Linear Schrödinger Eq

1) $u(t,x) : I \times \mathbb{R}^n \longrightarrow \mathbb{C}$ $= ?$ s.t. $I = [0,T]$

$$i\partial_t u(t,x) - \Delta_x u(t,x) = 0 \quad , \quad u(0,x) = u_0(x) \qquad (1)$$

Let $\hat{u}(t,\xi) := \int e^{-2\pi i x\cdot\xi} u(t,x)\,dx$

* If $|\partial_t u(t,x)| \le v(x)$ unif. on I, then

$$\partial_t \hat{u}(t,\xi) = \widehat{\partial_t u}(t,\xi)$$

i.e. $$\frac{\partial}{\partial t}\int e^{-2\pi i x\cdot\xi} u(t,x)\,dx = \int e^{-2\pi i x\cdot\xi} \frac{\partial}{\partial t} u(t,x)\,dx$$

(by L.D.C. and Mean Value Thm)

Also, if $\Delta_x u(t,x) \in L^1$ (with $\Delta_x = \frac{1}{4\pi}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$)

then $$\widehat{\Delta_x u}(t,\xi) = \frac{|2\pi i\,\xi|^2}{4\pi} = -\pi|\xi|^2\,\hat{u}(t,\xi)$$

Thus (1) $\Rightarrow$

$$i\,\widehat{\partial_t u}(t,\xi) + \pi|\xi|^2\,\hat{u}(t,\xi) = 0$$

$$\Leftrightarrow \partial_t \hat{u}(t,\xi) - \pi i|\xi|^2\,\hat{u}(t,\xi) = 0, \quad \hat{u}(0,\xi) = \hat{u}_0(\xi)$$

$$\frac{d}{dt}\left[e^{-\pi i t|\xi|^2}\,\hat{u}(t,\xi)\right] = 0 \Rightarrow e^{-\pi i t|\xi|^2}\underset{\hat{u}(t)}{} = \hat{u}_0(\xi)$$

$$\hat{u}(t,\xi) = e^{\pi i t|\xi|^2}\hat{u}_0(\xi) \Longrightarrow u(t,x) = U(t)\,u_0(x)$$

Let $u(t,x) = U(t)\,u_0(x)$. A

<u>Claim!</u> Let $u_0 \in S(\mathbb{R}^n)$, then $u(t,\cdot) = U(t)u_0$

satisfies $\quad i\partial_t u(t,x) - \Delta_x u(t,x) = 0$, $\quad u(0,x) = u_0(x)$

<u>Pf</u> • $\forall t \in I \quad u(t,\cdot) \in S(\mathbb{R}^n)$ $\quad (\Leftarrow u_t(x) := u(t,x)$, then $u_t \in S(\mathbb{R}^n))$

$$\hat{u}_t(\xi) = e^{\pi i t|\xi|^2}\hat{u}_0(\xi) \in S \Rightarrow u_t \in S$$

$$\Rightarrow u(t,x) = \int e^{2\pi i x\cdot\xi}\hat{u}(t,\xi) = \int e^{2\pi i x\cdot\xi} e^{\pi i t|\xi|^2}\hat{u}_0(\xi)\,d\xi$$

Now $\left|\frac{\partial}{\partial t}\left(e^{2\pi i x\cdot\xi}\underbrace{\hat{u}_0(\xi)\,e^{\pi i t|\xi|^2}}_{\hat{u}(t,\xi)}\right)\right| \le \pi|\xi|^2|\hat{u}_0(\xi)| \in S(\mathbb{R}^n) \subset L^1$

$\overset{LDC}{\Longrightarrow}$ $\quad \partial_t u(t,x) = \int e^{2\pi i x\cdot\xi}\,\partial_t\hat{u}(t,\xi)\,d\xi$

(as $|\partial_t\hat{u}(t,\xi)| \le \pi|\xi|^2\hat{u}_0(\xi) \in L^1$ unif in t)

• $\widehat{\Delta u}(t,\xi) = -\pi|\xi|^2\,\hat{u}(t,\xi)$

as long as $\Delta u(t,\cdot) \in L^1$ but $u(t,\cdot) \in S \Rightarrow$

$\Rightarrow \Delta u(t,\cdot) \in S$

That $\quad [i\partial_t u(t,\cdot) - \Delta u(\cdot)]^\wedge(\xi) = i\partial_t\hat{u}(t,\xi) + \pi|\xi|^2\hat{u}(t,\xi) = 0$

(ii)

$$i\partial_t u - \Delta u = f(t,x), \quad u(0,x) = u_0(x)$$

<u>Formal solution</u>

$$i\partial_t\hat{u}(t,\xi) - \pi|\xi|^2\hat{u}(t,\xi) = \hat{f}(t,\xi), \quad \hat{u}(0,\xi) = \hat{u}_0(\xi)$$

$$\partial_t\hat{u}(t,\xi) + i\pi|\xi|^2\hat{u}(t,\xi) = \hat{f}(t,\xi) \qquad | \times e^{-\pi i t|\xi|^2}$$

$$\frac{d}{dt}\left[e^{-\pi i t|\xi|^2}\hat{u}(t,\xi)\right] = e^{-\pi i t|\xi|^2}\hat{f}(t,\xi)$$

$$e^{-\pi i t|\xi|^2}\hat{u}(t,\xi) = \hat{u}_0(\xi) + \int_0^t e^{-\pi i s|\xi|^2}\hat{f}(s,\xi)\,ds$$

$$\hat{u}_t(\xi) = \hat{u}(t,\xi) = e^{\pi i t|\xi|^2}\hat{u}_0(\xi) + \int_0^{t} e^{\pi i (t-s)|\xi|^2} \hat{f}(s,\xi)\, d\xi$$

$$'' \; e^{2\pi i t|\xi|^2}\int_0^t e^{-2\pi i \ldots}$$

$$\Rightarrow \quad u_t = \mathcal{U}(t)\, u_0 + \int_0^t \mathcal{U}(t-s) f_s \, ds$$

or $u(t,x) = [\mathcal{U}(t)\, u_0](x) + \int_0^t [\mathcal{U}(t,s) f(s,\cdot)](x)\, ds \qquad (4)$

Clm: If $f \in S(\mathbb{R}\times\mathbb{R}^n)$, $u_0 \in S(\mathbb{R}^n)$ then $u(t,x)$ solves (IV)

Pf. $\hat{u}_t \in S$ as $\hat{f}(s,\xi) \in S(\mathbb{R}\times\mathbb{R}^n)$; in fact

Also $|\partial_t \hat{u}(t,\xi)| \leq \pi|\xi|^2|\hat{u}_0(\xi)| + |\hat{f}(t,\xi)| + t|\xi|^2 + \int_0^t s|\xi|^2|\hat{f}(s,\xi)|$

$$\Rightarrow \widehat{\partial_t u}(t,\xi) = \partial_t[\hat{u}(t,\xi)]$$

$$\Rightarrow \forall t \; [i\partial_t u - \Delta_x u]^\wedge(\xi) = i\partial_t\hat{u} + \pi|\xi|^2\hat{u}(t,\xi) = \hat{f}(t,\xi) \in L^1$$

$$\Rightarrow i\partial_t u - \Delta_x u = f \quad \checkmark$$

Note The function $u_t = \mathcal{U}(t)\, u_0 + \int_0^t \mathcal{U}(t-s) f_s\, ds$

make sense if $u_0 \in L^2$, $f_s(x) = f(s,x) \in L^2(\mathbb{R}^n)$ and $s \mapsto f_s : I \to L^2(\mathbb{R}^n)$ is cont $\Rightarrow$ integrable!

($\hookrightarrow$ study later) Then $u_t \in L^2$ and $\Rightarrow \in S'(\mathbb{R}^n)$

and $i\partial_t u - \Delta u = f_t$ holds in the dist. sense $\leq C\int |\phi|(1+|x|)$

Note $f \in L^2 \Rightarrow u_f \in S'$ i.e. $\phi \to \int f\phi : S \to \mathbb{R}$ ... $\leq C\int|\phi|^2 \leq C'\int|\phi|$ ... $(1+|x|)$ ...

Note We've proved that if $z \in i\mathbb{R}\setminus\{0\}$, then for $\phi \in S(\mathbb{R})$

$$\int_{\mathbb{R}} e^{-\pi z x^2} \hat{\phi}(x)\,dx = \frac{1}{\sqrt{z}} \int e^{-\pi \frac{x^2}{z}} \phi(x)\,dx, \quad (8.1)$$

where $\sqrt{z}$ is defined on $\mathbb{C}\setminus[-\infty,0]$ so that $\sqrt{z}>0$ if $z>0$ and $\sqrt{\pm i} = e^{\pm\frac{\pi i}{4}}$.

Let T be an $n\times n$ real symmetric matrix, with signature σ.

$\sigma = k_+ - k_-$, where $k_+ = \#$ of positive eigen-values

$k_- = \#$ of negative — " ——

Note $\exists\, U \in SO(n)$ so that $S = UTU^{-1}$ is diagonal (with the same signature)

Let $G_T(x) = e^{-\pi i \langle Tx, x\rangle}$; where $\langle\, ,\rangle$ is the dot product on $\mathbb{R}^n$.

" generalized Gaussian "

Note $G_T \in C^\infty(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n) \subseteq S'(\mathbb{R}^n)$.

Prop 8.1. $\hat{G}_T = e^{\frac{-\pi i \sigma}{4}} |\det T|^{-\frac{1}{2}} G_{-T^{-1}}$ in the distributional sense ; i.e.

$$\int e^{-\pi i\langle Tx,x\rangle} \hat{\phi}(x)\,dx = e^{\frac{-\pi i \sigma}{4}} |\det T|^{-\frac{1}{2}} \int e^{\pi i \langle T^{-1}x,x\rangle} \phi(x)\,dx$$

, for all $\phi \in S(\mathbb{R}^n)$. (8.2)

Proof

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- $n=1$, $\langle Tx,x\rangle = tx^2$; then $G_T(x)=e^{-\pi i t|x|^2}$ and (8.2) follows from (8.1) with $z=it$

- $T = \begin{pmatrix} t_1 & & & & 0 \\ & t_k & & & \\ & & s_1 & & \\ 0 & & & & s_\ell \end{pmatrix}$ diagonal ; $t_i>0$, $s_j<0$, $k-\ell=\sigma$.

Let $\phi(x)=\phi_1(x_1)\ldots\phi_n(x_n) \Rightarrow \hat{\phi}(y)=\hat{\phi}_1(y_1)\ldots\hat{\phi}_n(y_n)$

Since $\langle Tx,x\rangle = \sum_{i=1}^{k} t_i x_i^2 + \sum_{j=1}^{\ell} s_j x_j^2$ both integrals are a product of 1-dim integral, thus (8.2) again reduces to (1.1)

- Suppose (8.2) holds for T.

<u>Claim</u> Then (8.2) holds for $S=UTU^{-1}$.

<u>Pf. of Claim</u>

$$\int e^{-\pi i\langle Sx,x\rangle}\hat{\phi}(x)\,dx = \int e^{-\pi i\langle T\underbrace{U^{-1}x}_{x:=U^{-1}x},\,U^{-1}x\rangle}\hat{\phi}(x)\,dx = \int e^{-\pi i\langle Tx,x\rangle}\hat{\phi}(Ux)\,dx$$

$$= \int e^{-\pi i\langle Tx,x\rangle}\widehat{\phi\circ U}(x)\,dx = e^{\frac{-\pi i\sigma}{4}}|\det T|^{-\frac12}\int e^{+\pi i\langle T^{-1}x,x\rangle}\phi(\underbrace{Ux}_{})\,dx$$

$$= e^{\frac{-\pi i\sigma}{4}}|\det T|^{-\frac12}\int e^{\pi i\langle T^{-1}U^{-1}x,U^{-1}x\rangle}\phi(x)\,dx = e^{\frac{-\pi i\sigma}{4}}|\det S|^{-\frac12}\int e^{\pi i\langle Sx,x\rangle}\phi(x)\,dx$$

- Since $\exists U$ st. UTU^{-1} is diagonal, (8.2) holds for all T symm, real, $\det T\neq 0$. $\square$

<u>Note</u> By analytic con't (8.2) extends complex, symmetric T such that $\operatorname{Im} T \geq 0$ (positive, semi-definite)

Complex Interpolation

Phragen- Lindelöt method

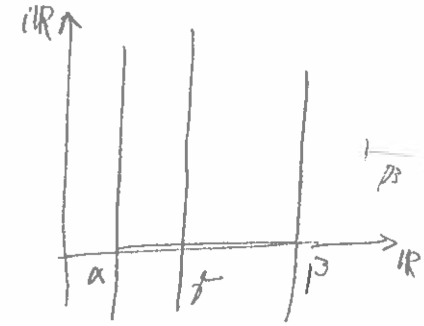
Lemma (3 lines lemma). Let $f(z)$ holom. on $S_{\alpha\beta} = \{\alpha \le \mathrm{Re}\, z \le \beta\}$.

Assume $|f(z)| \le A$ for $\mathrm{Re}\, z = \alpha$ & $|f(z)| \le \beta$ for $\mathrm{Re}\, z = \beta$.

Then if $\gamma = \theta\alpha + (1-\theta)\beta$, then $|f(z)| \le A^{\theta} B^{1-\theta}$ for $\mathrm{Re}\, z = \gamma$ (8.3)

as long as $|f(z)| \le \exp(C \exp(\tau|z|))$

with $\tau < \frac{\pi}{\beta - \alpha}$

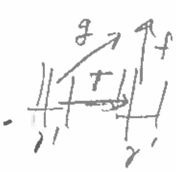


Pf (Sketch)

- Reduction to $\alpha = -\frac{\pi}{2}$, $\beta = \frac{\pi}{2}$; get $g(z) = f(Tz)$ with

  $T(z) = t + sz$ s.t. $T: -\frac{\pi}{2} \mapsto \alpha$, $\frac{\pi}{2} \to \beta$ $\left(s = \frac{\beta - \alpha}{\pi}\right)$

  Then $|g(z)| \le A$ if $\mathrm{Re}\, z = -\frac{\pi}{2}$ & $|g(z)| \le B$ if $\mathrm{Re}\, z = \frac{\pi}{2}$.

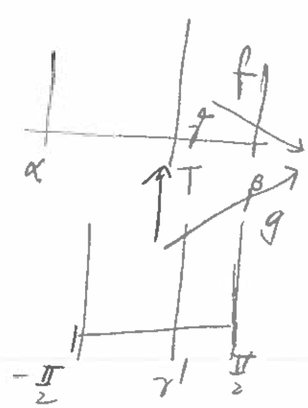


  $\Longrightarrow |g(z)| \le A^{\theta} B^{1-\theta}$ if $\mathrm{Re}\, z = \gamma' = \theta\left(-\frac{\pi}{2}\right) + (1-\theta)\frac{\pi}{2}$

  $\Longrightarrow |f(w)| \le A^{\theta} B^{1-\theta}$ if $w = T(z)$, $\mathrm{Re}\, w = \gamma = T(\gamma')$

  Also $|g(z)| \le \exp(\exp \tau'|z|)$ with $|\tau'| < 1$

  $\Longleftrightarrow |f(w)| \le \exp(\exp \tau|z|)$ with $|\tau| < \frac{\pi}{\beta - \alpha}$



- WLOG assume $\alpha = -\frac{\pi}{2}$, $\beta = \frac{\pi}{2}$

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Reduction to $A=1$, by taking $f(z) \to \frac{1}{A} f(z)$

& to $B=1$, by taking $f(z) \to e^{C(z+\frac{\pi}{2})}$ with approp C.

So WLOG can assume $A=B=1$, $\alpha=-\frac{\pi}{2}$, $\beta=\frac{\pi}{2}$, $\tau<1$

For $\varepsilon>0$ define $h_\varepsilon(z) = \exp(-\varepsilon(e^{i\gamma z}+e^{-i\gamma z}))$, with some $\tau<\gamma<1$.

Then $\mathrm{Re}(e^{\gamma z}+e^{-\gamma z}) = (e^{\gamma y}+e^{-\gamma y})\cos(\gamma x)$ for $z=x+iy$

$\geq \delta(e^{\gamma y}+e^{-\gamma y})$ with $\delta=\cos(\gamma\frac{\pi}{2})>0$

$\Rightarrow |h_\varepsilon(z)| \leq \exp(-\varepsilon\delta(e^{\gamma y}+e^{-\gamma y}))$

$\Rightarrow |f(z)h_\varepsilon(z)| \to 0$ as $|z|\to\infty$ ($\Leftrightarrow |y|\to\infty$)

$\Rightarrow$ If N is suff. large then $|f(z)h_\varepsilon(z)|\leq 1$ on the boundary of $R=\{-\frac{\pi}{2}\leq \mathrm{Re}\, z\leq\frac{\pi}{2},\ -N\leq \mathrm{Im}\, z\leq N\}$

$\Rightarrow$ by Max modulus principle, that $|f(z)h_\varepsilon(z)|\leq 1$ $\forall z$.

- Fix z; then $\lim\limits_{\varepsilon\to 0}|f(z)h_\varepsilon(z)| = |f(z)|\leq 1$ $\forall z$ $\square$

<u>Note</u> Condition is sharp; take $f(z)=e^{e^{iz}}$.

H.A.

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<u>Theorem</u> (Riesz–Thorin)

Let $1 \le p_0 < p_1 \le \infty$, Let T be a lin op which is defined on $L^{p_0} + L^{p_1}$ and assume

$$\|Tf\|_{L^{q_0}} \le A_0 \|f\|_{L^{p_0}}$$

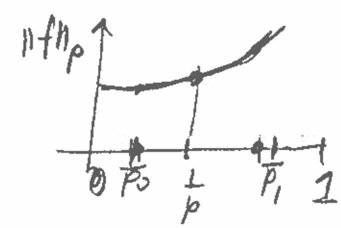
$$\|Tf\|_{L^{q_1}} \le A_1 \|f\|_{L^{q_1}}$$

If $\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$ and $\frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}$ then for $f \in L^p$ one has $\|Tf\|_q \le A_0^{\theta} A_1^{1-\theta} \|f\|_p$. $\quad (\| \ \|_p := \| \ \|_{L^p})$.

<u>Note</u>
- If $T = I$, and we have that $f \in L^{p_0} \cap L^{p_1}$ then $f \in L^p$, and $\|f\|_p \le \|f\|_{p_0}^{\theta} \|f\|_{p_1}^{1-\theta}$
- log-convexity of $\| \ \|_p$; $\log \|f\|_p \le \theta \log \|f\|_{p_0} + (1-\theta) \log \|f\|_{p_1}$

<u>Thm</u> (Hausdorff–Young) Let $f \in L^p$ for $1 \le p \le 2$.

Then $\|\hat{f}\|_q \le \|f\|_p$ where $\frac{1}{q} + \frac{1}{p} = 1$.



<u>Note</u> $p = 2 \Leftarrow$ Plancherel, as $q = 2$

$p = 1$ from. def. as $q = \infty$.

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Note: $\|f\|_p = \sup_{\|g\|_q \le 1} \int fg$, where $\frac{1}{p}+\frac{1}{q}=1$

Pf (Sketch)

- $|\int fg| \le \|f\|_p \|g\|_q \le \|f\|_p \quad \forall g : \|g\|_q \le 1$
- Let f be a simple function, i.e. $f = \sum_{i=1}^{m} t_i \mathbb{1}_{E_i}$; $E_i \cap E_j = \phi$

then $f = \sum_{i=1}^{n} |t_i| \omega_i \mathbb{1}_{E_i} = |f| \cdot \sum_{i=1}^{m} \omega_i 1_{E_i}$

$= |f| \cdot \omega$

- WLOG can assume $\|f\|_p = 1$. with $|\omega| \equiv 1$

Then $1 = \|f\|_p^p = \int |f|^p = \int f \cdot (|f|^{p-1}\omega) = \int f \cdot g$

$\|g\|_q^q = \int |g|^q = \int |f|^{(p-1)q} = \int |f|^p = 1 \Rightarrow \|g\|_q = 1$

then $\|f\|_p = \int f \cdot g$ with $\|g\|_q = 1$. ; note g is simple

- If f is not simple, $\varepsilon > 0$, $\exists\ \phi$ simple such that $\|f-\phi\|_p < \varepsilon$

$\exists \|g\|_q = 1$ s.t. $\int \phi g = \|\phi\|_p \ge \|f\|_p - \varepsilon$

$\Rightarrow |\int fg| \ge |\int \phi g| - |\int (f-\phi) g| \ge \|f\|_p - \varepsilon - \|f-\phi\|_p$

$\ge \|f\|_p - 2\varepsilon$.

$\Rightarrow \sup_{\|g\|_q \le 1} |\int fg| \ge \|f\|_p - 2\varepsilon$

$\square$

Pf (Hausdorff - Young)

Assume $\|f\|_p = 1$, and f is simple. Write $Tf := \hat{f}$,

Let $F = |f|^p$ and $f = |f| \cdot \omega = |F|^{\frac{1}{p}} \omega$

To show that $\|Tf\|_q \le 1$ it is enough to show that

$\int (Tf)\, g \leq 1$ for all $\|g\|_{p'} \leq 1$, g simple.

Write $G = |g|^{p'}$; $g = G^{\frac{1}{p'}} \cdot \varrho$, $|\varrho| = 1$.

Then
$$\int (Tf) \cdot g = \int_{\mathbb{R}^n} T(F^{\frac{1}{p}} \cdot \omega)\,(G^{\frac{1}{p'}} \cdot \varrho)\, dx$$

Now, define
$$\Phi(z) = \int_{\mathbb{R}^n} T(F^{z} \cdot \omega)\,(G^{z} \cdot \varrho)\, dx$$

- Since f is simple $f = \sum t_i 1_{A_i}$, $F^z\omega = \sum |t_i|^{pz} \omega_i 1_{A_i}$
  $g = \sum s_j 1_{B_j}$; $G^z \varrho = \sum |s_j|^{p'z} \varrho_j 1_{B_j}$

  $\Longrightarrow$ $\phi(z)$ is a finite linear comb. of exponentials hence is an entire function
- Let $\operatorname{Re} z = \frac{1}{2}$. Then $\|F^z\omega\|_2^2 = \int |F^{2z}\omega^2|\, dx =$
  $= \int |F| = 1$

  $\Longrightarrow$ $\|T(F^z\omega)\|_2 \leq 1$, similarly $\|G^z\varrho\|_2^2 = \int |G| = 1$

  $\Longrightarrow$ $|\phi(z)| \leq 1$.
- Let $\operatorname{Re} z = 1$, Then $\|F^z\omega\|_1 = \int |F| = 1 \Rightarrow \|T(F^z\omega)\|_\infty \leq 1$
  , also $\|G^z\omega\|_1 = \int |G| = 1$

  $\Longrightarrow$ $|\phi(z)| \leq 1$.
- By 3-lines thm $\Longrightarrow$ $|\phi(\frac{1}{p})| \leq 1$ but $\phi(\frac{1}{p}) = \int (Tf) \cdot g$. $\square$