Def A <u>tempered function</u> is a fund. $f \in L^1_{loc}(\mathbb{R}^n)$ s.t.

$$\int |f(x)| (1+|x|)^{-N} dx < \infty \quad , \text{ for some } N > 0$$

Note, f tempered and $\phi \in S \Rightarrow \int |\phi f| < \infty$

. $\phi \to \int f\phi : S \to \mathbb{C}$ is cont., and linear

($\phi_k \xrightarrow{S} 0 \Rightarrow \int f\phi_k \to 0$ (why ?))

$$\left|\int f\phi_k\right| \le \int |f(x)| (1+|x|)^{-N} \cdot (1+|x|)^N \phi_k \, dx$$

$$\le \left(\int |f(x)| (1+|x|)^{-N}\right) \sup_x \underbrace{(1+|x|^N)\phi_k}_{\downarrow 0}$$

Define for f tempered, ~~$\hat{g}$ s.t.~~ $g := \hat{f}$ s.t.

(*) $$\int g \cdot \phi \, dx := \int f \cdot \hat{\phi} \quad \text{ie. } \int \hat{f} \cdot \phi := \int f \hat{\phi}$$

Makes sen, as if $f \in S$ the (*) holds for $\hat{f}$.

Problem Is g tempered ?

Def 1) a tempered distribution is a $\overset{\text{cont.}}{\text{linear}}$ map: $u : S(\mathbb{R}^n) \to \mathbb{C}$,

$\Updownarrow$ i.e. $\phi_k \xrightarrow{S} 0$ then $u(\phi_k) \to 0$.

2) $u : S(\mathbb{R}^n) \to \mathbb{C}$, st $\exists N$ for which

$$|u(\phi)| \le C \sum_{|\alpha|, |\beta| \le N} \sup_x |x^\alpha \partial_x^\beta \phi(x)|$$

Ex 1. $\delta_0(\phi) = \phi(0)$; $\delta_a(\phi) = \phi(a)$

2. $(D^\alpha_x \delta_0)(\phi) := (-1)^{|\alpha|} D^\alpha \phi(0)$ etc. <u>Note</u> $u_\phi(\psi) = \int \phi\psi =$
$\int D_j\phi \cdot \psi = -\int \phi D_j \psi$, $\int D^\alpha\phi \cdot \psi = \int \phi D^\alpha \psi (-1)$
$u_{D^\alpha\phi}(\psi) = (-1)^{|\alpha|} u_\phi(D^\alpha \psi)$

3. $\mu \in M(\mathbb{R}^n)$: $\mu(\phi) = \int \phi \, d\mu$;

4. $a(x)$ is a temp. function : $M_a f(x) = a(x) f(x)$

5. $g(x)$ is a temp. f., then $I_g(\phi) := \int g(x)\phi(x)\,dx$

Def Let $u \in S'(\mathbb{R}^n)$, then $\mathcal{F}u = \hat{u}$ is define by

$$\langle \hat{u}, \phi \rangle := \langle u, \hat{\phi} \rangle$$

Ex $\hat{\delta}_a(\phi) := \delta_a(\hat{\phi}) = \hat{\phi}(a) = \int e^{-2\pi i x \cdot a} \phi(x)\,dx$

Lemma $u \in S' \Rightarrow \hat{u} \in S'$

Pf: $|\hat{u}(\phi)| = |u(\hat{\phi})| \le \sum_{|\alpha|+|\beta|\le N} \sup_\xi |\xi^\alpha D^\beta \hat{\phi}| \le C \sum_{|\alpha|+|\beta|\le N} \sup_\xi |\widehat{D^\alpha x^\beta \phi}(\xi)|$

$\le C \sum_{|\alpha|+|\beta|\le N} \| D^\alpha x^\beta \phi \|_{L^1} \le C' \sum_{|\alpha|+|\beta|\le N} \| x^\beta D^\alpha \phi \|_{L^1} \le C \sum_{|\alpha|+|\beta|\le N+n} \| x^\beta D^\alpha \phi \|_{L^\infty}$

Let $\psi \in S$, $\psi = D^\alpha x^\beta \phi$, then

$\|\psi\|_1 = \int |\psi(x)|\,dx = \int \psi(x)(1+|x|)^{n+1}(1+|x|)^{-n-1}\,dx$

$\le C \sup \psi(x)(1+|x|)^{n+1}$

<u>Note</u> $S'(\mathbb{R}^n)$ is a lin. space. We say $u_n \xrightarrow{S'} u$ if

$\forall \phi \in S(\mathbb{R}^n)$; $u_n(\phi) \to u(\phi)$ (weak top.)

Fourier multiplier operators Let $a(\xi) \in L^\infty(\mathbb{R}^n)$, define

$$a(D)f := \mathcal{F}^{-1} M_a \mathcal{F} f \quad \text{where } M_a g(\xi) = a(\xi) g(\xi))$$

Claim $f \in L^2 \Rightarrow a(D) f \in L^2$ and $\|a(D)f\|_{L^2} \leq \|a\|_{L^\infty} \|f\|_{L^2}$

Note

- $\widehat{a(D)f}(\xi) = M_a \hat{f}(\xi) = a(\xi)\hat{f}(\xi)$
- If $p(\xi) = \sum_{|\alpha| \leq n} c_\alpha \xi^\alpha$ pol. of degree n

  and if $f \in S$, then since
  $$\widehat{D^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$$
  we have
  $$\widehat{\left(\tfrac{1}{2\pi i} D\right)^\alpha f}(\xi) = \xi^\alpha \hat{f}(\xi)$$
  $$\Rightarrow \widehat{p\left(\tfrac{D}{2\pi i}\right) f} = p(\xi)\hat{f}(\xi) \quad \text{so } p(\xi)\hat{f}(\xi) \text{ is}$$
  indeed a multiplier op. (with a $2\pi i$-factor mis

$$p(D) = P\left(\frac{D}{2\pi i}\right) \quad \text{diff. operator}$$

OR better $P(D) = \sum_{|\alpha| \le n} c_\alpha D^\alpha$ "diff. op"

then $\widehat{P(D)f}(\xi) = \sum_{|\alpha| \le n} c_\alpha \widehat{D^\alpha f}(\xi) = \sum_{|\alpha| \le n} c_\alpha (2\pi i \xi)^\alpha \hat{f}(\xi)$

$= p(2\pi i \xi)\hat{f}(\xi)$ however $p(\xi)$ is not bdd;

- $a, b \in L^\infty \Rightarrow a(D)\, b(D) f = ab(D) f$

  also $a(D)^* = \bar{a}(D)$

Prop 6.1. If $a \in S(\mathbb{R}^n)$, then $a(D)f = K_a * f$; with

$$K_a(x) = \hat{a}(-x) \in S(\mathbb{R}^n)$$

Pf
- assume $f \in S$ (first)

$$a(D)f(x) = \int e^{2\pi i x \xi} a(\xi)\hat{f}(\xi)\, d\xi =$$

$$= \iint e^{2\pi i (x-y)\xi} a(\xi) f(y)\, dy\, d\xi = \int \hat{a}(y-x) f(y)\, dy$$

Let $K_a(x) = \hat{a}(-x)$

$$= \int K_a(x-y) f(y)\, dy = K_a * f(x)$$

- Let $f \in L^2$ and let $f_k \in S$ s.t. $f_k \xrightarrow{L^2} f$

  Then $\hat{f}_k \xrightarrow{L^2} \hat{f} \Rightarrow a\hat{f}_k \xrightarrow{L^2} a\hat{f} \Rightarrow a(D) f_k \xrightarrow{L^2} a(D) f$

  But $\| K_a * f_k - K_a * f \|_{L^2} = \| K_a * (f_k - f) \|_{L^2} \le \| K_a \|_{L^1} \| f_k - f \|_{L^2} \to 0$

  thus $a(D) f_k = K_a * f_k \xrightarrow{L^2} K_a * f \Rightarrow K_a * f = a(D) f$

3. If not just $a \in L^\infty$ but $\hat{a} \in L^1$, then

we have $a(D)f = K_a * f$ with $K_a(x) = \hat{a}(-x)$

this extends to $f \in L^q$ s.t. $\| a(D)f \|_{L^q} = \| K_a * f \|_{L^q} \le \| K_a \|_{L^1} \| f \|_{L^q}$
for all $1 \le q \le \infty$.
$= \| \hat{a} \|_{L^1} \| f \|_{L^q}$

Example Schrödinger group

- For $t \in \mathbb{R}$, define the op: $u(t) = e^{-\pi i t |\xi|^2}$ by
$\widehat{u(t)f}(\xi) = e^{-\pi i t |\xi|^2} \hat{f}(\xi)$ so $u(t) = a(D)$
with $a(\xi) = e^{it|\xi|^2}$

Prop 6.3 $u(0) = I$, $u(t+s) = u(t)u(s)$ and $\| u(t) f \|_{L^2} = \| f \|_{L^2}$ $\forall t$

"$u(t)$" is a 1-par. group of unitary operators on L^2

Pf: $\| u(t) f \|_2 \le \| f \|_2$ but $\| f \|_2 = \| u(-t) u(t) f \|_2 \le \| u(t) f \|_2$.

Prop 6.4 $u(t) f = f * K_t$ with $K_t = \dfrac{e^{\pm \frac{i n \pi}{4}}}{|t|^{\frac{n}{2}}} e^{\pm \frac{\pi i |x|^2}{t}}$

$\pm = \operatorname{sgn} t$

Pf: Let $f \in S$. Then..

$$u(-t) f(x) = \int e^{2\pi i x \cdot \xi} e^{\pi i t |\xi|^2} \hat{f}(\xi)\, d\xi \overset{\text{By L.D.C.}}{=} \lim_{\varepsilon \to 0^+} \int e^{\pi i t |\xi|^2} e^{-\pi \varepsilon |\xi|^2} e^{2\pi i x \cdot \xi} \hat{f}(\xi)\, d\xi$$

$$= \lim_{\varepsilon \to 0^+} \underbrace{\int e^{-\pi(\varepsilon - it)|\xi|^2} e^{2\pi i x \cdot \xi} \hat{f}(\xi)\, d\xi}$$

$$= \lim_{\varepsilon \to 0^+} \int \int e^{-\pi(\varepsilon - it)|\xi|^2} e^{2\pi i (x-y) \cdot \xi} f(y)\, d\xi\, dy = \lim_{\varepsilon \to 0^+} \int K_{\varepsilon - it}(x-y) f(y)\, dy$$

For $\mathrm{Re}\, z > 0$, define $K_z(x) = \int e^{-\pi z |\xi|^2} e^{2\pi i x\cdot \xi}\, d\xi$

note: $\left| e^{-\pi z |\xi|^2} \right| = e^{-\pi \mathrm{Re} z\, |\xi|^2} \in S(\mathbb{R}^n)$.

If $z \in \mathbb{R}$, $z > 0$, then $K_z(x) = \int \underbrace{e^{-\pi z |\xi|^2}} e^{2\pi i x\cdot \xi}\, d\xi$

$$= z^{-\frac{n}{2}} \int e^{-\pi |\xi|^2} e^{2\pi i (z^{-\frac{1}{2}} x)\cdot \xi}\, d\xi = z^{-\frac{n}{2}} e^{-\pi \frac{|x|^2}{z}}$$

$\xi' = z^{\frac{1}{2}} \xi \quad d\xi = z^{-\frac{n}{2}} d\xi'$

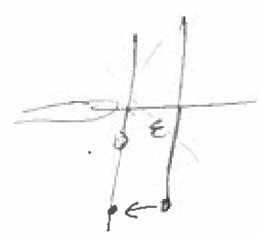
as for the Gaussian, $g(\xi) = e^{-\pi |\xi|^2}$ we saw that $\hat{g}(x) = e^{-\pi |x|^2} = g(x)$

Now $z \to K_z(x)$ is holomorphic on $\mathrm{Re}\, z > 0$ (by L.D.C)

$$\Rightarrow g_z(x) = z^{-\frac{n}{2}} e^{\frac{-\pi |x|^2}{z}} \Rightarrow g_{\varepsilon - it} = (\varepsilon - it)^{-\frac{n}{2}} e^{-\frac{\pi |x|^2}{\varepsilon - it}}$$

Let $t > 0$, then $(\varepsilon - it)^{-\frac{1}{2}} \to e^{+\frac{i\pi}{4}} |t|^{-\frac{1}{2}}$

$t < 0$, then $(\varepsilon - it)^{-\frac{1}{2}} \to e^{-\frac{i\pi}{4}} |t|^{-\frac{1}{2}}$



$$\Rightarrow \lim_{\varepsilon \to 0} (\varepsilon - it)^{-\frac{n}{2}} = e^{\pm \frac{in\pi}{4}} |t|^{-\frac{n}{2}} e^{i\pi \frac{|x|^2}{t}}$$

Thus $K_{-t}(x) = \dfrac{e^{\pm\frac{in\pi}{4}}}{|t|^{\frac{n}{2}}} e^{i\pi \frac{|x|^2}{t}}$

For $t \neq 0$

Cor: $\| e^{it\Delta} f \|_\infty \le C |t|^{-\frac{n}{2}} \| f \|_1$ as $\| K_t \|_\infty \le C |t|^{-\frac{n}{2}}$.

So If $f \in L^1 \cap L^2$ then $e^{it\Delta} f \to 0$ uniformly.

Application: The linear Schrödinger equation

Let for $f \in \mathcal{S}(\mathbb{R}^n)$; $\Delta f = +\frac{1}{4\pi} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} = +\frac{1}{4\pi} \left(\sum_{j=1}^{n} D_j^2\right) f$

- $\frac{1}{4\pi}$ — does not matter

- "$-$" matters as $\langle \Delta f, f \rangle = +\frac{1}{4\pi} \sum_{j=1}^{n} \langle D_j^2 f, f \rangle =$

$$= \frac{-1}{4\pi} \sum_{j=1}^{n} \langle D_j f, D_j f \rangle = \frac{-1}{4\pi} \sum_{j=1}^{n} \|D_j f\|_{L^2}^2 \leq 0$$

So $-\Delta$ is a positive operator.

- linear Schrödinger PDE (smooth version)

$I = [0, T]$, $\left.\begin{array}{l} f \in C_0^\infty(I \times \mathbb{R}^n) \\ u_0 \in C_0^\infty(\mathbb{R}^n) \end{array}\right\}$ best data

Find $u(t,x) = ?$ $\quad u(t,x) : I \times \mathbb{R}^n \to \mathbb{C}$ s.t.

$$\left.\begin{array}{l} i\partial_t u(t,x) - \Delta_x u(t,x) = f(t,x) \\ u(0,x) = u_0(x) \end{array}\right\} \text{IVP}$$

Note evolution equation $\quad u_t(x) := u(t,x)$ given at $t = 0$

1. let $f(t,x) \equiv 0$.
$i\,\partial_t u(t,x) = \Delta_x u(t,x)$, $u(0,x) = u_0(x)$

formally, let $\widehat{u}_t(\xi) = \int e^{-2\pi i x \cdot \xi} u(t,x)\, dx$

then taking the $\mathcal{F}$-trf of both sides, we get

$$i \frac{d}{dt} \widehat{u}_t(\xi) = \int e^{-2\pi i x \xi} \, i\partial_t u(t,x)\, dx$$

int. by parts

$$= \int e^{-2\pi i x \cdot \xi} \left(\frac{-1}{4\pi} \sum_j D_j^2\right) u(t,x)\, dx$$

$-4\pi^2|\xi|^2$

$$i\frac{d}{dt}\hat{u}_t(\xi) = i\frac{\partial}{\partial t}\hat{u}(t,\xi) = \widehat{\Delta_x u}(t,\xi) = +\frac{1}{4\pi}|2\pi i\xi|^2\hat{u}(t,\xi)$$

For fixed ξ: $\quad i\frac{d}{dt}\hat{u}_t(\xi) = -\pi|\xi|^2\,\hat{u}_t(\xi)$

So Δ_x – negative op. "negative op"

Let $\quad \gamma(t) = \hat{u}_t(\xi), \quad \gamma(0) = \hat{u}_0(\xi)$

we have a linear ODE

$$\gamma'(t) = +i\pi|\xi|^2\gamma(t), \quad \gamma(0) = \hat{u}_0(\xi)$$

$$\Rightarrow \gamma(t) = e^{+i\pi t|\xi|^2}\gamma(0)$$

$$\Rightarrow \hat{u}_t(\xi) = e^{+\pi i t|\xi|^2}\hat{u}_0(\xi) = e^{-}$$

$K_t(x) \sim |t|^{-\frac{n}{2}} e^{-i\frac{|x|^2}{t}}$

$t>0$

oscillatory!!

avg in weak sense

$$\Rightarrow u_t(x) = U(t)\,u_0(x) \quad \text{— "formally"}$$

Check $\quad u_0 \in S(\mathbb{R}^n) \Rightarrow U(t)u_0(x) = K_t * u_0(x) = \int K_t(x-y)u_0(y)\,dy$

Define $\quad u(t,x) = U(t)\,u_0(x) \Rightarrow \hat{u}(t,\xi) = e^{\pi i t|\xi|^2}\hat{u}_0(\xi) \quad \forall t$

$\in S$

Verify $\quad$ 1) $x \mapsto u(t,x) \in S(\mathbb{R}^n) \quad \forall t \quad \Rightarrow u(t,x) \in S$

$\widehat{\Delta_x u(t,\cdot)}(\xi) = -\pi|\xi|^2\hat{u}(t,\xi)$

2) $\widehat{\partial_t u}(t,\xi) = \frac{\partial}{\partial t}\hat{u}(t,\xi) = \pi i|\xi|^2\hat{u}(t,\xi)$

$\widehat{\Delta u}(t,\xi) = \pi i|\xi|^2\hat{u}(t,\xi)$

$\Rightarrow [i\partial_t u(t) + \Delta u(t,\cdot)]^\wedge(\xi)$

$\Rightarrow i\partial_t u(t,x) - \Delta u(t,x) = \ldots$

$\int K_t(x)\,dx$

next time

Need! $\quad$ 1. $u_0(S) \to e^{\pi i t|\xi|^2}\hat{u}_0 \in S \Rightarrow U(t)\,u_0 \in S$

$\Rightarrow \forall t: \; x \mapsto u(t,x) \in S$

2. $\left[x \mapsto \frac{\partial}{\partial t}u(t,x)\right]^\wedge(\xi) = \frac{\partial}{\partial t}\hat{u}(t,\xi)$

problem

$\left(\frac{K_{t+h}(y) - K_t(y)}{h}\right)u_0(x-y)e^{-2\pi i x\xi}\,dx\,dy$

$$\int\left(\lim_{h\to 0}\frac{u(t+h,x) - u(t,x)}{h}\right)\cdot e^{-2\pi i x\xi}\,dx = \lim_{h\to 0}\int\left(\frac{u(t+h,x) - u(t,x)}{h}\right)e^{-2\pi i x\xi}\,dx$$