| \leq (1+|x|)^{-2}$ [$|x-n|$]

$$\sum_{n=1}^{N} (1+|x-n|)^{-2} \leq 2\left(1 + \sum_{k=1}^{\lceil n/2 \rceil} k^{-2}\right) \leq 1$$

$$\Rightarrow \int \left(\sum_{n=1}^{N} |\phi(x-n)| \right)^{-p} \leq C \int_{-N}^{2N} 1 + \int_{|x| \geq N} |x|^{-2} \leq CN$$

Heuristics $\hat{\phi}_n = [(\tau_n \phi) \cdot e^{2\pi i n x}]^{\wedge} =$

$\phi_n(x) = \phi(x-n) e^{2\pi i n x}$ behaves like C_0^∞-functions of disjoint support, thus

$$\left\| \sum_{n=1}^{N} \phi_n \right\|_p^p \approx \sum_{n=1}^{N} \|\phi_n\|_p^p \approx N$$

$\hat{\tau_n \phi}$

Khinchine's ineq. ("randomization")

(Ω, μ) is a prob. space, if $\mu \geq 0$ is a meas, s.t. $\mu(\Omega) = 1$

A function $\omega : \Omega \to \mathbb{R}$ is a random-variable

If $a \leq b$ then $\mu\{x \in \Omega;\ a \leq \omega(x) \leq b\}$ is the prob. that ω takes its value between a, b. ; denoted by $\text{Prob}\{\omega \in [a,b]\} = \text{Prob}\{a \leq \omega \leq b\}$

Expectation: $\mathbb{E}(\omega) := \int_\Omega \omega \, d\mu$ "expected value"

Independent Random Var: $\omega_1, \dots, \omega_n$ are indep. random. variables

if $\forall\ I_1 = [a_1, b_1], \dots, I_n = [a_n, b_n]$

$$\mu\{x; \omega_1(x) \in I_1, \dots, \omega_n(x) \in I_n\} = \mu\{\omega_1 \in I_1\} \cdot \ldots \cdot \mu\{\omega_n \in I_n\}$$

We say $\omega : \Omega \to \mathbb{R}$ is discrete if it takes only finitely many values.

Lemma $\omega_1, \dots, \omega_n : \Omega \to \mathbb{R}$ discrete random variables ; then

(i) ω_1 and $\omega_2 \cdot \ldots \cdot \omega_n$ are indep.

(ii) $\mathbb{E}(\omega_1 \cdot \omega_2 \cdot \ldots \cdot \omega_n) = \mathbb{E}(\omega_1)\, \mathbb{E}(\omega_2) \cdot \ldots \cdot \mathbb{E}(\omega_n)$

Pf: Let $T \subseteq \mathbb{R}$ finite s.t. $\mathcal{R}(\omega_i) \subseteq T = \{t_1, \dots, t_m\}$

(i) $\forall\ \varepsilon_1, \dots, \varepsilon_n \in T$ we have $\mu\{\omega_1 = t_1, \dots, \omega_n = t_n\} = \mu\{\omega_1 = t_1\} \dots \mu(\omega_n = t_n)$

Since $\{\omega_2 \cdot \ldots \cdot \omega_n = \tau\}$ is a disjoint union of sets $\{\omega_2 = \varepsilon_2, \dots, \omega_n = \varepsilon_n\}$

$$\mu\{\omega_1 = \varepsilon_1, \omega_2 \cdots \omega_n = \tau\} = \mu\{\omega_1 = \varepsilon_1, \omega_2 \cdots \omega_n = \tau\}$$

(ii) By (i) and induction, enough to prove $\mathbb{E}(\omega_1 \cdot \omega_2) = \mathbb{E}(\omega_1) \cdot \mathbb{E}(\omega_2)$

as then $\mathbb{E}(\omega_1 \cdot (\omega_2 \cdots \omega_n)) = \mathbb{E}(\omega_1)\, \mathbb{E}(\omega_2 \cdots \omega_n) = \mathbb{E}(\omega_1)\, \mathbb{E}(\omega_2) \dots \mathbb{E}(\omega_n)$

<u>Thm</u> (Khinchin's ineq.) Let $\omega_1 \ldots \omega_N : \Omega \to \pm 1$ be indep. random var.

Then $\forall a_1 \ldots a_N \in \mathbb{R}$ $\forall 1 \le p < \infty$, we have

$$\mathbb{E}\left(\left|\sum_{n=1}^{N} a_n \omega_n\right|^p\right) \approx \left(\sum_{n=1}^{N} |a_n|^2\right)^{\frac{p}{2}} \qquad (1)$$

<u>Note</u> $\mathbb{E}\left(\left|\sum_{n=1}^{N} a_n \omega_n\right|^2\right) = \int_\Omega \left(\sum_{n=1}^{N} a_n \omega_n\right)^2 d\mu =$

$= \int_\Omega \sum_{n,m=1}^{N} a_n a_m \omega_n \cdot \omega_m \, d\mu$; but if $n \neq m$ $\int_\Omega \omega_n \cdot \omega_m = \int_\Omega \omega_n \int_\Omega \omega_m = 0$.

Thus (1) says, that

$$\mathbb{E}\left(\left|\sum a_n \omega_n\right|^p\right) \approx \mathbb{E}\left(\left|\sum_n a_n \omega_n\right|^2\right)^{\frac{p}{2}}$$

$$\Leftrightarrow \left\|\sum_{n=1}^{N} a_n \omega_n\right\|_{L^p(\Omega,\mu)} \approx \left\|\sum_{n=1}^{N} a_n \omega_n\right\|_{L^2(\Omega)}$$

ω_n's are completely orthogonal — $p \to \infty$ becomes much stronger!

<u>Proof</u>

$$\mathbb{E}\left(e^{t \sum_{n \le N} a_n \omega_n}\right) = \mathbb{E}\left(\prod_{n \le N} e^{t a_n \omega_n}\right) = \prod_{n \le N} \left(\mathbb{E}\, e^{t a_n \omega_n}\right)$$

(indep. random var.; $\omega_n = -1, 1$)

$$= \prod_{n \le N} \frac{1}{2}\left(e^{t a_n} + e^{-t a_n}\right)$$

Use $\frac{1}{2}(e^x + e^{-x}) \le e^{\frac{x^2}{2}}$

$$\Rightarrow \mathbb{E}\left(e^{t \cdot \sum_{n \le N} a_n \omega_n}\right) \le e^{\frac{t^2}{2} \sum_{n \le N} a_n^2}$$

HA | $t>0$

Let $\Omega_\lambda = \mu\left(\sum_{n\le N} a_n\omega_n \ge \lambda\right)$; $x\in\Omega_\lambda \Rightarrow e^{t\sum a_n\omega_n(x)} \ge e^{t\lambda}$

$$\Rightarrow \int_{\Omega_\lambda} e^{t\sum a_n\omega_n} \ge \mu(\Omega_\lambda)\, e^{t\lambda} \qquad \le e^{\frac{t^2}{2}\sum a_n^2}$$

$$\Rightarrow \mu(\Omega_\lambda) \le e^{-t\lambda + \frac{t^2}{2}\sum a_n^2} \qquad \forall \lambda>0,\ t>0$$

Choose t optimally: $-\lambda + t\sum_{n\le N} a_n^2 = 0$ so $t = \dfrac{\lambda}{\sum_{n\le N} a_n^2}$

$$\mu(\Omega_\lambda) \le e^{-\frac{\lambda^2}{2\sum a_n^2}} \Rightarrow \mu\left(\left|\sum a_n\omega_n\right| \ge \lambda\right) \le 2e^{-\left(\frac{\lambda^2}{2\sum a_n^2}\right)} \quad (*)$$

Distributional functions. Let $f = \sum_{n\le N} a_n\omega_n$

$$\int_\Omega |f|^p\,d\mu = \int_\Omega \int_0^{|f|^p} 1\,dt\,d\mu = \int_\Omega\int_0^\infty \mathbb{1}_{\{f(x)\ge t\}}\,dt\,d\mu$$

$$= \int_\Omega \mathbb{1}_{\{|f(x)|^p \ge t\}}\,d\mu\,dt = \int_\Omega^\infty \mu\{x; |f(x)| \ge t^{\frac{1}{p}}\}\,dt =$$

let $\lambda = t^{\frac{1}{p}}$ so $t = \lambda^p$ $dt = \lambda^{p-1}\,d\lambda$

$$= \int_0^\infty \mu\{|f(x)| \ge \lambda\}\,\lambda^{p-1}\,d\lambda$$

$e^{-t^2}\, t^{p-1}$

$$\Rightarrow \int_\Omega \left|\sum a_n\omega_n\right|^p d\mu \le 2p\int^\infty e^{-\left(\frac{\lambda^2}{2\sum a_n^2}\right)}\,\lambda^{p-1}\,d\lambda =$$

let $t = \dfrac{\lambda}{\sqrt{\sum a_n^2}}$ $\quad \lambda^{p-1}d\lambda = (\sum a_n^2)^{\frac{p}{2}}\, t^{p-1}\,dt$ $\quad d\lambda = (\sum a_n^2)^{\frac{1}{2}}\,dt$ $\quad = C_p\left(\sum a_n^2\right)^{\frac{p}{2}}$ ✓

"Hölder's ineq"

$$\sum_{n\le N} |a_n|^2 \overset{f\cdot f}{=} \mathbb{E}\left(\left|\sum_{n\le N} a_n\omega_n\right|^2\right) \le \left(\mathbb{E}\left|\sum a_n\omega_n\right|^p\right)^{\frac{1}{p}} \mathbb{E}\left(\left|\sum a_n\omega_n\right|^{p'}\right)^{\frac{1}{p'}}$$

$$\lesssim \left(\sum |a_n|^2\right)^{\frac{1}{2}} \left(\mathbb{E}\left(\left|\sum a_n\omega_n\right|^p\right)\right)^{\frac{1}{p}}$$

$$\Rightarrow \left(\sum |a_n|^2\right)^{\frac{1}{2}} \lesssim \mathbb{E}\left(\left|\sum a_n\omega_n\right|^p\right)^{\frac{1}{p}}$$

□

Hausdorff-Young Let $\phi \in C_0^\infty$, $(k_n)_{n=1}^N$ s.t. $\phi(\cdot - k_n) = \phi_n$ have disjoint support. Let $\omega_1, \dots, \omega_N : \Omega \to \{-1,1\}$ indep. random var.

$$\Rightarrow \left\|\sum_{n\le N} \omega_n \phi_n\right\|_p^p = \sum \|\phi_n\|_p^p = CN \Rightarrow \left\|\sum \omega_n\phi_n\right\|_p = CN^{\frac{1}{p}}$$

$\left\|\sum \omega_n\hat{\phi}_n\right\|_{p'} \gtrsim N^{\frac{1}{2}} \gg N$

Now consider
$$\left\|\sum_{n\le N} \omega_n \hat{\phi}_n\right\|_{p'}^{p'} = \int_{\mathbb{R}} \left|\sum_{n\le N} \omega_n \hat{\phi}_n(\xi)\right|^{p'} d\xi$$

$\|\hat{f}\|_{p'} \le \|f\|_p \quad p<2$

$$= \int_{\mathbb{R}} \left|\sum_{n\le N}{}^{'} \omega_n \underbrace{e^{2\pi i k_n \cdot \xi}\hat{\phi}(\xi)}_{a_n}\right|^{p'} d\xi$$

Average this over $x \in \Omega$ i.e. consider

$$\mathbb{E}\left(\left\|\sum \omega_n \hat{\phi}_n\right\|_{p'}^{p'}\right) = \int_\Omega \left\|\sum \omega_n(x)\hat{\phi}_n\right\|_{p'} d\mu$$

$$= \int_{\mathbb{R}} \int_{\Omega} \underbrace{\left|\sum_{n\le N}{}^{'} \omega_n e^{2\pi i k_n\cdot\xi}\hat{\phi}(\xi)\right|^{p'} d\mu}_{\approx \left(N|\hat{\phi}(\xi)|^2\right)^{\frac{p'}{2}}} d\xi \lesssim N^{\frac{p'}{2}} \|\hat{\phi}\|_{p'}^{p'}$$

$\lesssim N^{\frac{p'}{2}} \Rightarrow \exists\, \omega_n = \pm 1$ s.t. $\left\|\sum \omega_n \hat{\phi}_n\right\|_{p'} \gtrsim N^{\frac{1}{2}}$ □