H. A.

Thm 4.1 $T: S \to S$ lin., s.t. $TM_j = M_j T$, $TD_j = D_j T$ $\forall j$

then $T = c \cdot I$

Lem. 4.1. $f \in S$ & $f(y) = 0 \Rightarrow \exists f_1, \dots, f_n \in S$ s.t. $f(x) = \sum_{j=1}^{n} (x_j - y_j) f_j$

Cor: $F: S \mapsto S$, Four. trf., $Rf(x) := f(-x)$, then

$$RF^2 = I$$

Pf: • Let $T = RF^2$. Then $TM_j = M_j T$, $TD_j = D_j T \Rightarrow T = cI$

• Let $f(x) = e^{-\pi|x|^2}$. Then $Tf = f \Rightarrow c = 1$

Four. Inversion on S Let $f \in S$, then

• $RF^2 f(x) = f(x) \Rightarrow F(Ff)(x) = f(-x)$

• Write $\hat{f} := Ff \Rightarrow \int e^{-2\pi i x \cdot \xi} \hat{f}(\xi) = f(-x) \Rightarrow f(x) = \int e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi$

Four. Inversion for $f \in L^1$ s.t. $\hat{f} \in L^1$ Let $g(x) = e^{-\pi|x|^2}$

Consider formally Then $\hat{g}(\xi) = e^{-\pi|\xi|^2}$, $\int \hat{g}(\xi) d\xi = 1$

$$\int \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi = \iint f(y) \underbrace{e^{2\pi i (x-y)\cdot \xi}}_{\delta(x-y)} d\xi dy \quad \text{not abs. conv.}$$

So make it abs. conv. by embedding $= g(\varepsilon x)$

Note $g(\varepsilon x) \to g(0) = 1$, $\forall x$ as $\varepsilon \to 0$

$\mathcal{S}(\mathbb{R}^n)$ – Schw. Sp

$f \in \mathcal{S}$ if $\quad \| f \|_{\alpha\beta} = \| x^\alpha D^\beta f \|_\infty < \infty, \ \forall \alpha, \beta$

Prop 3.2 $\mathcal{S}$ is an algebra

Prop 3.3 • $M_\alpha : \mathcal{S} \to \mathcal{S}$ con't

• $D_\beta : \mathcal{S} \to \mathcal{S}$ con't

Note . $f \in \mathcal{S} \iff \forall \beta \, \forall N \quad | D^\beta f | \le C (1 + |x|^2)^{-N}$

. $f \in \mathcal{S} \implies x^\alpha D^\beta f \to 0$ as $|x| \to \infty$

Prop 3.4 $\quad C_0^\infty \subseteq \mathcal{S}$ is dense (in the top. of $\mathcal{S}$)

Pf $\quad \| \phi_k f - f \|_{\alpha\beta} \le \underbrace{\| \phi_k x^\alpha D^\beta f - x^\alpha D^\beta \phi_k f \|_\infty}_{(\mathrm{I})} + \underbrace{\| \phi_k x^\alpha D^\beta f - x^\alpha D^\beta f \|_\infty}_{(\mathrm{II})}$

(I) Leibnitz rule $\le \frac{C}{k} \sum_{0 < \gamma \le \beta}' \| x^\alpha D^{\beta - \gamma} f \|_\infty$

(II) $\le \phi_k (1 + |x|^2)^{-1} \to 0$

Lemma 3.5 $\quad$ Let $y \in \mathbb{R}^n$, $f \in \mathcal{S}$ st. $\phi(y) = 0$. Then $\exists \, \phi_j \in \mathcal{S}$

s.t. $\quad f(x) = \sum (x_j - y_j) f_j(x)$

Pf: • $g(t) := f(y + t(x-y))$; $g(1) = \int_0^1 g'(t) \, dt \Rightarrow \sum_j (x_j - y_j) g_j(x)$

$= \sum (x_j - y_j) \ldots (x)$

• $x \ne y$: $\quad h_j(x) = \phi(x) \dfrac{x_j - y_j}{|x-y|^2}$

• $f_j(x) = \varphi(x) \, g_j(x) + (1 - \varphi(x)) \, h_j(x)$, φ cut-off at y

• $f_j \in \mathcal{S}$, $\quad f = \sum (x_j - y_j) f_j$

Cl 4/2

$$\int \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi = \lim_{\varepsilon\to 0} \int \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, g(\varepsilon\cdot\xi)\, d\xi \quad \text{(by LDC as } \hat{f}\in L^1)$$

$$= \lim_{\varepsilon\to 0} \iint f(y)\, e^{2\pi i (x-y)\cdot\xi}\, g(\varepsilon\xi)\, d\xi\, dy \quad \text{(by Fubini)}$$

$$= \lim_{\varepsilon\to 0} \int f(y)\, \varepsilon^{-n} g\left(\frac{x-y}{\varepsilon}\right) dy \;\; (= f * g_\varepsilon) \; ; \text{ let } z = \frac{x-y}{\varepsilon} \text{ so } y = x - \varepsilon z, \; dy = \varepsilon^n dz$$

$$= \lim_{\varepsilon\to 0} \int f(x-\varepsilon z)\, g(z)\, dz \overset{LDC}{=} f(x)\int g(z)\,dz = f(x)$$

$\forall z$: $f(x - \varepsilon z) \to f(x)$ as $\varepsilon \to 0$ $\square$

Note $\int f(x)\hat{\Phi}(x)\,dx = \int \hat{f}(\xi)\phi(\xi)\,d\xi \le C\|\phi\|_\infty \le C\|\hat{\Phi}\|_1$

Since $F : S \mapsto S$ is 1-1 and onto, we have

$\int f(x)\phi(x)\,dx \le C\|\phi\|_1$ Suppose $\exists E$ bdd, s.t. $m(E)>0$ and $\forall x\in E$ $f(x) \ge c_1 > c$

Then $\int f(x) 1_E(x) \ge c \|1_E\|_1$

Let $\phi_k \in S(\mathbb{R}^n)$ s.t. $\phi_k \xrightarrow{L^1} 1_E$, $0\le\phi_k\le 1 \Rightarrow \int f(x)\phi_k(x) \to \int f(x)1_E(x)$

Approx of identity: Let $\phi\in S$, $\phi\ge 0$, $\int_{\mathbb{R}^n}\phi = 1$, and let $\phi_\varepsilon(x) = \varepsilon^{-n}\phi\left(\frac{x}{\varepsilon}\right)$

$\{\phi_\varepsilon\}$ called an approx of identity; $\int\phi_\varepsilon = 1$ $\forall\varepsilon$

Lemma 4.2. 1. If $f\in C_0$ (con't, $f(x)\to 0$ as $|x|\to\infty$) then $f*\phi^\varepsilon \to f$ uniformly as $\varepsilon\to 0$

$\phi_\varepsilon \overset{?}{\longrightarrow} \delta_0$ Dirac delta mass

2. If $1\le p<\infty$, then $f*\phi^\varepsilon \xrightarrow{L^p} f$ (HW)

Cor 4.1 Note $f, g \in S \Rightarrow \int \hat{f} g = \int f \hat{g}$

(1) If $f \in L^1$ and $\hat{f} = 0 \Rightarrow f = 0$

(2) Plancherel (1st version) If $f, g \in S$ then

$$\int f(x)\overline{g}(x)\,dx = \int \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi \quad \text{i.e. } \langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$$

Pf Let $\tilde{f}(x) := \overline{f(-x)}$, then $\hat{\tilde{f}} = \dots = \overline{\hat{f}}$

First notice $\int \hat{f} \cdot g = \int f \cdot \hat{g}$, then

$$\int f(x)\,\overline{g(x)}\,dx = \int \hat{\hat{f}}(-x)\,\overline{g(x)} = \int \hat{\hat{f}}(x)\,\overline{g(-x)}\,dx$$

$$= \int \hat{\hat{f}}(x)\,\tilde{g}(x)\,dx = \int \hat{f}(x)\,\hat{\tilde{g}}(x)\,dx = \int \hat{f}(x)\,\overline{\hat{g}(x)}$$

Plancherel (2nd version) There is a unique bdd lin. op: $F: L^2 \to L^2$

s.t. $Ff = \hat{f}$ when $f \in S$. Moreover

1. F is a unitary operator (i.e. $\|Ff\|_{L^2} = \|f\|_{L^2}$ $\forall f \in L^2$
   $Ff = 0 \Rightarrow f = 0$)
2. $Ff = \hat{f}$, for $f \in L^1 \cap L^2$

Pf: . $S \subseteq L^2$ is dense, as $S \subseteq_\varepsilon C_0$ is dense in L^2 - theorem.

(using e.g. $C \subseteq L^2$ dense)

- If $f_k \xrightarrow{L^2} f$, then $\{\mathcal{F}f_k\}$ C-seq. $\Rightarrow \mathcal{F}f := \lim \mathcal{F}f_k$
- $\|\mathcal{F}f\|_2 = \lim \|\mathcal{F}f_k\|_2 = \lim \|f_k\|_2 = \|f\|_2 \Rightarrow \mathcal{F}$ is isometry
- $M = F(L^2)$ closed & dense

(also $\langle \mathcal{F}f, \mathcal{F}g\rangle = \langle f, g\rangle$)

Background from real anal.

1. Approx. of Id
2. C-S, Hölder, $\|f\|_p$, Minkowski
3. Convolution $\|f*g\|_p \le \|f\|_p \|g\|_1$, $1 \le p \le \infty$

Pf (Plancherel II)

Lemma Let $f \in L^1_{loc}$. Then $\exists \{g_k\} \subseteq C_0^\infty$ st.

if $p \in [1,\infty]$, $f \in L^p$ then $g_k \xrightarrow{L^p} f$. If $f \in C_0$ then $g_k \xrightarrow{u} f$

Pf: Let $\psi \in C_0^\infty$, $\psi \ge 0$, $\int \psi = 1$, and let ϕ be a cut-off.

Let $\varepsilon_k \searrow 0$ and define: $g_k(x) = \phi(\frac{x}{k})(\psi_\varepsilon * f)(x)$

$f \in L^p$

- $\|\psi_\varepsilon * f\|_p \le \|f\|_p \|\psi_\varepsilon\|_1 = \|f\|_p$

$\Longrightarrow \|g_k - f*\psi_\varepsilon\|_{L^p} \le \|f*\psi_{\varepsilon_k}\|_{L^p\{|x|\ge k\}} \longrightarrow 0$ by LDC

- $\|f - f*\psi_{\varepsilon_k}\|_p \longrightarrow 0$ by Approx of Id.

Pf (con't)

Let $f \in L^1 \cap L^2$, Let $\{g_k\} \subseteq C_0^\infty$ st. $g_k \xrightarrow{L^1} f$, $g_k \xrightarrow{L^2} f$,

then $F g_k = \hat{g}_k$ as $g_k \in S$ and $\hat{g}_k \xrightarrow{unif} \hat{f}$ also $\hat{g}_k \xrightarrow{L^2} Ff$

$\Longrightarrow Ff = \hat{f}$ (why?) b/c

H. A. 2/4/5

<u>Note</u> This extends the $\mathcal{F}$-trf to L^1+L^2.

If $g \in L^1+L^2$ then $g = f_1+f_2$; define $Fg = Ff_1 + Ff_2$

- $g = f_1+f_2 = h_1+h_2 \Rightarrow f_1-h_1 = h_2-f_2 \in L^1 \cap L^2$

$\Rightarrow F(f_1-h_1) = F(h_2-f_2) \Rightarrow F(f_1+f_2) = F(h_1+h_2)$

$\Rightarrow$ well-def.

- $g \in L^1+L^2$, $Fg=0 \Rightarrow g=0$

<u>Pf:</u> $F(f_1+f_2)=0 \Rightarrow Ff_1 = F(-f_2) \in L^2 \Rightarrow f_1 \in L^2 \Rightarrow f_1 = -f_2$

- We'll write $Ff = \hat{f}$ for $f \in L^1+L^2$

- $f \in L^1+L^2 \Leftarrow f \in L^p \quad \forall \; 1 \le p \le 2$.

<u>Indeed:</u> Let $f \in L^p$, let $f_1 = f \cdot \mathbb{1}_{\{x;\, |f(x)| \le 1\}}$, $f_2 = f \cdot \mathbb{1}_{\{x,\, |f(x)|>1\}}$

Then $f = f_1+f_2$; $|f_1(x)| \le 1 \Rightarrow |f_1(x)|^2 \le |f_1(x)|^p \le |f(x)|^p$ as $p \le 2$

$\Rightarrow f_1 \in L^2$

$|f_2(x)| > 1 \Rightarrow |f_2(x)| \le |f_2(x)|^p \le |f(x)|^p \Rightarrow f_2 \in L^1$

<u>Note</u> • Since $S(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ is dense most prop. of F-trf remain true for $f \in L^1+L^2$; e.g. $\widehat{f \circ T} = |\det(T)|^{-1} \hat{f} \circ T^{-t}$

- $h_\alpha(x) = |x|^{-\alpha} \in L^1+L^2$ if $\frac{n}{2} < \alpha < n$, but $h_\alpha \notin L^p \quad \forall \, 1 \le p \le 2$.

We extend the definition of F trf to gen. funct. or <u>distribution</u>