Harm. Anal. : Schwarz space

$f: \mathbb{R}^n \to \mathbb{C}$ is Schw. funct, we say $f \in S$ if $\underbrace{\forall \alpha, \beta \text{ ind.}}_{}$ $f \in C^\infty$ and

$$\sup_x |x^\alpha D^\beta f| < \infty$$

and def. the norm $\|f\|_{\alpha\beta} := \|x^\alpha D^\beta f\|_\infty$

Converge. in S ; $\{f_k\} \in S$ conv. to $f \in S$ if $\|f_k - f\|_{\alpha\beta} \to 0$ as $k \to \infty$ $\forall \alpha, \beta$

Note • $C_0^\infty \subseteq S$ • $e^{-\pi|x|^2} \in S$ • $p(x) e^{-\pi|x|^2} \in S$

Properties

1. $f, g \in S \Rightarrow cf, f \pm g, f \cdot g \in S$ (HW) ← cont in the top. of S
2. $f \in S \Rightarrow D^\beta f \in S$, $f \in S \Rightarrow p(x) f \in S$ $p(x)$ pol.

i.e. $f_k \xrightarrow{S} f$, $g_k \xrightarrow{S} g$ $\Rightarrow$ $f_k g_k \xrightarrow{S} f \cdot g$

$D^\beta f_k \to D^\beta f$

Remark The foll. are equiv.

(i) $f \in S \iff \forall \beta\ \forall N \in \mathbb{N} : |D^\beta f| \leq C_{\beta,N} (1 + |x|^2)^{-N}$

• $f \in S \iff \forall \alpha, \beta$ $x^\alpha D^\beta f(x) \to 0$ as $|x| \to \infty$

Pf $\Leftarrow$ $|x_j| \leq |x| \leq 1 + |x|^2 \Rightarrow |x^\alpha| \leq (1 + |x|^2)^{|\alpha|}$ (ii) $|x|^2 |x^\alpha D^\beta f| \leq C$

$\Rightarrow \underset{(1+x_1^2 + \dots + x_n^2)^N}{\overset{\shortparallel}{(1+|x|^2)^N}} = \sum_{|\alpha| \leq 2N} c_\alpha x^\alpha \quad \left(= \sum_{|\alpha| \leq N} c_\alpha x^{2\alpha} \right)$ $|x^\alpha D^\beta f| \leq \frac{C}{1+|x|^2} \to 0$

<u>Prop</u> $C_0^\infty \subseteq S$ in dense, i.e. $\forall f \in S$ $\exists \{f_k\} \subseteq C_0^\infty$ s.t. $f_k \xrightarrow{S} f$.

<u>Pf</u> Let $f_k = \phi_k f$, $\phi_k = \phi(x/k)$. Then want to sh

$$\|\phi_k f - f\|_{\alpha\beta} = \|x^\alpha (D^\beta(\phi_k f) - x^\alpha D^\beta f)\|_\infty \to 0 \text{ as } k \to \infty$$

$$= \underbrace{\|\phi_k (x^\alpha D^\beta f) - x^\alpha D^\beta f\|_\infty}_{(I)} + \underbrace{\| x^\alpha D^\beta (\phi_k f) - x^\alpha \phi_k D^\beta f\|_\infty}_{(II)}$$

We have $|x^\alpha D^\beta f| \le \frac{C_{\alpha\beta}}{1+|x|^2} \Rightarrow (I) \le \frac{C_{\alpha\beta}}{1+k^2} \to 0$

(II) $\|\sum_{0 \le \gamma < \beta} x^\alpha D^\gamma f \cdot D^{\beta-\gamma} \phi_k \|_\infty \le \sum_{0 \le \gamma < \beta} \|x^\alpha D^\gamma f\|_\infty \frac{c}{k} \to 0$

$\square$

<u>Thm</u> S is dense in L^p for all $1 \le p < \infty$.

<u>Note</u> $f \in L^p$ if $\|f\|_\infty := \left(\int |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty$

<u>Pf</u> Let $f \in L^p$ then $f\phi_k \xrightarrow{L^p} f$ by LDC.

$\Rightarrow$ Can assume $f \in L^p$ and supp f compact $\Rightarrow f \in L^1$

$$\left(\int_{|x| \le R} |f(x)| \, dx \le \int_{B_R \cap \{|f| \le 1\}} |f(x)| + \int_{B_R \cap \{|f| > 1\}} |f(x)| \le |B_R| + \int |f(x)|^p dx \right.$$

$$\le |B_R| + \|f\|_p^p < \infty$$

Note Math 8100: $C_0(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ is dense in L^p

$1 \le p \le \infty$

Lemma $f \in L^p, g \in L^1 \Rightarrow f * g \in L^p$ and $\|f * g\|_{L^p} \le \|f\|_{L^p} \|g\|_{L^1}$

Pf

- $p=2$. can assume $f \ge 0, g \ge 0$ as $\|f\|_{L^p} = \| |f| \|_{L^p}$.

 Can assume $\|g\|_1 = 1$ (or $\|g\|_1 = 0 \Leftrightarrow g = 0 \Rightarrow$ triv)

 So $g \ge 0$, $\int_{\mathbb{R}^n} g = 1$

$$|f * g(x)|^2 = \left| \int f(x-y) g(y)\, dy \right|^2 = \left| \int f(x-y) g(y)^{\frac{1}{2}} g(y)^{\frac{1}{2}} \right|^2 \le$$

$$\overset{C-S}{\le} \int f(x-y)^2 g(y) \cdot \int g(y) = \int f(x-y)^2 g(y)\, dy$$

$$\Rightarrow \|f * g\|_2^2 \le \iint f(x-y)^2 g(y)\, dy\, dx = \|f\|_2^2$$

- $1 \le p < \infty$ $\quad |f * g(x)|^p = \left| \int f(x-y) g(y)^{\frac{1}{p}} g(y)^{\frac{1}{q}} dy \right|^p$; $\frac{1}{p} + \frac{1}{q} = 1$

$$\overset{\text{Hölder}}{\le} \left(\int f(x-y)^p g(y)\, dy \right) \left(\int g(y)\, dy \right)^{\frac{p}{q}} \le \int f(x-y)^p g(y)\, dy \Rightarrow \checkmark$$

Lemma $C_0^\infty \subseteq C_0$ dense in L^p for all $1 \le p < \infty$.

Pf: Let $\phi_\varepsilon(x) = \varepsilon^{-n} \phi\left(\frac{x}{\varepsilon}\right)$, $f \in C_0$ Let $f_\varepsilon = f * \phi_\varepsilon \in C_0^\infty$

Then $|(f - f_\varepsilon)(x)|^p \le \left(\int |f(x) - f(x-y)| \phi_\varepsilon(y)\, dy \right)^p$

$$= \left(\int |f(x) - f(x-y)| \phi_\varepsilon(y)^{\frac{1}{p}} \phi_\varepsilon(y)^{\frac{1}{q}} \right)^2 \le \int |f(x) - f(x-y)|^p \phi_\varepsilon(y)\, dy$$

$\Rightarrow \|f - f_\varepsilon\|_p^p \le \int \phi_\varepsilon(y) \left(\int |f(x) - f(x-y)|^p dx \right) dy$ Let $\delta > 0$; du ε: $|f(x) - f(y)| \le d$

Fourier inversion

$\widehat{D^\beta f}(\xi) = (2\pi i \xi)^\beta \hat{f}(\xi)$ ← ibbl

$\widehat{x^\alpha f}(\xi) = (-2\pi i)^{-|\alpha|} D^\alpha \hat{f}(\xi)$ ← der

↳ int. by parts

$\int x f(x) e^{-2\pi i x \cdot \xi}$

$\int f(x)$

Lemma 3. $f \in S \Rightarrow \hat{f} \in S$

Pf: $\| \widehat{x^\alpha D^\beta f} \|_\infty \le \| x^\alpha D^\beta f \|_1 \le C'_{\alpha\beta}$ (← why)

$\Rightarrow \quad \widehat{x^\alpha \underbrace{D^\beta f}_{g}} = (-2\pi i D)^{-|\alpha|} \widehat{D^\beta f}(\xi) = 2\pi i D^\alpha$

$\widehat{x^\alpha g}(\xi) = (2\pi i)^{-|\alpha|} D_\xi^\alpha \hat{g}(\xi)$; $\hat{g}(\xi) = \widehat{D^\beta f}(\xi) = (2\pi i)^{|\beta|} \xi^\beta \hat{f}(\xi)$

$\Rightarrow \quad \widehat{x^\alpha D^\beta f}(\xi) = (2\pi i)^{-|\alpha|} (-2\pi i)^{|\beta|} D_\xi^\alpha \xi^\beta \hat{f}(\xi)$

$\Rightarrow \quad \forall \alpha, \beta \quad | D_\xi^\alpha (\xi^\beta \hat{f}(\xi)) | \le C_{\alpha,\beta}$

Trick Use the fact that $D^\beta x^\alpha f \in S \Rightarrow \| D^\beta x^\alpha f \|_1 \le C_{\alpha\beta}$

$\Rightarrow \| \xi^\beta D^\alpha \hat{f} \|_\infty \le C_{\alpha,\beta}$

$\rightarrow \hat{f} \in S(\mathbb{R}^n)$

Lemma $f, g \in S \Rightarrow f * g \in S$ $\square$

Pf $D^\alpha (f * g) = (D^\alpha f) * g$, $D^\alpha f \in S$, $g \in S$

$(D^\alpha f) * g(x) = \int D^\alpha f(x-y) g(y)\, dy \le$

$\le \int_{|y| \le \frac{|x|}{2}} \underbrace{|D^\alpha f(x-y)|}\, \underbrace{|g(y)|}_{\le C} dy + \int_{|y| \ge \frac{|x|}{2}} |D^\alpha f(x-y)|\, |g(y)|$

$N \geq 2n$

$$|I| \leq \int_{|y| \leq \frac{|x|}{2}} (1+|x-y|)^{-N} dy \leq C\left(1+\frac{|x|}{2}\right)^{-N}\left(\frac{|x|}{2}\right)^{n} \leq \left(1+\frac{|x|}{2}\right)^{-n} \leq C_n (1+|x|)^{-n}$$

$$|II| \leq \int_{|y| \geq \frac{|x|}{2}} (1+|y|)^{-2n-1} dy \leq \int_{|y| \geq \frac{|x|}{2}} \left(1+\frac{|x|}{2}\right)^{-n} (1+|y|)^{-n-1} \leq \left(1+\frac{|x|}{2}\right)^{-n} \int (1+|y|)^{-n-1}$$

$$\lesssim \left(1+\frac{|x|}{2}\right)^{-n}$$

□

<u>Fourier inversion</u> Suppe $f \in L^1$, $\hat{f} \in L^1$ then for a.e. x

$$f(x) = \int \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

<u>Pf</u>: • $G(x) = e^{-\pi|x|^2}$, $G_\varepsilon(\xi) = G(\varepsilon\xi) = e^{-\pi\varepsilon^2|\xi|^2}$

by scaly $\hat{G}_\varepsilon(\xi) = \varepsilon^{-n}\hat{G}(\varepsilon^{-1}\xi) = \varepsilon^{-n} e^{-\pi\frac{|\xi|^2}{\varepsilon^2}}$

• Duality $\exists f, g \in L^1$ th. $G_\varepsilon(x) = \varepsilon^{-n} G(\frac{x}{\varepsilon})$, $\hat{G}_\varepsilon(\xi) = \hat{G}(\varepsilon\xi) \to 1$ as $\varepsilon \to 0$

$$\int \hat{f}(x) g(x) dx = \int f(x) \hat{g}(x) dy$$ note: Both Int abs a

<u>Pf</u> $\iint f(y) e^{-2\pi i x\cdot y} g(x) dy\, dx \overset{\text{Fubini}}{=} \text{---}$

• Damping, As $\hat{f} \in L_1$

$$I_\varepsilon(x) = \int \hat{f}(\xi) e^{2\pi i x\cdot\xi} G(\varepsilon\xi) d\xi \longrightarrow \int \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi = I(x)$$

<u>Now</u>: ~~Gnf~~ $\left[e^{2\pi i x\cdot\xi} G(\varepsilon\xi)\right]^\wedge(y) = G_\varepsilon(y-x) \Rightarrow$

CC 3/6

$$\Rightarrow \quad I_\varepsilon = \int f(y)\left[e^{2\pi i x\cdot\xi} G_\varepsilon\right]^\wedge(y)\, dy =$$

$$= \int f(y)\, \hat{G}_\varepsilon(y-x)\, dy = (f * \hat{G}_\varepsilon)(x)$$

We just need to show that $f * \hat{G}_\varepsilon \xrightarrow{L^1} f$ ~~for a.e. x~~.

Alegbraic pf of Fourier inversion

Lemma Let $f \in C^\infty$ and $f(0)=0$ then $\exists\, f_j \in C^\infty$ s.t.

$$f(x) = \sum_{j=1}^n x_j f_j(x)$$

Pf: $f(x) = f(x) - f(0) = g(1) - g(0) = \int_0^1 \frac{d}{dt} g(t)\, dt = \int_0^1 \frac{d}{dt} f(tx)\, dt =$

let $g(t) = f(tx)$

$t|x| \gtrsim |x|^\varepsilon$

$t|x| \leq |x|^\varepsilon \Rightarrow t \leq |x|^{\varepsilon-1}$

$$= \int_0^1 \sum_{j=1}^n \partial_{x_j} f(tx)\, x_j = \sum_{j=1}^n x_j f_j(x) \quad \Big| \quad \phi_j(x) = \int_0^1 D_j\phi(ty)\, dt$$

$\lesssim (1+|x|)^{-n}$

Cor: If $\phi \in C^\infty$ and $\phi(y)=0$ then $\phi(x) = \sum (x_j - y_j)\, \phi_j(x)$

Lem Let $T: S \to S$ lin. op, st. $T D_j \phi = D_j T\phi \quad \forall \phi \in S$

$$T x_j \phi = x_j T\phi$$

$\Rightarrow T\phi = c\cdot\phi$ for some const c.

Pf: (1) $\phi(x) = 0 \Rightarrow T\phi(x) = 0$

(2) $(\phi - \phi(x))(x) = 0$

$\phi_0(x) = 1$

$\phi(x) = 0 \Rightarrow (T\phi)(x) = 0$

$(\phi - \phi(x)\phi_0)(x) = 0$

Let ϕ_0 be a fun $\phi_0(x) = 1$

$\phi(x) - c = 0$

$(T\phi - \phi(x) T\phi_0)(x) = 0$

$T(\phi(x) - c) = 0$

$\Rightarrow (\phi - \phi(x)\phi_0)(x) = 0$

$\Rightarrow (T\phi - \phi(x) T\phi)(x) \quad T\phi(x) - \phi(x)\,(T\phi_0)(x) \quad T\phi(x) = \phi(x)\,(T\phi_0)(x) \Rightarrow T\phi(x) = c(x)\,\phi(x)$